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Entanglement-Assisted Quantum Negacyclic BCH Codes

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Abstract
The entanglement-assisted quantum error correcting codes (EAQECCs) are a simple and important class of quantum codes. The entanglement-assisted formalism can transform arbitrary classical linear codes into EAQECCs by using pre-shared entanglement between the sender and the receiver. In this paper, by decomposing the defining set of negacyclic BCH codes, we construct a class of new EAQECCs with length $n = \frac{q^{4m}-1}{q^2-1}$.

Keywords Negacyclic codes · BCH codes · EAQECCs

1 Introduction

Quantum error-correcting codes (QECCs) play an important role in quantum information and computation. As we all know, constructing good QECCs is a crucial subject of research [1–8] all the time. Recently, such theory has been extended to EAQECCs. Customarily, an entanglement-assisted quantum error correcting code (EAQECC) can be denoted as $[[n, k, d; c]]_q$, which encodes k information qubits into n channel qubits with the help of c pairs of maximally entangled states and corrects up to $\lfloor \frac{d-1}{2} \rfloor$ errors, where d is the minimum distance of the code. If $c = 0$, then it is called a q-ary standard $[[n, k, d]]$ quantum code. The performance of an EAQECC is measured by its rate $\frac{k}{n}$ and net rate $\frac{k-c}{n}$.

Brun et al. [9] proposed an entanglement-assisted stabilized formalism, which overcame the barrier of the dual-containing condition in constructing standard quantum codes from classical codes. They proved that if shared entanglement is available between the sender

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and the receiver in advance, non-dual-containing classical quaternary codes can be used to construct EAQECCs. Since then, more and more scholars begin to study EAQECCs [10–14].

Hsieh et al. [15] constructed some EAQECCs with good parameters from quasicyclic low-density parity-check codes. Fujiwara et al. [16] used low-density parity-check codes to construct some good parameters' EAQECCs with different lengths soon afterwards. In Refs. [17] and [18], Li et al. proposed the concept about decomposing the defining set of BCH cyclic codes, transformed the problem of calculating the number of share pairs into determining a special subset of the defining set of a BCH code, and constructed some EAQECCs with good parameters. Afterwards, Lü and Li made a further study on constructing of EAQECCs by using primitive quaternary BCH codes with length $n = 4^m - 1$ in Ref. [19]. Recently, Chen et al. [20] generalized their method to apply in negacyclic codes, and obtained four classes of optimal EAQECCs. Lü et al. [21] constructed six classes of q-ary entanglement-assisted quantum MDS codes based on classical negacyclic MDS codes.

Most of them committed themselves to the construction of entanglement-assisted quantum MDS codes, while the larger length case has received less attention. This reality inspires us to construct EAQECCs with the larger length. In this paper, we obtain a class of new EAQECCs by negacyclic BCH codes with length $n = \frac{q^{4m}-1}{q^2-1}$, where q is odd and $m \geq 2$. Speaking specifically, we construct a class of EAQECCs with parameters as follows:

(1) (i) If $\delta \leq \frac{q^{2m}+3}{2}$,

$$\begin{cases} [[n, n-4m\left\lceil(\delta-\frac{3}{2})(1-q^{-2})\right\rceil+4m\varepsilon-mq^2+7m, \geq\delta; 4m]]_q, if\, \varepsilon = 2 or 3, \\ [[n, n-4m\left\lceil(\delta-\frac{3}{2})(1-q^{-2})\right\rceil+mq^2+m, \geq\delta; 4m]]_q, if\, \varepsilon = 4. \end{cases}$$

(ii) If $\frac{q^{2m}+5}{2} \leq \delta \leq q^{2m}$,

$$[[n, n-4m\left\lceil(\delta-\tfrac{3}{2})(1-q^{-2})\right\rceil+mq^2+m, \geq\delta; 4m]]_q,$$

where $q = 3$ and $m \geq 2$.

(2)

$$\begin{cases} [[n, n-4m\lceil(\delta-\frac{3}{2})(1-q^{-2})\rceil+4m, \geq\delta; 4m]]_q, if\, \varepsilon < 6, \\ [[n, n-4m\lceil(\delta-\frac{3}{2})(1-q^{-2})\rceil+4\varepsilon m-mq^2+7m, \geq\delta; 4m]]_q, if\, 6 \leq \varepsilon \leq 11, \end{cases}$$

where $q = 5$ and $m \geq 2$.

(3) $[[n, n-4m\lceil(\delta-\frac{3}{2})(1-q^{-2})\rceil+4m, \geq\delta; 4m]]_q$, where $q \geq 7$ is a power of an odd prime p and $m \geq 2$.

This paper is organized as follows. In Section 2, some basic background and results about negacyclic codes and BCH codes are reviewed. In Section 3, we briefly review some basic definitions and results of EAQECCs. In Section 4, we construct a class of EAQECCs with new parameters. In Section 5, we give an example to illustrate the significance of results in this paper. Section 6 concludes the paper.

2 Preliminaries

Let q be a power of an odd prime p and $\mathbb{F}_{q^2}$ be a finite field with q^2 elements. For any element $a \in \mathbb{F}_{q^2}$, we denote the conjugate a^q of a by $\overline{a}$. Given two vectors $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ and $\mathbf{b} = (b_0, b_1, \ldots, b_{n-1}) \in \mathbb{F}_{q^2}^n$, their Hermitian inner product is defined as

$$\langle \mathbf{a}, \mathbf{b} \rangle = a_0\overline{b}_0 + a_1\overline{b}_1 + \cdots + a_{n-1}\overline{b}_{n-1} \in \mathbb{F}_{q^2}.$$

The vectors $\mathbf{a}$ and $\mathbf{b}$ are called orthogonal with respect to the Hermitian inner product if $\langle \mathbf{a}, \mathbf{b} \rangle = 0$. A q^2-ary linear code $\mathcal{C}$ of length n is a nonempty subspace of the vector space $\mathbb{F}_{q^2}^n$. For a q^2-ary linear code $\mathcal{C}$, the Hermitian dual code of $\mathcal{C}$ is defined as

$$\mathcal{C}^{\perp_h} = \{\mathbf{a} \in \mathbb{F}_{q^2}^n | \langle \mathbf{a}, \mathbf{b} \rangle = 0 \text{ for all } \mathbf{b} \in \mathcal{C}\}.$$

A q^2-ary linear code $\mathcal{C}$ of length n is called Hermitian self-orthogonal if $\mathcal{C} \subseteq \mathcal{C}^{\perp_h}$, and it is called Hermitian self-dual if $\mathcal{C} = \mathcal{C}^{\perp_h}$. If a q^2-ary linear code $\mathcal{C}$ of length n satisfies the property that

$$(-c_{n-1}, c_0, \ldots, c_{n-2}) \in \mathcal{C}, \text{forall}(c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C},$$

then $\mathcal{C}$ is said to be a negacyclic code of length n over $\mathbb{F}_{q^2}$. Customarily, a codeword $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1})$ in $\mathcal{C}$ is identified with its polynomial representation $c(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1}$. It is well known that a q^2-ary negacyclic code of length n is precisely an ideal of the quotient ring $\mathbb{F}_{q^2}[x]/\langle x^n + 1 \rangle$ and $\mathcal{C}$ can be generated by a monic divisor $g(x)$ of $x^n + 1$. The polynomial $g(x)$ is called the generator polynomial of the code $\mathcal{C}$ and the dimension of $\mathcal{C}$ is $n - k$, where $k = \deg(g(x))$.

In the following, we assume q is a power of an odd prime p with $\gcd(n, p) = 1$, where n is a positive integer. Let β be a primitive $2n$-th root of unity in some extension field of $\mathbb{F}_{q^2}$ and $\eta = \beta^2$. Then η is a primitive n-th root of unity. Hence,

$$x^n + 1 = \prod_{j=0}^{n-1}(x - \beta\eta^j) = \prod_{j=0}^{n-1}(x - \beta^{1+2j}).$$

Let $\Omega = \{1 + 2j | 0 \leq j \leq n - 1\}$. For each $i \in \Omega$, let C_i be the q^2-cyclotomic coset modulo $2n$ containing i,

$$C_i = \{i, iq^2, iq^4, \ldots, iq^{2(m_i-1)}\},$$

where m_i is the smallest positive integer such that $iq^{2m_i} \equiv i \mod 2n$. Each C_i corresponds to an irreducible divisor of $x^n + 1$ over $\mathbb{F}_{q^2}$. Let $\mathcal{C}$ be a negacyclic code of length n over $\mathbb{F}_{q^2}$ with generator polynomial $g(x)$. Then the set $Z = \{i \in \Omega | g(\delta^i) = 0\}$ is called the defining set of $\mathcal{C}$. Obviously, the defining set of $\mathcal{C}$ must be a union of some q^2-cyclotomic cosets modulo $2n$ and $\dim(\mathcal{C}) = n - |Z|$.

For a negacyclic code of length n over $\mathbb{F}_{q^2}$, it is easy to verify that its Hermitian dual code is still a negacyclic code. Therefore, the Hermitian dual code $\mathcal{C}^{\perp_h}$ of $\mathcal{C}$ is still an ideal of $\mathbb{F}_{q^2}[x]/\langle x^n + 1 \rangle$. Hence, if Z is the defining set of $\mathcal{C}$, then its Hermitian dual code $\mathcal{C}^{\perp_h}$ has defining set $Z^{\perp_h} = \{z \in \Omega | -qz \bmod 2n \notin Z\}$. Note that $Z^{-q} = \{-qz \bmod 2n | z \in Z\}$. Then $\mathcal{C}$ contains its Hermitian dual code if and only if $Z \cap Z^{-q} = \emptyset$ from Lemma 2.2 in Ref. [22].

Let q be a power of an odd prime p with $\gcd(n, p) = 1$ and β be a primitive $2n$-th root of unity. A negacyclic BCH code of length n over $\mathbb{F}_{q^2}$ with designed distance δ is a negacyclic code with generator polynomial

$$g(x) = \prod_{j \in Z}(x - \beta^j),\ where\ Z = \cup_{j=b}^{b+\delta-2} C_{1+2j}\ and\ b\ is\ some\ integer.$$

Let $\mathcal{C}_{(n,q^2,b,\delta)}$ denote the negacyclic BCH codes of length n with generator polynomial $g(x)$. If $b = 0$, then we abbreviate $\mathcal{C}_{(n,q^2,b,\delta)}$ as $\mathcal{C}_{(n,q^2,\delta)}$. Similarly to BCH codes, negacyclic BCH codes have the following property.

Theorem 1 [24] *(The BCH bound for negacyclic codes) Assume that* $\gcd(n, q) = 1$. *Let $\mathcal{C}$ be a negacyclic code of length n over $\mathbb{F}_{q^2}$, and let its generator polynomial $g(x)$ have elements $\{\beta^{1+2j} | 0 \leq j \leq d-2\}$ as the roots, where β is a primitive $2n$-th root of unity. Then the minimum distance of $\mathcal{C}$ is at least d.*

3 Review of EAQECCs

In this section, we give some basic definitions and results of EAQECCs. For more details about EAQECCs theory, please refer to Refs. [9–21] therein.

Suppose that H is an $(n-k) \times n$ parity check matrix of $\mathcal{C}$ over $\mathbb{F}_{q^2}$. Then $\mathcal{C}^{\perp_h}$ has an $n \times (n-k)$ generator matrix $H^\dagger$, where $H^\dagger$ is the conjugate transpose matrix of H over $\mathbb{F}_{q^2}$.

Similarly to the CSS construction of stabilizer quantum codes, there is the following construction method for EAQECCs in Refs. [9] and [10].

Theorem 2 [9, 10] *If $\mathcal{C} = [n, k, d]_{q^2}$ is a classical code over $\mathbb{F}_{q^2}$ and H is its parity check matrix, then $\mathcal{C}^{\perp_h}$ stabilizes an entanglement-assisted code with parameters $[[n, 2k-n+c, d; c]]_q$, where $c = rank(HH^\dagger)$ is the number of maximally entangled states required and $H^\dagger$ is the conjugate matrix of H over $\mathbb{F}_{q^2}$.*

4 Construction of Entanglement-Assisted Quantum BCH Codes

In Ref. [20], the authors gave the following definition and lemma which can determine the number of entangled states by decomposing the defining set of negacyclic codes.

Definition 1 [20] Let $\mathcal{C}$ be a negacyclic code of length n with defining set Z. Assume that $Z_1 = Z \cap (-qZ)$ and $Z_2 = Z \setminus Z_1$, where $-qZ = \{n - qx | x \in Z\}$. Then $Z = Z_1 \cup Z_2$ is called a decomposition of the defining set of $\mathcal{C}$.

Lemma 1 [20] *Let $\mathcal{C}$ be a negacyclic code of length n over $\mathbb{F}_{q^2}$, where $\gcd(n, q) = 1$. Suppose that Z is the defining set of the negacyclic code $\mathcal{C}$ and $Z = Z_1 \cup Z_2$ is a decomposition of Z. Then the number of entangled states required is $c = |Z_1|$. In order to construct the entanglement-assisted quantum BCH codes, we firstly give two lemmas below.*

Lemma 2 [23] $\mathcal{C}^{\perp_h}_{(n,q^2,\delta)} \subseteq \mathcal{C}_{(n,q^2,\delta)}$ *if and only if* $2 \leq \delta \leq \delta_{max}$, *where* $\delta_{max} = \frac{q^{2m+1}-q}{q^2-1}$.

Lemma 3 [23] *Let $n = \frac{q^{4m}-1}{q^2-1}$ and $m \geq 2$. Let i be an integer such that $0 \leq i \leq q^{2m}$ and $i \not\equiv \frac{q^2-1}{2} \mod q^2$. If $q^2 \equiv 1 \mod 4$, then*

$$|C_{1+2i}| = \begin{cases} m, & i = \frac{q^{2m}-1}{4}, \frac{3q^{2m}+1}{4}, \\ 2m, & otherwise. \end{cases}$$

In addition, $1+2i$ is not a coset leader in the following cases:

$$i \in \left\{b \cdot \frac{q^{2m}-1}{q^2-1} : b \in \left[\frac{q^2+3}{4}, \frac{q^2-3}{2}\right]\right\} \bigcup \left\{\frac{q^{2m}+1}{2} + b \cdot \frac{q^{2m}-1}{q^2-1} : b \in \left[\frac{q^2+3}{4}, \frac{q^2-1}{2}\right]\right\}.$$

4.1 The Number of Entangled States

Theorem 3 *Let $n = \frac{q^{4m}-1}{q^2-1}$, where q is a power of an odd prime p and $m \geq 2$. Then we have $|C_{1+2(\frac{q^{2m+1}-q}{q^2-1}-1)}| = |C_{\frac{2q^{2m+1}-q^2-2q+1}{q^2-1}}| = 2m$ and $|C_{\frac{2q^{4m}-2q^{2m+2}+q^3+2q^2-q-2}{q^2-1}}| = 2m$.*

Proof On the one hand, since
$(q^{2m}-1)(q^2-1) - 4(q^{2m+1}-q^2-q+1) = q^{2m+2} - 4q^{2m+1} - q^{2m} + 3q^2 + 4q - 3$, if $q = 3$, then $(q^{2m}-1)(q^2-1) < 4(q^{2m+1}-q^2-q+1)$; if $q \geq 5$, then $(q^{2m}-1)(q^2-1) > 4(q^{2m+1}-q^2-q+1)$. Therefore, $\frac{q^{2m+1}-q}{q^2-1} - 1 \neq \frac{q^{2m}-1}{4}$.

On the other hand,

$$\frac{3q^{2m}+1}{4} - (\frac{q^{2m+1}-q}{q^2-1} - 1) = \frac{3q^{2m+2} - 4q^{2m+1} - 3q^{2m} + 5q^2 + 4q - 5}{4(q^2-1)},$$

if $q = 3$, then $\frac{3q^{2m}+1}{4} < \frac{q^{2m+1}-q}{q^2-1} - 1$; if $q \geq 5$, then $\frac{3q^{2m}+1}{4} > \frac{q^{2m+1}-q}{q^2-1} - 1$. Therefore, $\frac{q^{2m+1}-q}{q^2-1} - 1 \neq \frac{3q^{2m}+1}{4}$. From Lemma 3, we have $|C_{1+2(\frac{q^{2m+1}-q}{q^2-1}-1)}| = 2m$ immediately.

From $\frac{2q^{4m}-2q^{2m+2}+q^3+2q^2-q-2}{q^2-1} = 2n - q(1+2(\frac{q^{2m+1}-q}{q^2-1}-1))$ and $|C_{1+2(\frac{q^{2m+1}-q}{q^2-1}-1)}| = 2m$, we have $|C_{\frac{2q^{4m}-2q^{2m+2}+q^3+2q^2-q-2}{q^2-1}}| = 2m$ immediately. □

4.2 The Dimension of EAQECCs with $\delta = \frac{q^{2m+1}-q}{q^2-1} + t$

Lemma 4 [23] *Let $n = \frac{q^{4m}-1}{q^2-1}$, where q is odd and $m \geq 2$.*

(i) If $\delta \leq \frac{q^{2m}+3}{2}$, then we define $\varepsilon = \left\lfloor \frac{(\delta-2)(q^2-1)}{q^{2m}-1} \right\rfloor$ and $\mathcal{C}_{(n,q^2,\delta)}$ has dimension

$$k = \begin{cases} n - 2m\left\lceil (\delta - \frac{3}{2})(1-q^{-2}) \right\rceil, if\, \varepsilon < \left\lfloor \frac{q^2-1}{4} \right\rfloor, \\ n - 2m\left\lceil (\delta - \frac{3}{2})(1-q^{-2}) \right\rceil + m(2\varepsilon - \frac{q^2-3}{2}), if \left\lfloor \frac{q^2-1}{4} \right\rfloor \leq \varepsilon \leq \frac{q^2-3}{2}, \\ n - 2m\left\lceil (\delta - \frac{3}{2})(1-q^{-2}) \right\rceil + m\frac{q^2-3}{2}, if\, \varepsilon > \frac{q^2-3}{2}. \end{cases}$$

(ii) *If* $\frac{q^{2m}+5}{2} \leq \delta \leq q^{2m}$, *then we define* $\varepsilon = \left\lfloor \frac{(\delta - \frac{q^{2m}+5}{2})(q^2-1)}{q^{2m}-1} \right\rfloor$ *and* $\mathcal{C}_{(n,q^2,\delta)}$ *has dimension*

$$k = \begin{cases} n - 2m\left\lceil (\delta - \frac{3}{2})(1-q^{-2}) \right\rceil + m\frac{q^2-3}{2}, if \varepsilon < \left\lfloor \frac{q^2-1}{4} \right\rfloor, \\ n - 2m\left\lceil (\delta - \frac{3}{2})(1-q^{-2}) \right\rceil + 2m\varepsilon, if \left\lfloor \frac{q^2-1}{4} \right\rfloor \leq \varepsilon \leq \frac{q^2-3}{2}, \\ n - 2m\left\lceil (\delta - \frac{3}{2})(1-q^{-2}) \right\rceil + m(q^2-3), if \varepsilon > \frac{q^2-3}{2}. \end{cases}$$

In order to calculate the dimension of EAQECCs with $n = \frac{q^{4m}-1}{q^2-1}$ *and* $\delta = \frac{q^{2m+1}-q}{q^2-1} + t$, *where* q *is a power of an odd prime* p, $m \geq 2$ *and* $1 \leq t \leq \frac{(q^{2m-1}+1)(q+3)}{2(q+1)}$, *we need to determine the range of* δ *for distinct* q *from Lemma 4. So we give the following theorem firstly.*

Theorem 4 *Let* $n = \frac{q^{4m}-1}{q^2-1}$ *and* $\delta = \frac{q^{2m+1}-q}{q^2-1} + t$, *where* q *is a power of an odd prime* p, $m \geq 2$ *and* $1 \leq t \leq \frac{(q^{2m-1}+1)(q+3)}{2(q+1)}$.

(i) *If* $q = 3$, *then* $\delta_{min} < \frac{q^{2m}+3}{2}$ *and* $\frac{q^{2m}+5}{2} < \delta_{max} < q^{2m}$.

(ii) *If* $q \geq 5$, *then* $\delta_{max} < \frac{q^{2m}+3}{2}$.

Proof Since $\delta = \frac{q^{2m+1}-q}{q^2-1} + t$ and $1 \leq t \leq \frac{(q^{2m-1}+1)(q+3)}{2(q+1)}$, we have

$$\frac{q^{2m+1}-q}{q^2-1} + 1 \leq \delta \leq \frac{3q^{2m+1} + 2q^{2m} - 3q^{2m-1} + q^2 - 3}{2q^2-2}.$$

From

$$\frac{q^{2m}+3}{2} - \delta_{max} = \frac{q^{2m}+3}{2} - \frac{3q^{2m+1} + 2q^{2m} - 3q^{2m-1} + q^2 - 3}{2q^2-2} = \frac{q^{2m-1}(q^3 - 3q^2 - 3q + 3) + 2q^2}{2q^2-2},$$

if $q = 3$, then $\frac{q^{2m}+3}{2} - \delta_{max} < 0$; if $q \geq 5$, then $\frac{q^{2m}+3}{2} - \delta_{max} > 0$.
When $q = 3$, we have

$$q^{2m} - \delta_{max} = q^{2m} - \frac{3q^{2m+1} + 2q^{2m} - 3q^{2m-1} + q^2 - 3}{2q^2-2} = \frac{q^{2m-1}(2q^3 - 3q^2 - 4q + 3) - q^2 + 3}{2q^2-2} > 0,$$

and

$$\frac{q^{2m}+3}{2} - \delta_{min} = \frac{q^{2m}+3}{2} - (\frac{q^{2m+1}-q}{q^2-1} + 1) = \frac{q^{2m+2} - 2q^{2m+1} - q^{2m} + q^2 + 2q - 1}{2(q^2-1)} > 0,$$

i.e., $\delta_{min} < \frac{q^{2m}+3}{2}$ and $\frac{q^{2m}+5}{2} < \delta_{max} < q^{2m}$. □

From the discussion above, we can determine the dimension of the negacyclic BCH codes with length $n = \frac{q^{4m}-1}{q^2-1}$, where q is a power of an odd prime p and $m \geq 2$.

Theorem 5 *Let* $n=\frac{q^{4m}-1}{q^2-1}$ *and* $\delta=\frac{q^{2m+1}-q}{q^2-1}+t$, *where* $q=3$, $m\geq 2$ *and* $1\leq t\leq \frac{(q^{2m-1}+1)(q+3)}{2(q+1)}$.

(i) *If* $\delta\leq\frac{q^{2m}+3}{2}$, *then* $\varepsilon=\left\lfloor\frac{(\delta-2)(q^2-1)}{q^{2m}-1}\right\rfloor$ *and* $\mathcal{C}_{(n,q^2,\delta)}$ *has dimension*

$$k=\begin{cases} n-2m\left\lceil(\delta-\frac{3}{2})(1-q^{-2})\right\rceil+m(2\varepsilon-\frac{q^2-3}{2}), if\, \varepsilon=2\ or\ 3,\\ n-2m\left\lceil(\delta-\frac{3}{2})(1-q^{-2})\right\rceil+m\frac{q^2-3}{2}, if\, \varepsilon=4.\end{cases}$$

(ii) *If* $\frac{q^{2m}+5}{2}\leq\delta\leq q^{2m}$, *then* $\varepsilon=\left\lfloor\frac{(\delta-\frac{q^{2m}+5}{2})(q^2-1)}{q^{2m}-1}\right\rfloor$ *and* $\mathcal{C}_{(n,q^2,\delta)}$ *has dimension* $k=n-2m\left\lceil(\delta-\frac{3}{2})(1-q^{-2})\right\rceil+m\frac{q^2-3}{2}$.

Proof From Lemma 4 and Theorem 4, we have the following results:

(i) If $\frac{q^{2m+1}-q}{q^2-1}+1\leq\delta\leq\frac{q^{2m}+3}{2}$, then $\varepsilon=\left\lfloor\frac{(\delta-2)(q^2-1)}{q^{2m}-1}\right\rfloor$. Therefore, we have

$$\frac{q^{2m+1}-q^2-q+1}{q^{2m}-1}\leq\varepsilon\leq 4.$$

Since $\frac{q^2-1}{4}-\varepsilon_{min}=\frac{q^{2m+2}-4q^{2m+1}-q^{2m}+3q^2+4q-3}{4(q^{2m}-1)}<0$, then $\varepsilon_{min}>\left\lfloor\frac{q^2-1}{4}\right\rfloor$. From $\frac{q^2-3}{2}-\varepsilon_{min}=\frac{q^2-3}{2}-\frac{q^{2m+1}-q^2-q+1}{q^{2m}-1}=\frac{q^2+2q+1}{2(q^{2m}-1)}>0$, we have $\varepsilon_{min}<\frac{q^2-3}{2}$. Besides, $\varepsilon_{max}=4>\frac{q^2-3}{2}=3$. Therefore, $\mathcal{C}_{(n,q^2,\delta)}$ has dimension as below:

$$k=\begin{cases} n-2m\left\lceil(\delta-\frac{3}{2})(1-q^{-2})\right\rceil+m(2\varepsilon-\frac{q^2-3}{2}), if\, \varepsilon=2\ or\ 3,\\ n-2m\left\lceil(\delta-\frac{3}{2})(1-q^{-2})\right\rceil+m\frac{q^2-3}{2},\ if\ \varepsilon=4.\end{cases}$$

(ii) If $\frac{q^{2m}+5}{2}\leq\delta\leq\frac{q^{2m+2}+q^{2m}+6}{2(q^2-1)}$, then $\varepsilon=\left\lfloor\frac{(\delta-\frac{q^{2m}+5}{2})(q^2-1)}{q^{2m}-1}\right\rfloor$. Therefore, we have

$$0\leq\varepsilon\leq\frac{2q^{2m}-5q^2+11}{2(q^{2m}-1)}.$$

From $\frac{q^2-1}{4}-\varepsilon_{max}=\frac{q^2-1}{4}-\frac{2q^{2m}-5q^2+11}{2(q^{2m}-1)}=\frac{q^{2m+1}+q^{2m}+q^4-21}{4(q^{2m}-1)}>0$, we have $\varepsilon_{max}<\frac{q^2-1}{4}$.

Therefore, $\mathcal{C}_{(n,q^2,\delta)}$ has dimension $k=n-2m\left\lceil(\delta-\frac{3}{2})(1-q^{-2})\right\rceil+m\frac{q^2-3}{2}$. □

Theorem 6 *Let* $n=\frac{q^{4m}-1}{q^2-1}$, $\varepsilon=\left\lfloor\frac{(\delta-2)(q^2-1)}{q^{2m}-1}\right\rfloor$ *and* $\delta=\frac{q^{2m+1}-q}{q^2-1}+t$, *where* $q\geq 5$ *is odd,* $m\geq 2$ *and* $1\leq t\leq\frac{(q^{2m-1}+1)(q+3)}{2(q+1)}$.

(i) *If* $q=5$, *then* $\mathcal{C}_{(n,q^2,\delta)}$ *has dimension*

$$k=\begin{cases} n-2m\left\lceil(\delta-\frac{3}{2})(1-q^{-2})\right\rceil, if\ \varepsilon<6,\\ n-2m\left\lceil(\delta-\frac{3}{2})(1-q^{-2})\right\rceil+m(2\varepsilon-\frac{q^2-3}{2}), if\ 6\leq\varepsilon\leq 11.\end{cases}$$

(ii) *If $q \geq 7$, then $\mathcal{C}_{(n,q^2,\delta)}$ has dimension $k = n - 2m\left\lceil (\delta - \frac{3}{2})(1 - q^{-2})\right\rceil$.*

Proof From Lemma 4 and Theorem 4, we can define $\delta = \frac{q^{2m+1}-q}{q^2-1} + t$ and $1 \leq t \leq \frac{(q^{2m-1}+1)(q+3)}{2(q+1)}$. Then we have

$$\frac{q^{2m+1}-q}{q^2-1} + 1 \leq \delta \leq \frac{3q^{2m+1} + 2q^{2m} - 3q^{2m-1} + q^2 - 3}{2q^2 - 2}.$$

From $\varepsilon = \left\lfloor \frac{(\delta-2)(q^2-1)}{q^{2m}-1} \right\rfloor$, we have

$$\frac{q^{2m+1} - q^2 - q + 1}{q^{2m} - 1} \leq \varepsilon \leq \frac{3q^{2m+1} + 2q^{2m} - 3q^{2m-1} - 3q^2 + 1}{2(q^{2m} - 1)}.$$

On the one hand, $\frac{q^2-1}{4} - \varepsilon_{min} = \frac{q^{2m+2} - 4q^{2m+1} - q^{2m} + 3q^2 + 4q - 3}{4(q^{2m}-1)} > 0$, then $\varepsilon_{min} < \left\lfloor \frac{q^2-1}{4} \right\rfloor$; on the other hand,

$$\varepsilon_{max} - \frac{q^2-1}{4} = \frac{-q^{2m-1}(q^3 - 6q^2 - 5q + 6) - 5q^2 + 1}{4(q^{2m} - 1)}.$$

If $q = 5$, then $\varepsilon_{max} > \frac{q^2-1}{4}$; if $q \geq 7$, then $\varepsilon_{max} < \frac{q^2-1}{4}$. Therefore, when $q \geq 7$, we have $\varepsilon < \frac{q^2-1}{4}$, then $\mathcal{C}_{(n,q^2,\delta)}$ has dimension $k = n - 2m\left\lceil (\delta - \frac{3}{2})(1 - q^{-2})\right\rceil$.

As for $q = 5$, from $\frac{q^2-3}{2} - \varepsilon_{max} = \frac{q^{2m-1}(q^3 - 3q^2 - 5q + 3) + 2q^2 + 2}{2(q^{2m}-1)} > 0$, we have $\varepsilon_{max} < \frac{q^2-3}{2}$. Then $\mathcal{C}_{(n,q^2,\delta)}$ has dimension:

$$k = \begin{cases} n - 2m\left\lceil (\delta - \frac{3}{2})(1 - q^{-2})\right\rceil, if\ \varepsilon < 6, \\ n - 2m\left\lceil (\delta - \frac{3}{2})(1 - q^{-2})\right\rceil + m(2\varepsilon - \frac{q^2-3}{2}),\ if\ 6 \leq \varepsilon \leq 11. \end{cases}$$

□

4.3 The Construction of EAQECCs

Based on the discussions above, we give a theorem below.

Theorem 7 *Let $n = \frac{q^{4m}-1}{q^2-1}$, $q \geq 7$ is a power of an odd prime p, $m \geq 2$ and $\delta = \frac{q^{2m+1}-q}{q^2-1} + t$, where $q^2 \equiv 1 \mod 4$ and $1 \leq t \leq \frac{(q^{2m-1}+1)(q+3)}{2(q+1)}$. If $\mathcal{C}$ is a q^2-ary negacyclic code of length n with defining set $Z = \bigcup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-2+t} C_{1+2i}$, then there exist EAQECCs with parameters $[[n, n - 4m\lceil(\delta - \frac{3}{2})(1 - q^{-2})\rceil + 4m, \geq \delta; 4m]]_q$.*

Proof From Lemma 2, we can assume that the defining set of the negacyclic code $\mathcal{C}$ is $Z = \bigcup_{i=0}^{\delta-2} C_{1+2i}$, where $\delta = \frac{q^{2m+1}-q}{q^2-1} + t$, $q \geq 7$, $m \geq 2$ and $1 \leq t \leq \frac{(q^{2m-1}+1)(q+3)}{2(q+1)}$.

Then $\mathcal{C}$ is a negacyclic code with parameters $[n, n-2m\lceil(\delta-\frac{3}{2})(1-q^{-2})\rceil, \geq \delta]_{q^2}$ from Theorems 1 and 6. Therefore, we have the following result:

$$\begin{aligned}
Z_1 &= Z \cap (-qZ) \\
&= ((\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1} C_{1+2i}) \cup (\cup_{i=\frac{q^{2m+1}-q}{q^2-1}}^{\frac{q^{2m+1}-q}{q^2-1}-2+t} C_{1+2i})) \\
&\quad \cap(-q(\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1} C_{1+2i}) \cup -q(\cup_{i=\frac{q^{2m+1}-q}{q^2-1}}^{\frac{q^{2m+1}-q}{q^2-1}-2+t} C_{1+2i})) \\
&= ((\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1} C_{1+2i}) \cap -q(\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1} C_{1+2i})) \\
&\quad \cup((\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1} C_{1+2i}) \cap -q(\cup_{i=\frac{q^{2m+1}-q}{q^2-1}}^{\frac{q^{2m+1}-q}{q^2-1}-2+t} C_{1+2i})) \\
&\quad \cup((\cup_{i=\frac{q^{2m+1}-q}{q^2-1}}^{\frac{q^{2m+1}-q}{q^2-1}-2+t} C_{1+2i}) \cap -q(\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1} C_{1+2i})) \\
&\quad \cup((\cup_{i=\frac{q^{2m+1}-q}{q^2-1}}^{\frac{q^{2m+1}-q}{q^2-1}-2+t} C_{1+2i}) \cap -q(\cup_{i=\frac{q^{2m+1}-q}{q^2-1}}^{\frac{q^{2m+1}-q}{q^2-1}-2+t} C_{1+2i})) \\
&= C_{\frac{2q^{2m+1}-q^2-2q+1}{q^2-1}} \cup C_{\frac{2q^{4m}-2q^{2m+2}+q^3+2q^2-q-2}{q^2-1}}. \qquad (1)
\end{aligned}$$

In order to get the result of (1), we have to show that

$$(\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1} C_{1+2i}) \cap -q(\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1} C_{1+2i}) = C_{\frac{2q^{2m+1}-q^2-2q+1}{q^2-1}} \cup C_{\frac{2q^{4m}-2q^{2m+2}+q^3+2q^2-q-2}{q^2-1}},$$
$$(\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1} C_{1+2i}) \cap -q(\cup_{i=\frac{q^{2m+1}-q}{q^2-1}}^{\frac{q^{2m+1}-q}{q^2-1}-2+t} C_{1+2i}) = \emptyset,$$
$$(\cup_{i=\frac{q^{2m+1}-q}{q^2-1}}^{\frac{q^{2m+1}-q}{q^2-1}-2+t} C_{1+2i}) \cap -q(\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1} C_{1+2i}) = \emptyset,$$

and

$$(\cup_{i=\frac{q^{2m+1}-q}{q^2-1}}^{\frac{q^{2m+1}-q}{q^2-1}-2+t} C_{1+2i}) \cap -q(\cup_{i=\frac{q^{2m+1}-q}{q^2-1}}^{\frac{q^{2m+1}-q}{q^2-1}-2+t} C_{1+2i}) = \emptyset.$$

Firstly, we demonstrate that

$$(\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1} C_{1+2i}) \cap -q(\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1} C_{1+2i}) = C_{\frac{2q^{2m+1}-q^2-2q+1}{q^2-1}} \cup C_{\frac{2q^{4m}-2q^{2m+2}+q^3+2q^2-q-2}{q^2-1}}.$$

From Lemma 2, $\mathcal{C}^{\perp_h}_{(n,q^2,\delta_{max})} \subseteq \mathcal{C}_{(n,q^2,\delta_{max})}$ means that

$$(\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-2} C_{1+2i}) \cap -q(\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-2} C_{1+2i}) = \emptyset.$$

And from $\mathcal{C}^{\perp_h}_{(n,q^2,\delta_{max}+1)} \nsubseteq \mathcal{C}_{(n,q^2,\delta_{max}+1)}$ in the proof of Lemma 2 in Ref. [23], we can get $C_{1+2(\delta_{max}-1)} \cap -q(\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1} C_{1+2i}) = C_{1+2(\delta_{max}-1)}$ immediately, where $C_{1+2(\delta_{max}-1)} = C_{\frac{2q^{2m+1}-q^2-2q+1}{q^2-1}}$, i.e.,

$$C_{\frac{2q^{2m+1}-q^2-2q+1}{q^2-1}} \cap -q(\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1} C_{1+2i}) = C_{\frac{2q^{2m+1}-q^2-2q+1}{q^2-1}}.$$

From the equation above, we can get

$$-qC_{\frac{2q^{2m+1}-q^2-2q+1}{q^2-1}} \cap (\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1} C_{1+2i}) = -qC_{\frac{2q^{2m+1}-q^2-2q+1}{q^2-1}} = C_{\frac{2q^{4m}-2q^{2m+2}+q^3+2q^2-q-2}{q^2-1}}.$$

Thus, the desired result $(\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1} C_{1+2i}) \cap -q(\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1} C_{1+2i}) = C_{\frac{2q^{2m+1}-q^2-2q+1}{q^2-1}} \cup C_{\frac{2q^{4m}-2q^{2m+2}+q^3+2q^2-q-2}{q^2-1}}$ follows.

Secondly, we testify

$$(\cup_{i=\frac{q^{2m+1}-q}{q^2-1}}^{\frac{q^{2m+1}-q}{q^2-1}-2+t} C_{1+2i}) \cap -q(\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1} C_{1+2i}) = \emptyset,$$

for $2 \le t \le \frac{(q^{2m-1}+1)(q+3)}{2(q+1)}$.

If $(\cup_{i=\frac{q^{2m+1}-q}{q^2-1}}^{\frac{q^{2m+1}-q}{q^2-1}-2+t} C_{1+2i}) \cap -q(\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1} C_{1+2i}) \neq \emptyset$, for $2 \le t \le \frac{(q^{2m-1}+1)(q+3)}{2(q+1)}$, i.e.,

$$(\cup_{i=2}^{t} C_{1+2(i+\frac{q^{2m+1}-q}{q^2-1}-2)}) \cap -q(\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1} C_{1+2i}) \neq \emptyset,$$

for $2 \le t \le \frac{(q^{2m-1}+1)(q+3)}{2(q+1)}$, then there exist two integers l and j, where $2 \le l \le \frac{(q^{2m-1}+1)(q+3)}{2(q+1)}$, $0 \le j \le \frac{q^{2m+1}-q}{q^2-1} - 1$, such that

$$1 + 2(\frac{q^{2m+1}-q}{q^2-1} - 2 + l) \equiv -q(1+2j)q^{2k} \bmod 2n,$$

for some $k \in \{0, 1\}$. We can seek contradictions as follows.

(i) When $k = 0$, we have $1 + 2(\frac{q^{2m+1}-q}{q^2-1} - 2 + l) \equiv -q(1+2j) \bmod 2n$, which is equivalent to

$$\frac{2q^{2m+1}-2q}{q^2-1} + q - 3 + 2l + 2jq \equiv 0 \bmod 2n, \tag{2}$$

where $n = \frac{q^{4m}-1}{q^2-1}$. From $2 \le l \le \frac{(q^{2m-1}+1)(q+3)}{2(q+1)}, 0 \le j \le \frac{q^{2m+1}-q}{q^2-1} - 1$, then $\frac{2q^{2m+1}-2q}{q^2-1} + q + 1 \le \frac{2q^{2m+1}-2q}{q^2-1} + q - 3 + 2l + 2jq \le \frac{2q^{2m+2}+3q^{2m+1}+2q^{2m}-3q^{2m-1}-q^3-4q^2+q}{q^2-1}$. Because of $\frac{2q^{2m+2}+3q^{2m+1}+2q^{2m}-3q^{2m-1}-q^3-4q^2+q}{q^2-1} < 2n$, (2) is not established.

(ii) When $k=1$, we have $1+2(\frac{q^{2m+1}-q}{q^2-1}-2+l)\equiv -q^3(1+2j) \bmod 2n$, which is equivalent to

$$\frac{2q^{2m+1}-2q}{q^2-1}+q^3-3+2l+2jq^3\equiv 0 \bmod 2n, \tag{3}$$

where $n=\frac{q^{4m}-1}{q^2-1}$. From $2\le l\le \frac{(q^{2m-1}+1)(q+3)}{2(q+1)}, 0\le j\le \frac{q^{2m+1}-q}{q^2-1}-1$, we have: $\frac{2q^{2m+1}-2q}{q^2-1}+q^3+1\le \frac{2q^{2m+1}-2q}{q^2-1}+q^3-3+2l+2jq^3\le \frac{2q^{2m+4}+3q^{2m+1}+2q^{2m}-3q^{2m-1}-q^5-2q^4+q^3-2q^2}{q^2-1}$. Because of $\frac{2q^{2m+4}+3q^{2m+1}+2q^{2m}-3q^{2m-1}-q^5-2q^4+q^3-2q^2}{q^2-1}<2n$, (3) is not established.

From the discussions above, we can see

$$(\cup_{i=\frac{q^{2m+1}-q}{q^2-1}}^{\frac{q^{2m+1}-q}{q^2-1}-2+t}C_{1+2i})\cap -q(\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1}C_{1+2i})=\emptyset,$$

for $2\le t\le \frac{(q^{2m-1}+1)(q+3)}{2(q+1)}$.

Next, we show that

$$(\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1}C_{1+2i})\cap -q(\cup_{i=\frac{q^{2m+1}-q}{q^2-1}}^{\frac{q^{2m+1}-q}{q^2-1}-2+t}C_{1+2i})=\emptyset.$$

Since

$$-q((\cup_{i=\frac{q^{2m+1}-q}{q^2-1}}^{\frac{q^{2m+1}-q}{q^2-1}-2+t}C_{1+2i})\cap -q(\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1}C_{1+2i}))=-q\emptyset=\emptyset,$$

it follows that

$$(\cup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-1}C_{1+2i})\cap -q(\cup_{i=\frac{q^{2m+1}-q}{q^2-1}}^{\frac{q^{2m+1}-q}{q^2-1}-2+t}C_{1+2i})=\emptyset.$$

Finally, we show that

$$(\cup_{i=\frac{q^{2m+1}-q}{q^2-1}}^{\frac{q^{2m+1}-q}{q^2-1}-2+t}C_{1+2i})\cap -q(\cup_{i=\frac{q^{2m+1}-q}{q^2-1}}^{\frac{q^{2m+1}-q}{q^2-1}-2+t}C_{1+2i})=\emptyset,$$

for $2\le t\le \frac{(q^{2m-1}+1)(q+3)}{2(q+1)}$. If $(\cup_{i=\frac{q^{2m+1}-q}{q^2-1}}^{\frac{q^{2m+1}-q}{q^2-1}-2+t}C_{1+2i})\cap -q(\cup_{i=\frac{q^{2m+1}-q}{q^2-1}}^{\frac{q^{2m+1}-q}{q^2-1}-2+t}C_{1+2i})\neq\emptyset$, i.e.,

$$(\cup_{i=2}^{t}C_{1+2(i+\frac{q^{2m+1}-q}{q^2-1}-2)})\cap -q(\cup_{i=2}^{t}C_{1+2(i+\frac{q^{2m+1}-q}{q^2-1}-2)})\neq\emptyset,$$

for $2\le t\le \frac{(q^{2m-1}+1)(q+3)}{2(q+1)}$, then there exist two integers l and j, where $2\le l, j\le \frac{(q^{2m-1}+1)(q+3)}{2(q+1)}$, such that

$$1+2(l+\frac{q^{2m+1}-q}{q^2-1}-2)\equiv -q[1+2(j+\frac{q^{2m+1}-q}{q^2-1}-2)]q^{2k} \bmod 2n,$$

for some $k\in\{0,1\}$. We can seek contradictions as follows.

(i) When $k=0$, we have $1+2(l+\frac{q^{2m+1}-q}{q^2-1}-2)\equiv -q[1+2(j+\frac{q^{2m+1}-q}{q^2-1}-2)] \bmod 2n$, which is equivalent to

$$\frac{2q^{2m+2}+2q^{2m+1}-2q^2-2q}{q^2-1}-3q-3+2l+2jq\equiv 0 \bmod 2n, \tag{4}$$

where $n=\frac{q^{4m}-1}{q^2-1}$. From $2\leq l,j\leq \frac{(q^{2m-1}+1)(q+3)}{2(q+1)}$, we have: $\frac{2q^{2m+2}+2q^{2m+1}+q^3-q^2-3q-1}{q^2-1}\leq \frac{2q^{2m+2}+2q^{2m+1}-2q^2-2q}{q^2-1}-3q-3+2l+2jq\leq \frac{3q^{2m+2}+5q^{2m+1}-q^{2m}-3q^{2m-1}-2q^3-2q^2}{q^2-1}$. Because of $\frac{3q^{2m+2}+5q^{2m+1}-q^{2m}-3q^{2m-1}-2q^3-2q^2}{q^2-1}<2n$, (4) is not established.

(ii) When $k=1$, we have $1+2(l+\frac{q^{2m+1}-q}{q^2-1}-2)\equiv -q^3[1+2(j+\frac{q^{2m+1}-q}{q^2-1}-2)] \bmod 2n$, which is equivalent to

$$2l+2jq^3\equiv \frac{2q^{4m}-2q^{2m+4}-2q^{2m+1}+3q^5+2q^4-3q^3+3q^2+2q-5}{q^2-1} \bmod 2n. \tag{5}$$

From $2\leq l,j\leq \frac{(q^{2m-1}+1)(q+3)}{2(q+1)}$, we have

$$4+4q^3\leq 2l+2jq^3\leq \frac{q^{2m+4}+2q^{2m+3}-3q^{2m+2}+q^{2m+1}+2q^{2m}-3q^{2m-1}+q^5+2q^4-3q^3+q^2+2q-3}{q^2-1}.$$

Then we have the following results:

(a) when $m=2$, since $(4+4q^3)(q^2-1)=4q^5-4q^3+4q^2-4>q^5+2q^4-3q^3+3q^2+2q-5$, (5) is not established;

(b) when $m\geq 3$, we have

$$\frac{2q^{4m}-2q^{2m+4}-2q^{2m+1}+3q^5+2q^4-3q^3+3q^2+2q-5}{q^2-1}>(2l+2jq^3)_{max}.$$

Therefore, (5) is not established.

From the discussions above, we can see

$$\Big(\bigcup_{i=\frac{q^{2m+1}-q}{q^2-1}}^{\frac{q^{2m+1}-q}{q^2-1}-2+t} C_{1+2i}\Big)\cap -q\Big(\bigcup_{i=\frac{q^{2m+1}-q}{q^2-1}}^{\frac{q^{2m+1}-q}{q^2-1}-2+t} C_{1+2i}\Big)=\emptyset,$$

for $2\leq t\leq \frac{(q^{2m-1}+1)(q+3)}{2(q+1)}$.

From Lemma 1 and Theorem 3, we have $c=4m$. From Theorem 2, there exist entanglement assisted quantum codes with parameters

$$[[n, n-4m\lceil(\delta-\frac{3}{2})(1-q^{-2})\rceil+4m, \geq\delta; 4m]]_q,$$

where $1\leq t\leq \frac{(q^{2m-1}+1)(q+3)}{2(q+1)}$. □

When $q=3$ or 5, we can construct EAQECCs in the following two theorems. The proof is similar to Theorem 7, so we omit it here.

Theorem 8 *Let* $q = 3$, $m \geq 2$, $n = \frac{q^{4m}-1}{q^2-1}$ *and* $\delta = \frac{q^{2m+1}-q}{q^2-1} + t$, *where* $q^2 \equiv 1 \mod 4$ *and* $1 \leq t \leq \frac{(q^{2m-1}+1)(q+3)}{2(q+1)}$. *If* $\mathcal{C}$ *is a* q^2*-ary negacyclic code of length n with defining set* $Z = \bigcup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-2+t} C_{1+2i}$, *then there exsit EAQECCs with parameters*

(i) *If* $\delta \leq \frac{q^{2m}+3}{2}$,

$$\begin{cases} [[n, n-4m\left\lceil(\delta-\frac{3}{2})(1-q^{-2})\right\rceil + 4m\varepsilon - mq^2 + 7m, \geq \delta; 4m]]_q, if\, \varepsilon = 2 \; or \; 3, \\ [[n, n-4m\left\lceil(\delta-\frac{3}{2})(1-q^{-2})\right\rceil + mq^2 + m, \geq \delta; 4m]]_q, if\, \varepsilon = 4. \end{cases}$$

(ii) *If* $\frac{q^{2m}+5}{2} \leq \delta \leq q^{2m}$, $[[n, n-4m\left\lceil(\delta-\frac{3}{2})(1-q^{-2})\right\rceil + mq^2 + m, \geq \delta; 4m]]_q$.

Theorem 9 *Let* $q = 5$, $m \geq 2$, $n = \frac{q^{4m}-1}{q^2-1}$ *and* $\delta = \frac{q^{2m+1}-q}{q^2-1} + t$, *where* $q^2 \equiv 1 \mod 4$ *and* $1 \leq t \leq \frac{(q^{2m-1}+1)(q+3)}{2(q+1)}$. *If* $\mathcal{C}$ *is a* q^2*-ary negacyclic code of length n with defining set* $Z = \bigcup_{i=0}^{\frac{q^{2m+1}-q}{q^2-1}-2+t} C_{1+2i}$, *then there exsit EAQECCs with parameters*

$$\begin{cases} [[n, n-4m\lceil(\delta-\frac{3}{2})(1-q^{-2})\rceil + 4m, \geq \delta; 4m]]_q, if \; \varepsilon < 6, \\ [[n, n-4m\lceil(\delta-\frac{3}{2})(1-q^{-2})\rceil + 4\varepsilon m - mq^2 + 7m, \geq \delta; 4m]]_q, if \; 6 \leq \varepsilon \leq 11. \end{cases}$$

Table 1 new EAQECCs from Theorem 8

q	m	δ	$[[n,k,d;c]]_q$
3	2	31	$[[820, 608, \geq 31; 8]]_3$
3	2	32	$[[820, 616, \geq 32; 8]]_3$
3	2	33	$[[820, 616, \geq 33; 8]]_3$
3	2	34	$[[820, 608, \geq 34; 8]]_3$
3	2	35	$[[820, 600, \geq 35; 8]]_3$
3	2	36	$[[820, 592, \geq 36; 8]]_3$
3	2	37	$[[820, 584, \geq 37; 8]]_3$
3	2	38	$[[820, 576, \geq 38; 8]]_3$
3	2	39	$[[820, 568, \geq 39; 8]]_3$
3	2	40	$[[820, 560, \geq 40; 8]]_3$
3	2	41	$[[820, 552, \geq 41; 8]]_3$
3	2	42	$[[820, 552, \geq 42; 8]]_3$
3	2	43	$[[820, 544, \geq 43; 8]]_3$
3	2	44	$[[820, 536, \geq 44; 8]]_3$
3	2	45	$[[820, 528, \geq 45; 8]]_3$
3	2	46	$[[820, 520, \geq 46; 8]]_3$
3	2	47	$[[820, 512, \geq 47; 8]]_3$
3	2	48	$[[820, 504, \geq 48; 8]]_3$
3	2	49	$[[820, 496, \geq 49; 8]]_3$
3	2	50	$[[820, 488, \geq 50; 8]]_3$
3	2	51	$[[820, 488, \geq 51; 8]]_3$

5 Example

Let $q = 3$ and $m = 2$. Then $n = \frac{q^{4m}-1}{q^2-1} = 820$, $1 \leq t \leq \frac{(q^{2m-1}+1)(q+3)}{2(q+1)} = 21$ and $31 \leq \delta = \frac{q^{2m+1}-q}{q^2-1} + t \leq 51$. Then from Theorem 8, we can construct some EAQECCs with new parameters in Table 1.

6 Conclusion

In this work, we have constructed a class of EAQECCs from negacyclic BCH codes over the finite fields $\mathbb{F}_{q^2}$ of length $n = \frac{q^{4m}-1}{q^2-1}$, where $q \geq 3$ is some odd prime power and $m \geq 2$. The construction is through cyclotomic cosets and ideal theory. It would be interesting to construct EAQECCs with different lengths from other types of linear codes.

References

1. Shor, P.W.: Scheme for reducing decoherence in quantum computer memory. Phys. Rev. A **52**(4), 2493–2496 (1995)
2. Steane, A.M.: Error correcting codes in quantum theory. Phys. Rev. Lett. **77**, 793–797 (1996)
3. Steane, A.M.: Simple quantum error-correcting codes. Phys. Rev. A **54**(6), 4741–4751 (1996)
4. Calderbank, A.R., Rains, E.M., Shor, P.W., Sloane, N.J.A.: Quantum error correction via codes over GF(4). IEEE Trans. Inf. Theory **44**(4), 1369–1387 (1998)
5. Grassl, M., Beth, T., Rötteler, M.: On optimal quantum codes. Int. J. Quantum Inf. **2**(1), 55–64 (2004)
6. Ketkar, A., Klappenecker, A., Kumar, S., Sarvepalli, P.K.: Nonbinary stabilizer codes over finite fields. IEEE Trans. Inf. Theory **52**(11), 4892–4914 (2006)
7. La Guardia, G.G.: Constructions of new families of nonbinary quantum codes. Phys. Rev. A **80**(1-11), 042331 (2009)
8. Aly, S.A., Klappenecker, A., Sarvepalli, P.K.: On quantum and classical BCH codes. IEEE Trans. Inf. Theory **53**(3), 1183–1188 (2007)
9. Brun, T., Devetak, I., Hsieh, M.-H.: Correcting quantum errors with entanglement. Science **314**(5798), 436–439 (2006)
10. Wilde, M.M., Brun, T.A.: Optimal entanglement formulas for entanglement-assisted quantum coding. Phys. Rev. A **77**(1-4), 064302 (2008)
11. Lai, C.-Y., Brun, T.A.: Entanglement increases the error-correcting ability of quantum error-correcting codes. Phys. Rev. A **88**(1-10), 012320 (2013)
12. Lü, L., Li, R., Guo, L., Fu, Q.: Maximal entanglement entanglement-assisted quantum codes constructed from linear codes. Quantum Inf. Process. **14**, 165–182 (2015)
13. Qian, J., Zhang, L.: On MDS linear complementary dual codes and entanglement-assisted quantum codes. Des. Codes Cryptogr. **86**, 1565–1572 (2018)
14. Guenda, K., Jitman, S., Gulliver, T.A.: Constructions of good entanglement-assisted quanutm error correcting codes. Des. Codes Cryptogr. **86**, 121–136 (2018)
15. Hsieh, M.-H., Brun, T.A., Devetak, I.: Entanglement-assisted quantum quasicyclic low-density parity-check codes. Phys. Rev. A **79**(1-7), 032340 (2009)
16. Fujiwara, Y., Clark, D., Vandendriessche, P., Boeck, M.D., Tonchev, V.D.: Entanglement-assisted quantum low-density parity-check codes. Phys. Rev. A **82**(1-19), 042338 (2010)
17. Li, R., Zuo, F., Liu, Y.: A study of skew symmetric q^2-cyclotomic coset and its application, vol. 12. (in Chinese) (2011)
18. Li, R., Xu, G., Lü, L.: Decomposition of defining sets of BCH codes and its applications, vol. 14. (in Chinese) (2013)

19. Lü, L., Li, R.: Entanglement-assisted quantum codes constructed from primitive quaternary BCH codes. Int. J. Quantum Inf. **12**(3), 1450015(1-14) (2014)
20. Chen, J., Huang, Y., Feng, C., Chen, R.: Entanglement-assisted quantum MDS codes constructed from negacyclic codes. Quantum Inf. Process. **16**(1-22), 303 (2017)
21. Lü, L., Li, R., Guo, L., Ma, Y., Liu, Y.: Entanglement-assisted quantum MDS codes from negacyclic codes. Quantum Inf. Process. **69**(1-23), 17 (2018)
22. Kai, X., Zhu, S., Li, P.: Constacyclic codes and some new quantum MDS codes. IEEE Trans. Inf. Theory **60**(4), 2080–2086 (2014)
23. Zhu, S., Sun, Z., Li, P.: A class of negacyclic BCH codes and its application to quantum codes. Des. Codes Cryptogr. **86**(10), 2139–2165 (2018)
24. Krishna, A., Sarwate, D.V.: Pseudocyclic maximum-distance-separable codes. IEEE Trans. Inf. Theory **36**(4), 880–884 (1990)