



# Efficient Entanglement Measure for Graph States

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## Abstract

In this paper, we study the multipartite entanglement properties of graph states up to seven qubits. Our analysis shows that the generalized concurrence measure (GCM) is more efficient than geometric entanglement measure (GEM) for measuring entanglement quantity in the multi-qubit graph states.

**Keywords** Entanglement · Graph state · Efficient measure

## 1 Introduction

Entanglement is one of the most striking features of quantum mechanics, which has many applications in quantum information theory such as quantum teleportation [1–5], quantum cryptography [6], quantum dense coding [7], and quantum computing [8–10]. Up to now, different measures have been introduced for measuring entanglement quantity in bipartite and multipartite systems, however, no one can claim which measure would be more efficient than the others yet [11]. Since each measure has advantages over the others in various conditions [11–16].

The entangled states have a fundamental role in multipartite systems for application in quantum information processing and communications as well. Graph states [17, 18] are a type of  $n$ -qubit pure state with various applications in quantum information processing [19–38]. An  $n$ -qubit graph state  $|G\rangle$  is a pure state associated to a graph  $G = (V, E)$ . The graph  $G = (V, E)$  gives both a recipe for preparing  $|G\rangle$  and mathematical characterization of  $|G\rangle$ . Graph states play several significant roles in quantum information theory, e.g. in quantum computation, quantum error-correction, quantum simulation, entanglement distillation protocols, multipartite purification schemes, GHZ or all-versus-nothing proofs of Bell's theorem. Graph states are essential for quantum communication protocols, including teleportation, entanglement-based quantum key distribution, secret sharing, and reduction of communication complexity. The graph state formalism is a useful concept which authorizes a detailed classification of  $n$ -qubit entanglement.

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The authors [39] detected that the serial application of a transformation is sufficient to generate the complete equivalence class of graph states under local unitary (LU) operations within the Clifford group; this simple transformation is Local Complementation (LC). All local complementary (LC) equivalent graph states have the same values of entanglement, accordingly only LC inequivalent graph states should be considered concerning with the entanglement [17, 18, 40]. The number of non-isomorphic and non-LC-equivalent connected graphs up to seven qubits is 45, which we here calculate their entanglement by two useful entanglement criteria.

So far, many papers have been published on graph states and hypergraph states in which the authors used geometric entanglement measure (GEM) for measuring the amount of entanglement [41–47]. In this work, we demonstrate that not only the GEM is not an efficient entanglement measure, but also it is actually an unsuitable measure for measuring entanglement quantity of multi-qubit graph states. This work is structured as follows: Section 2 is dedicated to the expression of basic concepts about the graph states. In Section 3, we present the generalized concurrence measure (GCM) and GEM which are used in this work as multipartite entanglement measures. Section 4 is devoted to the classification of graph states up to seven qubits under non-isomorphic and non-LC-equivalent connected graphs and multipartite entanglement measures. Finally, we conclude and summarize in the last section.

## 2 Graph States and Definitions

An  $n$ -qubit graph state corresponding to the graph  $G = (V, E)$  is given by the following equation [17, 18]

$$|G\rangle = \prod_{\{i,j\} \in E} CZ_{ij}|+\rangle_x^{\otimes n}, \quad (1)$$

where  $|+\rangle_x = \frac{1}{\sqrt{2}}\{|0\rangle + |1\rangle\}$  with basis vectors  $|0\rangle \doteq \binom{1}{0}$  and  $|1\rangle \doteq \binom{0}{1}$  is an eigenstate (eigenvector) of the Pauli matrix  $\sigma_x$  with eigenvalue  $+1$ . Now, for each edge connecting two qubits,  $i$  and  $j$ , it is applied the controlled-Z (CZ) gate between qubits  $i$  and  $j$

$$CZ = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| - |11\rangle\langle 11|. \quad (2)$$

In each  $n$ -qubit system, the number of simple graph states is  $2^{\binom{n}{2}}$ . So for a 3-qubit system, there are 8 graph states, for 4-qubit system there are 64 graph states etc., which their non-isomorphic and non-LC-equivalent connected graphs proposed by Hein et al. are plotted in Ref. [17, 18]. There are: an inequivalent 2-qubit graph, an inequivalent 3-qubit graph, two 4-qubit graphs (No. 3, No. 4), four 5-qubit graphs (No. 5 - No. 8), eleven 6-qubit graphs (No. 9 - No. 19), twenty six 7-qubit graphs (No. 20 - No. 45). Therefore, many graphs can be converted to each other by LU transformations or by permutations of the vertices. Consequently, they have the same value of entanglement [45].

**Definition 2.1 (graph isomorphism)** Two graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  are called isomorphic ( $G_1 \cong G_2$ ) if there is a bijection  $f : V_1 \rightarrow V_2$  is a mapping of a graph onto itself between a set of vertices such that  $\{a, b\} \in E_1$  if and only if (iff)  $\{f(a), f(b)\} \in E_2$  [48–50].

**Definition 2.2 (local complementation)** By local complementation (LC) of a graph  $G = (V, E)$  at some vertex of  $a \in V$  can obtain an LC-equivalent graph state as  $|\tau_a(G)\rangle = U_a^\tau(G)|G\rangle$ , in which

$$U_a^\tau(G) = \exp\left(-i\frac{\pi}{4}\sigma_x^a\right) \prod_{b \in N_a} \exp\left(i\frac{\pi}{4}\sigma_z^b\right), \quad (3)$$

is a local Clifford unitary [17, 18]. Also,  $N_a$  is neighbors of vertex  $a$  and  $\sigma_\alpha$  ( $\alpha = x, z$ ) are the Pauli matrices. Hence, two graph states  $|G\rangle$  and  $|G'\rangle$  are LC-equivalent iff the corresponding graphs are linked by a sequence of local complementations. In the other words, LC centered on a qubit  $a$  is visualized easily as a transformation of the subgraph of the  $a$ th qubit's neighbors, such that one edge between two neighbors of  $a$  is removed if the two neighbors are themselves connected, or one edge is added otherwise [42].

### 3 Multipartite Entanglement Measures

In order to investigate the entanglement depth of the graph states, we first review the entanglement measures. For multipartite systems, several measures of entanglement have been proposed [11]. Here, we use two important multipartite entanglement measures for measuring entanglement quantity of graph states, namely GCM and GEM. First, for an  $n$ -partite pure state  $|\psi\rangle$ , the GCM is defined as [51–54]

$$\text{GCM}(|\psi\rangle) = 2^{1-\frac{n}{2}} \times \left(2^n - 2 - \sum_{\alpha} \text{Tr}\rho_{\alpha}^2\right)^{\frac{1}{2}}, \quad (4)$$

where  $\alpha$  labels as all different subsystems of the  $n$ -partite system and  $\rho_{\alpha}$  are the corresponding reduced density matrices which determined by taking the partial trace of  $\hat{\rho} = |\psi\rangle\langle\psi|$ . For example, for a 3-qubit system, one requires to determine the reduced density matrices  $\rho_1 = \text{Tr}_{23}\hat{\rho}$ ,  $\rho_2 = \text{Tr}_{13}\hat{\rho}$ ,  $\rho_3 = \text{Tr}_{12}\hat{\rho}$ ,  $\rho_{12} = \text{Tr}_3\hat{\rho}$ ,  $\rho_{13} = \text{Tr}_2\hat{\rho}$ , and  $\rho_{23} = \text{Tr}_1\hat{\rho}$ . Second, let us consider GEM, which is defined for pure states as [55–60]

$$\text{GEM}(|\psi\rangle) = 1 - \max_{|\phi\rangle=|a\rangle|b\rangle|c\rangle} |\langle\phi|\psi\rangle|^2, \quad (5)$$

where  $|\phi\rangle = |a\rangle|b\rangle|c\rangle$  is the set of product states. This is the distance between the product state  $|\phi\rangle$  and another state as  $|\psi\rangle$  in terms of fidelity  $F_{\phi} = |\langle\phi|\psi\rangle|^2$ . For mixed states, this entanglement monotone can be extended using the convex roof.

### 4 Efficient Entanglement Measure

We note that up to seven qubits, there are 45 classes of graph states which are not equivalent under one-qubit unitary transformations [17, 18]. We first obtain 45 graph states corre-

sponding to 45 graphs which are all plotted in Ref. [17, 18] and numbered. However, to save space, we only address some of those. Explicitly,

$$\begin{aligned}
 |G_1\rangle &= \frac{1}{\sqrt{2}}\{|0\rangle_1|+\rangle_2 + |1\rangle_1|-\rangle_2\}, \\
 |G_2\rangle &= \frac{1}{\sqrt{2}}\{|0\rangle_1|+\rangle_2|+\rangle_3 + |1\rangle_1|-\rangle_2|-\rangle_3\}, \\
 |G_3\rangle &= \frac{1}{\sqrt{2}}\{|0\rangle_1|+\rangle_2|+\rangle_3|+\rangle_4 + |1\rangle_1|-\rangle_2|-\rangle_3|-\rangle_4\}, \\
 |G_5\rangle &= \frac{1}{\sqrt{2}}\{|0\rangle_1|+\rangle_2|+\rangle_3|+\rangle_4|+\rangle_5 + |1\rangle_1|-\rangle_2|-\rangle_3|-\rangle_4|-\rangle_5\}, \\
 |G_7\rangle &= \frac{1}{2}\{|+\rangle_1|0\rangle_2|+\rangle_3|0\rangle_4|+\rangle_5 + |+\rangle_1|0\rangle_2|-\rangle_3|1\rangle_4|-\rangle_5 \\
 &\quad + |-\rangle_1|1\rangle_2|-\rangle_3|0\rangle_4|+\rangle_5 + |-\rangle_1|1\rangle_2|+\rangle_3|1\rangle_4|-\rangle_5\}, \\
 |G_{12}\rangle &= \frac{1}{2}\{|+\rangle_1|0\rangle_2|+\rangle_3|0\rangle_4|+\rangle_5|+\rangle_6 + |+\rangle_1|0\rangle_2|-\rangle_3|1\rangle_4|-\rangle_5|+\rangle_6 \\
 &\quad + |-\rangle_1|1\rangle_2|-\rangle_3|0\rangle_4|+\rangle_5|-\rangle_6 + |-\rangle_1|1\rangle_2|+\rangle_3|1\rangle_4|-\rangle_5|-\rangle_6\}, \\
 |G_{15}\rangle &= \frac{1}{2}\{|+\rangle_1|+\rangle_2|+\rangle_3|0\rangle_4|+\rangle_5|0\rangle_6 + |-\rangle_1|+\rangle_2|-\rangle_3|0\rangle_4|-\rangle_5|1\rangle_6 \\
 &\quad + |+\rangle_1|-\rangle_2|+\rangle_3|1\rangle_4|-\rangle_5|1\rangle_6 + |-\rangle_1|-\rangle_2|-\rangle_3|1\rangle_4|+\rangle_5|0\rangle_6\}, \\
 |G_{20}\rangle &= \frac{1}{\sqrt{2}}\{|0\rangle_1|+\rangle_2|+\rangle_3|+\rangle_4|+\rangle_5|+\rangle_6|+\rangle_7 + |1\rangle_1|-\rangle_2|-\rangle_3|-\rangle_4|-\rangle_5|-\rangle_6|-\rangle_7\}, \quad (6)
 \end{aligned}$$

where  $|+\rangle_x = \frac{1}{\sqrt{2}}\{|0\rangle + |1\rangle\}$  and  $|-\rangle_x = \frac{1}{\sqrt{2}}\{|0\rangle - |1\rangle\}$  are eigenstates of the Pauli matrix  $\sigma_x$  with eigenvalues  $\pm 1$ . For example, in order to calculate the amount of entanglement in 3-qubit graph state (No. 2) by GCM and GEM, we obtain by direct numerical optimization  $GCM(|G_2\rangle)=1.22474$ , and  $GEM(|G_2\rangle)=0.5$  with the edges  $E_2 = \{\{1, 2\}, \{1, 3\}\}$ . There are 45 set of edges corresponding to these graphs which are reported in Appendix.

Next, we apply two entanglement measures (see (4) and (5)) for all 45 graph states and we enter numerical values in Tables 1 and 2. Hence, we have classified the 45 graphs into 27 and 7 classes, according to the GCM and GEM, respectively.

**Table 1** The classification of non-isomorphic and non-LC-equivalent connected graphs up to seven qubits based on generalized concurrence measure (GCM)

Class	GCM	Graph No.	Class	GCM	Graph No.
1	1	1	15	1.63936	14, 16
2	1.22474	2	16	1.64886	24
3	1.32288	3	17	1.65831	17, 25, 31
4	1.36931	5	18	1.67705	18, 26
5	1.39194	9	19	1.68634	27, 32
6	1.40312	20	20	1.69558	19, 28, 33
7	1.41421	4	21	1.70477	29, 34
8	1.50000	6	22	1.71391	30, 35, 36
9	1.54110	7, 10	23	1.72301	37
10	1.56125	21	24	1.73205	38, 39
11	1.58114	8, 11	25	1.74105	41
12	1.60078	12	26	1.75000	40, 42, 43, 45
13	1.62019	13, 15, 22	27	1.75891	44
14	1.62980	23			

**Table 2** The classification of non-isomorphic and non-LC-equivalent connected graphs up to seven qubits based on geometric entanglement measure (GEM)

Class	GEM	Graph No.
1	0.50000	1, 2, 3, 5, 9, 20
2	0.75000	4, 6, 7, 10-12, 15, 21-24, 31
3	0.86855	8
4	0.87500	13-14, 16-18, 25-30, 32-38, 43
5	0.91667	19
6	0.93428	39, 41, 45
7	0.93750	40, 42, 44

In the last step, we collect the classification results in Table 3. We here should comment about the results in the tables. The resolution power (RP) in Table 3 is computed using the following equation

$$RP = \frac{\eta[\chi]}{\eta[\kappa]} \times 100, \tag{7}$$

where  $\eta[\chi]$  denotes the number of classes based on GCM or GEM and  $\eta[\kappa]$  is the number of categories according to non-isomorphic and non-LC-equivalent connected graphs. Therefore, the RP in Table 3 is the ratio of the number of classifications taken from multipartite entanglement measures (see (4) and (5)) to the number of non-isomorphic and non-LC-equivalent connected graphs. Using the method of comparing the results of these classes with classification of graph states under non-isomorphic and non-LC-equivalent connected graphs, as we call RP method, we found that the GCM seems more efficient than GEM for multi-qubit graph states.

### 5 Conclusion

We propose two novel classifications for the entanglement in graph states up to seven qubits based on generalized concurrence measure (GCM) and geometric entanglement measure (GEM), and we also compare those with the classification under non-isomorphic and non-LC-equivalent connected graphs. Numerical values reveal that the 45 graph states are classified according to the GCM and GEM into 27 and 7 categories, respectively. Accordingly, our results propose that the GCM is a suitable entanglement measure for measuring entanglement quantity of multi-qubit graph states, and it is also more efficient than GEM. The suggested approach (RP method) can be employed to recognize the proper performance of each new measure proposed for measuring entanglement in multi-qubit graph states.

**Table 3** Classification of graph states up to seven qubits based on GCM and GEM

n	$\eta$ [GCM]	$\eta$ [GEM]	$\eta[\kappa]$	RP [GCM]	RP [GEM]
2	1	1	1	100%	100%
3	1	1	1	100%	100%
4	2	2	2	100%	100%
5	4	3	4	100%	75%
6	9	4	11	81.82%	36.36%
7	16	5	26	61.54%	19.23%
Up to 7	27	7	45	60%	15.55%

In the second column, the number of graph states with  $n = 2, 3, 4, 5, 6$  and  $7$  vertices is listed which are categorized by generalized concurrence measure ( $\eta$  [GCM]). In the third column, the number of graph states which are categorized by geometric entanglement measure ( $\eta$  [GEM]). The values in the fourth column (number of non-isomorphic and non-LC-equivalent connected graphs) were computed in Refs. [18] and [61] together with a database of representatives for each equivalence class ( $\eta[\kappa]$ ). In the fifth column and the last column give the results of the resolution power (RP)

We believe that the investigation of graph states is necessary for a better understanding of multipartite systems and validation of multipartite entanglement measures.

## Appendix

In this [Appendix](#), we give the set of edges corresponding to the non-isomorphic and non-LC-equivalent connected graphs up to seven qubits.

**Table 4** The set of edges corresponding to the non-isomorphic and non-LC-equivalent connected graphs

Graph No.	Edges
1	{{1, 2}}
2	{{1, 2}, {1, 3}}
3	{{1, 2}, {1, 3}, {1, 4}}
4	{{1, 2}, {2, 3}, {3, 4}}
5	{{1, 2}, {1, 3}, {1, 4}, {1, 5}}
6	{{1, 2}, {2, 3}, {3, 4}, {2, 5}}
7	{{1, 2}, {2, 3}, {3, 4}, {4, 5}}
8	{{1, 2}, {2, 3}, {3, 4}, {4, 5}, {1, 5}}
9	{{1, 2}, {1, 3}, {1, 4}, {1, 5}, {1, 6}}
10	{{1, 6}, {2, 6}, {3, 6}, {4, 5}, {5, 6}}
11	{{1, 6}, {2, 6}, {3, 5}, {4, 5}, {5, 6}}
12	{{1, 2}, {2, 3}, {3, 4}, {4, 5}, {2, 6}}
13	{{1, 2}, {2, 3}, {3, 4}, {4, 5}, {3, 6}}
14	{{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 6}}
15	{{1, 6}, {2, 4}, {3, 4}, {4, 5}, {5, 6}, {3, 6}}
16	{{1, 2}, {2, 3}, {3, 4}, {4, 5}, {2, 4}, {3, 6}}
17	{{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 1}, {1, 6}}
18	{{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 6}, {1, 6}}
19	{{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 6}, {1, 6}, {1, 3}, {4, 6}, {2, 5}}
20	{{1, 2}, {1, 3}, {1, 4}, {1, 5}, {1, 6}, {1, 7}}
21	{{1, 7}, {2, 7}, {3, 7}, {4, 7}, {5, 6}, {6, 7}}
22	{{1, 7}, {2, 7}, {3, 7}, {4, 6}, {5, 6}, {6, 7}}
23	{{1, 7}, {2, 7}, {3, 7}, {4, 5}, {5, 6}, {6, 7}}
24	{{1, 7}, {2, 7}, {3, 5}, {4, 5}, {5, 6}, {6, 7}}
25	{{1, 2}, {1, 7}, {3, 7}, {4, 7}, {5, 6}, {6, 7}}
26	{{1, 7}, {2, 7}, {3, 6}, {4, 5}, {5, 6}, {6, 7}}
27	{{1, 2}, {2, 7}, {2, 3}, {4, 3}, {5, 4}, {6, 5}}
28	{{1, 2}, {2, 3}, {3, 4}, {3, 5}, {5, 6}, {6, 7}}
29	{{1, 2}, {2, 3}, {3, 4}, {4, 5}, {3, 6}, {6, 7}}
30	{{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 6}, {6, 7}}
31	{{2, 3}, {3, 4}, {4, 5}, {5, 6}, {5, 7}, {1, 3}, {6, 3}}
32	{{1, 7}, {2, 7}, {3, 6}, {4, 5}, {5, 6}, {6, 7}, {5, 7}}
33	{{2, 3}, {3, 4}, {4, 5}, {6, 5}, {7, 6}, {3, 7}, {1, 3}}
34	{{2, 3}, {3, 4}, {4, 5}, {6, 5}, {7, 6}, {3, 6}, {1, 4}}
35	{{2, 3}, {3, 4}, {4, 5}, {6, 5}, {7, 6}, {3, 7}, {1, 6}}
36	{{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 6}, {4, 7}, {3, 5}}
37	{{1, 7}, {2, 3}, {3, 4}, {4, 5}, {5, 6}, {6, 7}, {3, 7}}
38	{{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 6}, {6, 7}, {1, 6}}
39	{{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 6}, {6, 7}, {1, 5}}
40	{{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 6}, {6, 7}, {1, 7}}
41	{{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 6}, {6, 7}, {1, 5}, {1, 6}}
42	{{1, 3}, {2, 3}, {3, 4}, {4, 5}, {5, 6}, {6, 7}, {1, 7}, {2, 6}}
43	{{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 6}, {5, 7}, {1, 4}, {3, 6}}
44	{{1, 4}, {2, 3}, {3, 4}, {4, 5}, {5, 6}, {6, 7}, {1, 7}, {2, 7}, {3, 5}}
45	{{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 6}, {6, 7}, {3, 7}, {2, 7}, {2, 5}, {4, 6}}

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