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Induced Representation of the (1 + 1)-Quantum Extended Galilei Algebra on the Bargmann Space-Time

Induced Representation on Bargmann Space-time

Mohammed Abdelwahhab Benbitour[1] **·**
Chaib Boussaid[1] **· Mohammed Tayeb Meftah**[1]

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Abstract We present a q-deformed local representations of the (1 +1) extended Galilei group acting on the space of wave-functions defined in the Bargmann space-time. This paper is a development of the work made by F. Bonechi et al on the induced representations of (1+1) quantum Galilei.

Keywords Quantum groups · Induced representation · Galilei group · Bargmann space

1 Introduction

The group theory approach plays a fundamental role in theoretical physics. The story goes back to the work of Wigner, who proposed the well-known classification of elementary particles according to the irreducible representations of the Poincaré group [7]. Later developments have come to light through the work of Wightman. Indeed, Wightman [8], using the technique of induced representations, gave an algebraic meaning to the notion of localization of elementary particles in space-time. Later, B. Mensky [5], using the same technique, managed to construct a relativistic quantum field theory without needing to define quantized fields [5, 9]. In his work, the local properties of the particle are described by an induced representation of the symmetry group, and the global properties are described by an irreducible representation, which can also be induced from a smaller symmetry. Then the quantum properties are a result of the intertwining of these two types of representation.

✉ Mohammed Tayeb Meftah
mewalid@yahoo.com

Mohammed Abdelwahhab Benbitour
mohamedbenbitour@gmail.com

1 Faculty of Mathematics and Matter Sciences, University of Ouargla, UKMO, Ouargla, 30000, Algeria

Moreover, in Ref. [5], taking the Poincaré group as the symmetry group, this approach was applied to free relativistic particles and in [9], by taking the Galilei group as the symmetry group, it was applied to the case of nonrelativistic particles. This approach uses two kinds of induced representations of the Galilei group. The first is the family of unitary, irreducible representations or momentum representation and the second kind of representation is the local representation which is induced from the stabilizer group.

More recently the quantum group approach has invaded theoretical physics and in this context Quantum Galilei algebras is a generalization of the symmetries associated to Galilei group, so we are tempted to generalize this approach to the case of quantum groups. To this end we need to give a local representation and a momentum representation of the Galilei quantum algebra.

In [2] F. Bonechi and co-authors constructed a q-deformed version of the local representation, or ”configuration representation”, of the Galilei quantum algebra and arrived at a ‘deformed’ quantum mechanics. The construction of a ”configuration” induced representation on the space of wave functions, defined on space-time, for the deformed algebra of the (1+1)-Galilei group presented two problems. First, in the monomial basis, the generators of the space-time coordinates are not isolated in a side, and second, the generators related to space and time coordinate do not close a subalgebra.

This paper is a development of [2], where induced representations of (1+1) quantum Galilei group are studied. In this paper we develop some aspects of the work made in [2], but we take a new alternative and we construct the configuration induced representation not on ordinary space-time but on Bargmann space. The 3-dimensional approach, used here, is based on a space so that the (1+1)-Galilei group elements acts on the extended space-time coordinates represented by (μ, x, t). Given that we can embed the usual configuration space (x, t) into a three dimensional space (μ, x, t) then the extra coordinate μ transforms under Galilean transformations as follow $\mu \rightarrow \mu + xv + \frac{1}{2}v^2 t$.

In Section 2, we present some aspects of the (1+1)-Galilei group, we also recall its projective representation acting on the Hilbert space of a non-relativistic quantum particle. In Section 3, we briefly recall the theory of induced representation and construct two essential representations, the momentum representation and the configuration representation. Finally, in Section 4 we give a short conclusion.

2 Extended Galilei Algebra

In a space-time with one dimension of space x and one dimension of time t , we define the $(1+1)$ −Galilei group, as being a Lie group of transformations. An arbitrary element g, of the Galilei group, can be parameterized by three variables (t, v, x), so we can formally write

$$g \equiv (t, v, x), \tag{1}$$

where t is a time translation, x and v are space translations and Galilean boosts respectively. The group law is then

$$(t', v', x')(t, v, x) = (t + t', v + v', x + x' + tv'). \tag{2}$$

In the context of this work the relevant representations of the Galilei group are projective, which are representations with group with an extra phase factor [6]

$$U(g)U(g') = e^{im\left(vx' + \frac{1}{2}v^2 t'\right)} U(gg'). \tag{3}$$

To avoid such projective representation, we consider the ordinary representation of an extension $\bar{G}$ of the group. This extension is realized using an auxiliary 1-parameter group, whose elements μ are defined so that they commute with all elements of the Galilei group. Elements of the extended group are then $\bar{g} \equiv (\mu, t, v, x)$ with the group law given by

$$(\mu', t', v', x')(\mu, t, v, x) = (\mu + \mu' + \frac{1}{2}v'^2 t + xv', t + t', v + v', x + x' + tv'). \quad (4)$$

This defines the $(1+1)$-extended Galilei group $\bar{G}$.

On the space $\mathcal{F}(\bar{G})$ of functions on $\bar{G}$, we define the bialgebra structure by mean of coproduct Δ, antipode S and counit ε, derived from the multiplication rule (4)

$$\begin{aligned} \Delta(t) &= t \otimes I + I \otimes t, \\ \Delta(x) &= x \otimes I + I \otimes x + v \otimes t, \\ \Delta(v) &= v \otimes I + I \otimes v, \\ \Delta(\mu) &= \mu \otimes I + I \otimes \mu + v \otimes x + \frac{1}{2}v^2 \otimes t, \end{aligned} \quad (5)$$

and

$$\begin{aligned} S(t) &= -t, \quad S(x) = -x + vt, \\ S(v) &= -v, \quad S(\mu) = xv - \frac{1}{2}v^2 t - \mu, \\ \varepsilon(.) &= 0. \end{aligned} \quad (6)$$

The quantum Galilei group can now be constructed as a deformation of the Lie bialgebra (5) by means of a non trivial 1−cocycle with values in $\Lambda^2 \left(Lie\bar{G}\right)$, which defines a Lie-Poisson structure on the space $\mathcal{F}(\bar{G})$

$$\{\mu, x\} = -2a\mu, \qquad \{\mu, v\} = av^2, \qquad \{x, v\} = 2av, \qquad \{t, .\} = 0, \quad (7)$$

where a is a deformation parameter. The quantum deformation of (5) in the direction of this last Poisson brackets gives us the quantum Galilei group $\mathcal{F}_q(\bar{G})$

$$\left[\hat{\mu}, \hat{x}\right] = -2a\hat{\mu}, \qquad \left[\hat{\mu}, \hat{v}\right] = a\hat{v}^2, \qquad \left[\hat{x}, \hat{v}\right] = 2a\hat{v}, \qquad \left[\hat{t}, .\right] = 0. \quad (8)$$

To simplify the notation, we will not use the notation with hat in the rest of the paper. After the definition of the structure of quantum Galilei group, the next task is the structure of its dual Hopf algebra, (the quantum Lie algebra). As is well known, we can construct a dual Hopf algebra by duality rule (or pairing), using the following standard rule

$$\langle XY, \Phi \rangle = \langle X \otimes Y, \Delta\Phi \rangle, \qquad \langle X, \Phi\Psi \rangle = \langle \Delta X, \Phi \otimes \Psi \rangle, \quad (9)$$

and the involution defined by

$$\left\langle X^*, \Phi \right\rangle = \left\langle X, S^{-1}\left(\Phi^*\right) \right\rangle. \quad (10)$$

The quantum (enveloping) algebra $\mathcal{U}_q\left(\bar{G}\right)$ can be expressed in terms of the generators $\{I, P, H, N\}$ so that, the bilinear form of the duality between $\mathcal{U}_q\left(\bar{G}\right)$ and $\mathcal{F}_q(\bar{G})$ is given by [1]

$$\left\langle I^\alpha P^\beta H^\gamma N^\delta, \mu^{\alpha'} x^{\beta'} t^{\gamma'} v^{\delta'} \right\rangle = \alpha!\beta!\gamma!\delta!\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}\delta_{\gamma,\gamma'}\delta_{\delta,\delta'}, \quad (11)$$

with the non vanishing commutations

$$[N, I] = ae^{-2aP} I^2, \qquad [N, P] = e^{-2aP} I, \qquad [N, H] = \frac{1}{2a}\left(1 - e^{-2aP}\right). \quad (12)$$

The generators of algebra are associated with transformations of coordinates μ, x, t as follows: the generator I to the transformation of μ, the generator P to the spatial translations, the generator H to the temporal translations and finally the generator N to the change of velocities"

3 Induced Representations

Given a subgroup K of G, any linear representation χ of the subgroup K (acting in a linear space L_χ) can induce a linear representation U_χ, noted $U_\chi = \chi(K) \uparrow G$, of G. This induced representation acts in the space $\mathcal{H}_\chi = \left\{\varphi : G \to L_\chi\right\}$. Functions φ are restricted by the equivariance condition

$$\varphi\left(kg\right) = \chi\left(k\right)\varphi\left(g\right). \tag{13}$$

The induced representation U_χ is defined as follows

$$\left(U_\chi(g)\varphi\right)\left(g'\right) = \varphi\left(g'g\right), \tag{14}$$

Equivalently, one may consider φ as a function on the quotient space $X = G/K$ of the group with respect to the subgroup K, on which the group G acts transitively. The subgroup K is the stabilizer for some origin $x_0 \in X$

$$K = \{k \in G; kx_0 = x_0\}. \tag{15}$$

If x_G is a representative of the coset gK , we have the system of factors [5]

$$g'x_G = \left(g'x\right)_G\left(g', g\right)_K, \tag{16}$$

where $\left(g', g\right)_K \in K$ (see [5] for more details). Then a vector in $\mathcal{H}_\chi$ is a function on X with values in L_χ, and the action becomes

$$\left(U_\chi(g')\varphi\right)(x) = \chi\left(k\right)\varphi\left(xg'\right), \tag{17}$$

with $k = \left(g, g'\right)_K$.

The induced representations scheme, can be easily reformulated in a Hopf algebra framework [2]. In general representations of a Lie group can be transferred to representations of its Lie algebra by a standard way. Take an element in the vicinity of the identity and extracted representation of the generators of the Lie algebra. The action (14) becomes

$$\left(U_\chi(1+\varepsilon.X)\varphi\right)(x) = \varphi\left(x\right) + \varepsilon U_\chi\left(X\right)\varphi\left(x\right) + O\left(\varepsilon^2\right). \tag{18}$$

Now, in a quantum algebra framework, to the subgroup K corresponds a quantum subalgebra $\mathcal{U}_q\left(K\right)$ of $\mathcal{U}_q\left(\bar{G}\right)$, dual of $\mathcal{F}_q\left(K\right)$ and acting on $\mathbb{C}$ by right (left) characters $\chi_K(k)$, $k \in \mathcal{U}_q\left(K\right)$ ($\mathbb{C}$ is the complex field). The induced representation $U_\chi\left(\bar{G}\right)$ acts on the representative space $\mathcal{H}_\chi = Hom_{\mathcal{U}_q(K)}\left(\mathcal{U}_q\left(\bar{G}\right), \mathbb{C}\right)$, whose elements ψ are those of $\mathcal{F}_q(\bar{G})$ that satisfy the reformulated equivariance condition (13)

$$\psi\left(Xk\right) = \psi\left(X\right)k. \tag{19}$$

3.1 Momentum Representation

In this subsection we consider the representation $\chi_{\alpha,\beta,\gamma}$ of the deformed subalgebra generated by $\{I, P, H\}$ (translation sector) in $\mathbb{C}$, given by

$$z \vdash I^p P^q H^r = z\alpha^p\beta^q\gamma^r, \quad z, \alpha, \beta, \gamma \in \mathbb{C}. \tag{20}$$

The representative space $\mathcal{H}_{\alpha,\beta,\gamma}$ is a subspace of $\mathcal{F}_q\left(\bar{G}\right)$ and the elements $\Phi \in \mathcal{H}_{\alpha,\beta,\gamma}$ satisfy the condition (19). One can deduce the general form of any element Φ, using the equivariance condition and a basis, $\mu^i x^j t^k v^l$ of $\mathcal{F}_q\left(\bar{G}\right)$. Let $\Phi = \sum_{i,j,k,l} c_{i,j,k,l}\mu^i x^j t^k v^l \in \mathcal{H}_{\alpha,\beta,\gamma}$ be an element of $\mathcal{H}_{\alpha,\beta,\gamma}$ then the pairing (11) gives

$$\left\langle I^p P^q H^r N^s, \Phi\right\rangle = p!q!r!s!c_{p,q,r,s}, \tag{21}$$

and the equivariance condition (19) gives

$$\begin{aligned}\left\langle I^p P^q H^r N^s, \Phi\right\rangle &= \left\langle N^s, \Phi\right\rangle \vdash I^p P^q H^r \\ &= s!c_{0,0,0,s}\alpha^p \beta^q \gamma^r.\end{aligned} \tag{22}$$

From (21) and (22) we have

$$c_{p,q,r,s} = c_{0,0,0,s}\frac{\alpha^p \beta^q \gamma^r}{p!q!r!}, \tag{23}$$

so for any element of $\mathcal{H}_{\alpha,\beta,\gamma}$ we obtain

$$\Phi = \sum_{i,j,k} \frac{(\alpha\mu)^i\,(\beta x)^j\,(\gamma t)^k}{i!j!k!} \sum_l c_l v^l, \tag{24}$$

where $c_{0,0,0,l} = c_l$. Finally the representative space $\mathcal{H}_{\alpha,\beta,\gamma}$ is the space of elements of the form

$$\Phi = e^{\alpha\mu} e^{\beta x} e^{\gamma t}\varphi(v), \tag{25}$$

where φ is a formal series of v.

The momentum induced representation corresponds to the induced action ($\vdash$) of the elements X of $\mathcal{U}_q\left(\bar{G}\right)$ on the space $\mathcal{H}_{\alpha,\beta,\gamma}$, this action is given by

$$\begin{aligned}\Phi \vdash X &= \sum_{p,q,r,s} c_s \frac{\alpha^p \beta^q \gamma^r}{p!q!r!}\mu^p x^q t^r v^s \vdash X \\ &= \sum_{p,q,r,s}\sum_{i,j,k,l} c_s \frac{\alpha^p \beta^q \gamma^r}{p!q!r!} X^{(p,q,r,s)}_{i,j,k,l}\mu^i x^j t^k v^l.\end{aligned} \tag{26}$$

Consider a general form for the action on the ordered monomials $\mu^p x^q t^r v^s$

$$\mu^p x^q t^r v^s \vdash X = \sum_{i,j,k,l} X^{(p,q,r,s)}_{i,j,k,l}\mu^i x^j t^k v^l, \tag{27}$$

the matrix elements $X^{(p,q,r,s)}_{i,j,k,l}$ can be calculated by the use of the pairing

$$X^{(p,q,r,s)}_{i,j,k,l} = \frac{\left\langle I^i P^j H^k N^l, \mu^p x^q t^r v^s \vdash X\right\rangle}{i!j!k!l!}. \tag{28}$$

Now, using a reformulation of (17)

$$\left\langle I^i P^j H^k N^l, \mu^p x^q t^r v^s \vdash X\right\rangle = \left\langle I^i P^j H^k N^l X, \mu^p x^q t^r v^s\right\rangle, \tag{29}$$

we deduce

$$X^{(p,q,r,s)}_{i,j,k,l} = \frac{\left\langle I^i P^j H^k N^l X, \mu^p x^q t^r v^s\right\rangle}{i!j!k!l!}. \tag{30}$$

Several cases are to be treated, $X \in \{I, P, H, N\}$. First we have the following interesting identities

$$\left[\left(e^{-2aP}\right)^n, N\right] = n\,(2a)\,I\left(e^{-2aP}\right)^{n+1}, \tag{31}$$

$$\left[I^n, N\right] = n\,(-a)\,e^{-2aP}I^{n+1}, \tag{32}$$

Proposition 1 *Using (31) and (32) we get*

$$N^n I = I N^n + (n)\, a e^{-2aP} I^2 N^{n-1} \tag{33}$$

$$N^n e^{-2aP} = e^{-2aP} N^n + \sum_{\sigma=1}^{n} (-a)^{\sigma} \frac{n!\,(\sigma+1)}{(n-\sigma)!} I^{\sigma} \left(e^{-2aP}\right)^{\sigma+1} N^{n-\sigma} \tag{34}$$

Proof We use the formula of iterated commutators

$$B^n A = A B^n + \sum_{k=1}^{n} \frac{n!\,(-1)^k}{k!\,(n-k)!} [\ldots [[A, B], B] \ldots B]\, B^{n-k}.$$

Then it is straightforward to have (33) because of $[[I, N], N] = 0$. Now using (33) and the identity, for $k-$ itterated commutators,

$$\left[\ldots \left[\left[e^{-2aP}, N\right], N\right] \ldots N\right] = a^k\, (k+1)! I^k \left(e^{-2aP}\right)^{k+1},$$

we have (34). □

Return to the matrix elements of the generators, by applying what has been obtained above we have

$$\begin{aligned} N^l P \;=\; & P N^l + l e^{-2aP} I N^{l-1} + \frac{1}{2} l(l-1) a \left(e^{-2aP}\right)^2 I^2 N^{l-2} \\ & + \sum_{\nu=1}^{l} \sum_{\sigma=1}^{l-\nu} (-a)^{\sigma} \frac{(l-\nu)!\,(\sigma+1)}{(l-\nu-\sigma)!} I^{\sigma+1} \left(e^{-2aP}\right)^{\sigma+1} N^{l-\sigma-1} \\ & + a \sum_{\nu=1}^{l} \sum_{\sigma=1}^{l-\nu} (-a)^{\sigma} \frac{(l-\nu)!\,(\sigma+1)}{(l-\nu-\sigma-1)!} I^{\sigma+2} \left(e^{-2aP}\right)^{\sigma+2} N^{l-\sigma-2}, \end{aligned} \tag{35}$$

$$\begin{aligned} N^l H \;=\; & H N^l + \frac{l}{2a} \left(1 - e^{-2aP}\right) N^{l-1} \\ & - \frac{1}{2a} \sum_{\nu=1}^{l} \sum_{\sigma=1}^{l-\nu} (-a)^{\sigma} \frac{(l-\nu)!\,(\sigma+1)}{(l-\nu-\sigma)!} I^{\sigma} \left(e^{-2aP}\right)^{\sigma+1} N^{l-\sigma-1}, \end{aligned} \tag{36}$$

$$N^l I = I N^l + l a e^{-2aP} I^2 N^{l-1}, \tag{37}$$

the matrix elements of the generator I are

$$I_{i,j,k,l}^{(p,q,r,s)} = p \delta_{i+1}^{p} \delta_{j}^{q} \delta_{k}^{r} \delta_{l}^{s} + a p\, (p-1)\, \delta_{i+2}^{p} \left[\frac{q!\,(-2a)^{q-j}}{j!\,(q-j)!}\right] \delta_{k}^{r} \delta_{l-1}^{s}, \tag{38}$$

of the generator P

$$\begin{aligned} P_{i,j,k,l}^{(p,q,r,s)} &= q\delta_i^p \delta_{j+1}^q \delta_k^r \delta_l^s + p\delta_{i+1}^p \left(\frac{(-2a)^{q-j}\, q!}{j!\,(q-j)!} \right) \delta_k^r \delta_{l-1}^s \\ &+ \frac{a}{2} p\,(p-1)\, \delta_{i+2}^p \delta_k^r \delta_{l-2}^s \left(\frac{(-4a)^{q-j}\, q!}{j!\,(q-j)!} \right) \\ &+ (-a)^{p-i}\, \delta_k^r \left. \frac{q!\,(-2a\,(p-i+1))^{q-j}}{j!\,(q-j)!} \right|_{q\geq j} \frac{p!}{i!} \delta_{l+i-1}^{s+p} \\ &\delta_r^k a \sum_{\nu=1}^{l} \sum_{\sigma=1}^{l-\nu} (-a)^\sigma \frac{(l-\nu)!\,(\sigma+1)}{(l-\nu-\sigma-1)!} (\sigma+i+2)! \\ &\frac{(l-\sigma-2)!}{i!l!} \left. \frac{q!\,(-2a\,(\sigma+2))^{q-j}}{j!\,(q-j)!} \right|_{q\geq j} \delta_{l-\sigma-2}^s \delta_{\sigma+i+2}^p, \end{aligned} \tag{39}$$

of the generator H

$$\begin{aligned} H_{i,j,k,l}^{(p,q,r,s)} &= r\delta_p^i \delta_q^j \delta_r^{k+1} \delta_s^l + \frac{1}{2a} \delta_p^i \delta_q^j \delta_r^k \delta_s^{l-1} - \frac{1}{2a} \left. \frac{(-2a)^{q-j}\, q!}{j!\,(q-j)!} \delta_p^i \delta_r^k \delta_s^{l-1} \right|_{q\geq j} \\ &- \frac{1}{2a} (-a)^{p-i}\, \delta_r^k \left. \frac{q!\,(-2a\,(p-i+1))^{q-j}}{j!\,(q-j)!} \right|_{q\geq j} \frac{p!}{i!} \delta_{s+p-i+1}^l \end{aligned} \tag{40}$$

and of the generator N

$$N_{i,j,k,l}^{(p,q,r,s)} = \frac{\left\langle I^i P^j H^k N^{l+1}, \mu^p x^q t^r v^s \right\rangle}{i!j!k!l!} = s\delta_i^p \delta_j^q \delta_k^r \delta_{l+1}^s. \tag{41}$$

The right actions of the generators of $\mathcal{U}_q\left(\bar{G}\right)$ are given by

$$\begin{aligned} \Phi \vdash I &= \left[\frac{\partial}{\partial \mu} + ave^{-2a\beta} \frac{\partial^2}{\partial \mu^2} \right] \Phi, \\ \Phi \vdash P &= \left\{ \frac{\partial}{\partial x} + ve^{-2a\beta} \frac{\partial}{\partial \mu} + \frac{a}{2} v^2 e^{-4a\beta} \frac{\partial^2}{\partial \mu^2} + \frac{1}{a} \ln\left(1 + a\alpha e^{-2\alpha\beta} v\right) \right\} \Phi, \\ \Phi \vdash H &= \left\{ \frac{\partial}{\partial t} + \frac{v}{2a} \left[1 - e^{-2a\beta} - \frac{e^{-2a\beta}}{1 + \alpha ave^{-2a\beta}} \right] \right\} \Phi, \\ \Phi \vdash N &= \frac{\partial}{\partial v} \Phi. \end{aligned} \tag{42}$$

Instead of functions $e^{\alpha\mu}e^{\beta x}e^{\gamma t}\varphi(v) \in F_q$, we can use formal power series $C[[v]]$ where the action of the generators become

$$\begin{aligned}
\varphi(v) &\vdash I = \alpha\left[1 + a\alpha e^{-2a\beta}v\right]\varphi(v), \qquad (43)\\
\varphi(v) &\vdash P = \left\{\beta + ve^{-2a\beta}\alpha + \frac{a}{2}v^2e^{-4a\beta}\alpha^2 + \frac{1}{a}\ln\left(1 + a\alpha e^{-2\alpha\beta}v\right)\right\}\varphi(v),\\
\varphi(v) &\vdash H = \left\{\gamma + \frac{v}{2a}\left[1 - e^{-2a\beta} - \frac{e^{-2a\beta}}{1 + \alpha a v e^{-2a\beta}}\right]\right\}\varphi(v),\\
\varphi(v) &\vdash N = \frac{\partial}{\partial v}\varphi(v).
\end{aligned}$$

3.2 Configuration Representation

Construction of an induced representation of local type turns out not possible for two facts, first the generators x and t are not isolated on a side of the chosen monomial basis, second $\{I^m N^n\}$ do not span a subalgebra. We use the Bargmann space, which is a generalization of the ordinary Newtonian space (x, t). The points of the Bargmann space are here labeled by three parameters (μ, x, t), time t, location x and an extra term μ, so that μ transforms under Galilean transformations as [3, 4]

$$\mu' = \mu + xv + \frac{1}{2}v^2 t. \qquad (44)$$

In this subsection, we consider the representation Λ_θ, in C, of the deformed subalgebra generated by $\{N\}$ given by

$$N^s \dashv z = z\theta^s, \quad z, \theta \in C. \qquad (45)$$

The representative space $\mathcal{H}_\theta$ is a subspace of $\mathcal{F}_q\left(\bar{G}\right)$ and the elements $\Psi \in \mathcal{H}_\theta$ are equivariant according to (19).

Let $\Psi = \sum_{i,j,k,l} c_{i,j,k,l}\mu^i x^j t^k v^l \in \mathcal{H}_\theta$ then the pairing (11) gives

$$\left\langle I^p P^q H^r N^s, \Psi\right\rangle = p!q!r!s!c_{p,q,r,s}, \qquad (46)$$

and the equivariance condition (19) give

$$\begin{aligned}
\left\langle I^p P^q H^r N^s, \Psi\right\rangle &= N^s \dashv \left\langle I^p P^q H^r, \Psi\right\rangle \qquad (47)\\
&= p!q!r!c_{p,q,r,0}\theta^s,
\end{aligned}$$

from (46) and (47) we have

$$c_{p,q,r,s} = c_{p,q,r,0}\frac{\theta^s}{s!}, \qquad (48)$$

so for any element of $\mathcal{H}_\theta$ we obtain

$$\Psi = \sum_l \frac{(\theta v)^l}{l!}\sum_{i,j,k} c_{i,j,k}\mu^i x^j t^k, \qquad (49)$$

where $c_{i,j,k,0} = c_{i,j,k}$. Finally the representative space $\mathcal{H}_\theta$ is the space of elements of the form

$$\Psi = e^{\theta v}\psi(\mu, x, t), \qquad (50)$$

where ψ is a formal series of μ, x and t.

The configuration induced representation corresponds to the induced action (⊣) of the elements X of $\mathcal{U}_q\left(\bar{G}\right)$ on the space $\mathcal{H}_\theta$, and is given by

$$X \dashv \Psi = \sum_{p,q,r,s} c_{p,q,r} \frac{\theta^s}{s!} X \dashv \mu^p x^q t^r v^s. \tag{51}$$

Consider a general form for the action on the ordered monomials $\mu^p x^q t^r v^s$, the matrix elements $X_{i,j,k,l}^{(p,q,r,s)}$ can be calculated by the use of the pairing

$$\tilde{X}_{i,j,k,l}^{(p,q,r,s)} = \frac{\left\langle I^i P^j H^k N^l, X \dashv \mu^p x^q t^r v^s \right\rangle}{i!j!k!l!}. \tag{52}$$

Now, with a reformulation of (17)

$$\left\langle I^i P^j H^k N^l, X \dashv \mu^p x^q t^r v^s \right\rangle = \left\langle X I^i P^j H^k N^l, \mu^p x^q t^r v^s \right\rangle, \tag{53}$$

we deduce the matrix elements of the generators I, P, H and N

$$\tilde{I}_{i,j,k,l}^{(p,q,r,s)} = p\delta_p^{i+1}\delta_q^j\delta_r^k\delta_s^l \tag{54}$$

$$\tilde{P}_{i,j,k,l}^{(p,q,r,s)} = q\delta_p^i\delta_q^{j+1}\delta_r^k\delta_s^l \tag{55}$$

$$\tilde{H}_{i,j,k,l}^{(p,q,r,s)} = r\delta_p^i\delta_q^j\delta_r^{k+1}\delta_s^l \tag{56}$$

$$\begin{aligned}
\tilde{N}_{i,j,k,l}^{(p,q,r,s)} &= s\delta_i^p\delta_j^q\delta_k^r\delta_{l+1}^s + \frac{1}{2a}\delta_i^p\delta_j^q\delta_{k-1}^r\delta_l^s - \frac{1}{2a}\delta_i^p\frac{q!\,(-2a)^{q-j}}{j!\,(q-j)!}\delta_{k-1}^r\delta_l^s \\
&\quad + p\,(p-1)\,\delta_{i+2}^p\frac{q!\,(-2a)^{q-j+1}}{(j-1)!\,(q-j+1)!}\delta_k^r\delta_l^s \\
&\quad - (-a)\,p\,(p-1)\,\delta_{i+1}^p\frac{q!\,(-2a)^{q-j}}{j!\,(q-j)!}\delta_k^r\delta_l^s.
\end{aligned} \tag{57}$$

Then the right action of the generators of $\mathcal{U}_q\left(\bar{G}\right)$ is

$$\begin{aligned}
I \dashv \Psi &= \frac{\partial}{\partial \mu}\Psi, \\
P \dashv \Psi &= \frac{\partial}{\partial x}\Psi, \\
H \dashv \Psi &= \frac{\partial}{\partial t}\Psi, \\
N \dashv \Psi &= \left[\frac{\partial}{\partial v} + \frac{t}{2a}\left(1 - e^{-2a\partial_x}\right) + (x-a)\,e^{-2a\partial_x}\frac{\partial^2}{\partial \mu^2}\right]\Psi.
\end{aligned} \tag{58}$$

In term of functions $\psi(\mu, x, t)$ the action of the generators becomes

$$\begin{aligned}
I \dashv \psi(\mu, x, t) &= \frac{\partial}{\partial \mu}\psi(\mu, x, t), \\
P \dashv \psi(\mu, x, t) &= \frac{\partial}{\partial x}\psi(\mu, x, t), \\
H \dashv \psi(\mu, x, t) &= \frac{\partial}{\partial t}\psi(\mu, x, t), \\
N \dashv \psi(\mu, x, t) &= \left[\theta + \frac{t}{2a}\left(1 - e^{-2a\partial_x}\right) + (x-a)\,e^{-2a\partial_x}\frac{\partial^2}{\partial \mu^2}\right]\psi(\mu, x, t).
\end{aligned} \tag{59}$$

The representative space $\mathcal{H}_\theta$ can be identified with the space of $C^\infty-$functions, and therefore can be endowed with a scalar product

$$\left(\psi, \psi'\right)_\theta = \int d\mu \int dx \int dt \psi(\mu, x, t)\overline{\psi'(\mu, x, t)}. \tag{60}$$

With this inner product (60) we have $\left(X \vdash \psi, \psi'\right) = \left(\psi, X^* \vdash \psi'\right)$ for any genarators.

4 Concluding Remarks

The problem of induced representations of the $(1 + 1)-$quantum Galilei group has been studied in [2]. In this work we obtained the momentum representation labeled by three parameters (α, β, γ) (43). However, in [2], this same representation is defined as a two parametric family induced from a one dimensional character labeled by two parameters (m, u) and given by $\omega_{m,u} = \exp\left[-i(m\hat{\mu} + u\hat{t})\right]$ ([2]). Instead of using the generators $\left\{M, K, K^{-1}, T, B\right\}$ as in [2] , which gives a non diagonal pairing, in this paper the quantum enveloping algebra is expressed in terms of the generators $\{I, P, H, N\}$ with relation (12). In this paper we have defined a representation of the deformed sub algebra generated by $\{I, P, H\}$ labeled by three parameters (α, β, γ) (43) and constructed from them a family of induced representations including those presented in [2]. The purpose of this paper is a contribution and some extent of the extended work made in [2]. We wish to show that the work made in [2] can be achieved by considering a 3-dimensional setting from the very beginning. Whereas in this paper, the construction of the configuration representation is new. As mentioned in the introduction, our work is part of the schema undertaken by M. B. Mensky for the Galilei, Poincaré and de Sitter groups. Our goal is to provide a platform for the construction of a deformed quantum mechanics based on the symmetry of the quantum Galilei algebra. The intertwining of the two representations will logically give us the counterpart of the propagator for the distorted symmetry and we hope to predict the effects of deformation of the symmetry. It remains an open task to investigate first if we obtain new irreducible representations, but this parts of the problem are not of great importance to us. On the other hand, our next purpose is the intertwining of the two constructed representations as done for the classical Galilei group in [5].

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