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Coherent Planar Symmetric Spacetimes Generated by Progressive Waves of Nambu–Goldstone Bosons

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Abstract Within a $SO(3,1)-$gauge invariant pseudo-orthonormal (Cartan) formalism, in the present paper, we are going to deal with the Einstein–Nambu–Goldstone system of equations, for a manifold with at least G_4 up to G_6 group of motion and a massless source-field excited along the $z-$direction. This is also equivalent with the pure radiation energy–momentum tensor coming from circularly polarized waves generated by a rotating magnetic field. The corresponding essential equation which establishes the connection between the spacetime geometry and the matter-field is solved in some physically interesting cases.

Keywords Einstein–Goldstone equations · Pure radiation · Mathieu functions · Nambu–Goldstone bosons

1 Introduction

In the last decades, the pure radiation metrics have been an active field of investigations and a classification of homogeneous pure radiation solutions has been performed in [1], based on the results publishes by Wils and Steele [2, 3].

Recently, it has been shown that all these spacetimes are satisfying the Einstein–Maxwell equations [4] and a conformal approach for the analysis of the non-linear stability of pure radiation cosmologies has been performed in [5].

In spite of their simple appearance, such spacetimes have peculiar properties as for example they may generate stable closed timelike curves [6].

Soon after Wils obtained what he claimed to be the only conformally flat, pure radiation metric which is not a plane wave [7], his result was generalized by Edgar and Ludwig [8].

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They also have shown that the class of metrics which are not plane waves and permits neither massless scalar fields, nor neutrino fields, can be completed with a subset of plane-fronted waves, subjected to special conditions on the Weyl tensor.

In terms of techniques, once Koutras [9] pointed out that the Wils's spacetime does not admit Killing vectors, the Geroch–Held–Penrose (GHP) formalism [10] has been considered as particularly suitable [11].

In what it concerns the so-called Nambu–Goldstone bosons which are geometrodynamically supporting our spacetime, these appear in models exhibiting a spontaneously broken continuous symmetry and remain massless as long as the symmetry is not also explicitly broken. Discovered by Yoichiro Nambu in the context of the Bardeen–Cooper–Schrieffer (BCS) superconductivity mechanism [12], this type of particles has been encountered in a large variety of domains. Today it is believed that studies on both Goldstone and pseudo-Goldstone modes have potentially far reaching future perspectives in high energy physics and cosmology.

2 The Einstein–Nambu–Goldstone System

The "would be" Goldstone particle, which gets eaten by the gauge field, is the well-known feature of the celebrated Higgs Mechanism. In its own rights, the Nambu–Goldstone field is characterized by the $SO(3,1)-$invariant Lagrangian density

$$\mathcal{L}[\phi] = -\,\frac{1}{2}\,g^{ik}\phi_{,i}\,\phi_{,k}\ , \tag{1}$$

on a Lorentzian base-manifold endowed with the signature $+2$ metric, $ds^2 = g_{ik}dx^i dx^k$.

In terms of pseudo-orthonormal tetrads, defining the bases $\mathcal{B}$ on $T(M_4)$, $E_a = E^i_a(x)\partial_i$, with $g(E_a, E_b) = \eta_{ab} = \text{diag}[1,1,1,-1]$, the line element becomes $ds^2 = \eta_{ab}\,\Omega^a\Omega^b$. The 1-forms $\Omega^a = \Omega^a_i(x)dx^i$ define the dual pseudo-orthonormal bases $\mathcal{B}^*$, on $T^*(M_4)$, so that $\langle\Omega^a,\ E_b\rangle = \delta^a_b$.

The Lagrangian density (1) achieves the manifestly $SO(3,1)-$gauge invariant expression

$$\mathcal{L}[\phi] = -\,\frac{1}{2}\,\eta^{ab}\phi_{|a}\phi_{|b}\,, \tag{2}$$

where $\phi_{|c} = E_c(\phi) = E^i_c\phi_{,i}$, with $c = \overline{1,4}$ and the corresponding Euler–Lagrange equation gets the form

$$\eta^{ab}\phi_{|ab} - \eta^{ab}\Gamma^c{}_{ab}\phi_{|c} = 0\,. \tag{3}$$

The connection coefficients build up the 1-forms $\Gamma^a{}_b = \Gamma^a{}_{bc}\Omega^c$ which decode the Cartan's first structure equation (with no torsion)

$$d\Omega^a + \Gamma^a{}_b \wedge \Omega^b = 0 \ \Rightarrow\ d\Omega^a = \Gamma^a_{[bc]}\Omega^b \wedge \Omega^c \tag{4}$$

with $1 \le b < c \le 4$ and $\Gamma_{d[bc]} = \Gamma_{dbc} - \Gamma_{dcb}$, where $\Gamma_{dbc} = \eta_{da}\Gamma^a{}_{bc}$.

In coordinate-free formulation, the curvature tensor components $R^a_{\ bcd}$ are given by the Cartan's second equation

$$\mathbf{R}^a{}_b = d\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b\ , \tag{5}$$

where the curvature 2-forms are defined as

$$\mathbf{R}^a{}_b = R^a{}_{bcd}\ \Omega^c \wedge \Omega^d\ ,$$

with $1 \leq c < d \leq 4$. The corresponding Ricci tensor components, scalar curvature and the Einstein tensor components will be given by the well-known relations $R_{ab} = \eta^{cd} R_{cadb}$, $R = \eta^{ab} R_{ab}$ and

$$G_{ab} = R_{ab} - \frac{1}{2}\,\eta_{ab}\,R\,.$$

Using the general expression of the conservative energy–momentum tensor, valid for any set of matter-fields governed by the invariant Lagrangian density (2), namely

$$T_{ab} = \phi_{|a}\phi_{|b} - \frac{1}{2}\,\eta_{ab}\,\phi^{|c}\phi_{|c}\,, \tag{6}$$

in the celebrated Einstein Equations $G_{ab} = \kappa_0 T_{ab}$, the whole Einstein–Nambu–Goldstone system reads

$$\begin{aligned} &R_{ab} - \frac{1}{2}\eta_{ab} R = \kappa_0 \left[\phi_{|a}\phi_{|b} - \frac{1}{2}\,\eta_{ab}\,\phi^{|c}\phi_{|c}\right]; \\ &\eta^{ab}\phi_{|ab} - \eta^{ab}\,\Gamma^{c}{}_{ab}\,\phi_{|c} = 0\,. \end{aligned} \tag{7}$$

In the followings, let us discuss the simplest clues of symmetry on choosing the metric. Thus, once the source-field is excited just along the $z-$direction, its independence of the planar coordinates x and y brings in the three generators of the Euclidean plane $\mathbf{R}^2$ isometries, $\hat{K}_1 = \partial_x$, $\hat{K}_2 = \partial_y$, $\hat{K}_3 = x\partial_y - y\partial_x$, which should be shared by the generated spacetime too, so that the metric reads

$$ds^2 = e^{2f(z,t)}\delta_{AB}\,dx^A dx^B + (dz)^2 - (dt)^2\,, \tag{8}$$

where $A, B = 1, 2$ and we have kept flat and orthogonal the two "active" directions $\{\partial_z,\ \partial_t\}$. One additional reason comes from the *massless* character of $\phi(z, t)$, that basically demands the activation of at least one of the two null directions and their respective coordinates

$$u = \frac{1}{\sqrt{2}}(t - z)\ , \quad v = \frac{1}{\sqrt{2}}(t + z)\,,$$

i.e. the metric (8) can be also written in the very suggestive form

$$ds^2 = e^{2f(u,v)}\delta_{AB}\,dx^A dx^B - 2\,du dv\ . \tag{9}$$

The essential Cartan components of the energy–momentum tensor T_{ab} being

$$\begin{aligned} &T_{11} = \frac{1}{2}\left[(\phi,_4)^2 - (\phi,_3)^2\right] = T_{22}\ , \\ &T_{33} = \frac{1}{2}\left[(\phi,_4)^2 + (\phi,_3)^2\right] = T_{44} \ \text{ and } \ T_{34} = \phi,_3\,\phi,_4\ \ , \end{aligned}$$

once we impose the progressive propagation of the Nambu–Goldstone wave, only the retarded null-coordinate "u" gets dynamized, so that the plane symmetric components vanish as well. The remaining three components achieve the highly algebraically symmetric expression

$$T_{33} = -T_{34} = T_{44} = \frac{1}{2}\left(\frac{d\phi}{du}\right)^2\,,$$

which casts the whole energy–momentum tensor of the progressively propagating Nambu–Goldstone field into the form of pure radiation

$$T_{ab} = \frac{1}{2}\left(\frac{d\phi}{du}\right)^2\left[\delta^4_a - \delta^3_a\right]\left[\delta^4_b - \delta^3_b\right]. \tag{10}$$

To see it clearly, in terms of the spacetime metric written as

$$ds^2 = \delta_{AB}\,\omega^A\omega^B - 2\omega^3\omega^4\,,$$

where

$$\omega^1 = \Omega^1\,,\ \omega^2 = \Omega^2\,,\ \omega^3 = \frac{1}{\sqrt{2}}\left(\Omega^4 - \Omega^3\right),\omega^4 = \frac{1}{\sqrt{2}}\left(\Omega^4 + \Omega^3\right);$$
$$e_1 = E_1\,,\ e_2 = E_2\,,\ e_3 = \frac{1}{\sqrt{2}}\left(E_4 - E_3\right),\ e_4 = \frac{1}{\sqrt{2}}\left(E_4 + E_3\right),$$

with $g_{\hat{a}\hat{b}} = g\left(e_{\hat{a}}, e_{\hat{b}}\right)$ given by the ± 1 values, $g_{AB} = \delta_{AB}$ and $g_{34} = -1$, we take the corresponding geometric object

$$\mathbf{T}[\phi] = T_{ab}\,\Omega^a \otimes \Omega^b = \frac{1}{2}\left(\frac{d\phi}{du}\right)^2\left[\Omega^4 - \Omega^3\right]\otimes\left[\Omega^4 - \Omega^3\right] = \left(\frac{d\phi}{du}\right)^2 du \otimes du\,,$$

which, with the proper energy density

$$w(\phi) = \frac{1}{2}\left(\frac{d\phi}{du}\right)^2,$$

turns into the well-known form, in null-tetrad bases,

$$\mathbf{T}[\phi] = 2w(\phi)\omega^3 \otimes \omega^3\,,$$

where $\omega^3 = du$ and $\omega^4 = dv$. Thus, the only non-vanishing covariant component of $\mathbf{T}[\phi]$ is $T_{\hat{3}\hat{3}}[\phi] = 2w(\phi)$ and the whole components can be written as

$$T_{\hat{a}\hat{b}} = 2w(\phi)\delta^3_{\hat{a}}\delta^3_{\hat{b}}\,.$$

This clearly shows a radiative field propagating at the speed of light along the z−direction, the dynamized null-coordinate being the retarded one, "u".

3 The Pseudo-orthonormal Frame Picture

For the sake of completeness, we give below the calculations on the geometrodynamics of the system, in terms of pseudo-orthonormal basis. Thus, defining the dual pseudo-orthonormal tetrad corresponding to (8), i.e.

$$\Omega^1 = e^f dx\,,\ \Omega^2 = e^f dy\,,\ \Omega^3 = dz\,,\ \Omega^4 = dt\,,$$

the first Cartan's Equations (4) give the essential 1-forms

$$\Gamma_{13} = f_{,3}\,\Omega^1\,,\ \Gamma_{23} = f_{,3}\,\Omega^2\,, \Gamma_{14} = f_{,4}\,\Omega^1\,, \Gamma_{24} = f_{,4}\,\Omega^2\,, \tag{11}$$

which are leading, through the second Cartan's Eq. (5), to the non-vanishing curvature 2-forms

$$\begin{aligned}
\mathbf{R}_{12} &= \left[(f_{,4})^2 - (f_{,3})^2\right]\Omega^1 \wedge \Omega^2\,,\\
\mathbf{R}_{13} &= -\left[f_{,33} + (f_{,3})^2\right]\Omega^1 \wedge \Omega^3 - [f_{,34} + f_{,3}\,f_{,4}]\,\Omega^1 \wedge \Omega^4\,,\\
\mathbf{R}_{23} &= -\left[f_{,33} + (f_{,3})^2\right]\Omega^2 \wedge \Omega^3 - [f_{,34} + f_{,3}\,f_{,4}]\,\Omega^2 \wedge \Omega^4\,,\\
\mathbf{R}_{14} &= -[f_{,34} + f_{,3}\,f_{,4}]\,\Omega^1 \wedge \Omega^3 - \left[f_{,44} + (f_{,4})^2\right]\Omega^1 \wedge \Omega^4\,,\\
\mathbf{R}_{24} &= -[f_{,34} + f_{,3}\,f_{,4}]\,\Omega^2 \wedge \Omega^3 - \left[f_{,44} + (f_{,4})^2\right]\Omega^2 \wedge \Omega^4\,.
\end{aligned} \tag{12}$$

With the essential components of the Einstein tensor

$$
\begin{aligned}
&G_{11} = f,_{33} - f,_{44} + (f,_3)^2 - (f,_4)^2 = G_{22}\ ;\\
&G_{33} = (f,_3)^2 - 2f,_{44} - 3\,(f,_4)^2\ ;\quad G_{44} = (f,_4)^2 - 2f,_{33} - 3\,(f,_3)^2\,;\\
&G_{34} = -2\,[f,_{34} + f,_3\, f,_4\,]\,,
\end{aligned}
\tag{13}
$$

the Einstein–Nambu–Goldstone system does fully read

$$
\begin{aligned}
&f,_{33} - f,_{44} + (f,_3)^2 - (f,_4)^2 = \frac{\kappa_0}{2}\left[(\phi,_4)^2 - (\phi,_3)^2\right];\\
&(f,_3)^2 - 2f,_{44} - 3\,(f,_4)^2 = \frac{\kappa_0}{2}\left[(\phi,_4)^2 + (\phi,_3)^2\right];\\
&(f,_4)^2 - 2f,_{33} - 3\,(f,_3)^2 = \frac{\kappa_0}{2}\left[(\phi,_4)^2 + (\phi,_3)^2\right];\\
&f,_{34} + f,_3\, f,_4 = -\,\frac{\kappa_0}{2}\,\phi,_3\,\phi,_4\ ;\\
&\phi,_{33} - \phi,_{44} + 2f,_3\,\phi,_3 - 2f,_4\,\phi,_4 = 0\,.
\end{aligned}
\tag{14}
$$

There are few comments now, shedding a brighter light onto the deep connection between Geometry and Physics. Firstly, the metric itself is radiating, since $G_{34} \neq 0$, as the field is exciting along the $z-$direction. However, that is not the genuine gravitational radiation, made of freely propagating gravitons, because the corresponding components of the Weyl tensor, even though diagonal and good-looking, i.e.

$$
\begin{aligned}
&C_{1212} = -\,\frac{1}{3}\,[f,_{44} - f,_{33}\,] = -C_{3434}\ ,\\
&C_{1313} = \frac{1}{6}\,[f,_{44} - f,_{33}\,] = C_{2323}\ ,\quad C_{1414} = -\,\frac{1}{6}\,[f,_{44} - f,_{33}\,] = C_{2424}\ ,
\end{aligned}
$$

do all vanish for either out-going or in-coming massless perturbations, the resulting planar symmetric manifold becoming conformally flat. Nevertheless, the essential off-diagonal sectional curvatures,

$$
R_{1314} = R_{1413} = R_{2423} = R_{2324} = \frac{\kappa_0}{2}\,\phi,_3\,\phi,_4\ ,
$$

are either progressively or regressively propagated, supporting the Umov 3-vector of the excited Nambu-Goldstone field,

$$
\Pi^3 = T^{34} = -T_{34} = -\phi,_3\,\phi,_4 = \pm\frac{1}{2}\left(\frac{d\phi}{du}\right)^2\ .
$$

Thence, in brief, each of the initially spacelike blades

$$
d\Sigma_2 = dz\wedge dx = \frac{1}{2}\,\varepsilon_{2\beta\gamma}\,dx^\beta\wedge dx^\gamma\ ,\ d\Sigma_1 = dy\wedge dz = \frac{1}{2}\,\varepsilon_{1\beta\gamma}\,dx^\beta\wedge dx^\gamma\ ,
$$

achieves a supplementary (coherent) curvature in the time-direction, which preserves as common sites the respective planar directions dx and dy, forming therefore the blades $d\sigma_1 = dx\wedge dt$, for $d\Sigma_2$ and $d\sigma_2 = dy\wedge dt$, for $d\Sigma_1$.

The other interesting feature of the system regards a geometry-governing equation, which comes from the fact the $T_{33} = T_{44}$, independently of weather the field is progressively or regressively propagating. Thus, equating the G_{33} and G_{44} components of the Einstein's tensor (13), one finds the important differential equation

$$
2f,_{44} + 4\,(f,_4)^2 = 2f,_{33} + 4\,(f,_3)^2\ ,
\tag{15}
$$

which does simply become the suggestive wave-equation

$$G,_{33} - G,_{44} = 0\ , \quad \text{where} \ \ G = e^{2f(z,t)} = g_{11}(z,t) = g_{22}(z,t)\ ,$$

for the two planar components of the metric tensor g, in coordinate bases. Thence, if $T_{33} = T_{44}$, the necessary condition for the planary symmetric metric

$$ds^2 = G(z,t)\delta_{AB}\, dx^A dx^B + (dz)^2 - (dt)^2$$

to be an exact solution to the corresponding Einstein Equations, is that the warp function G satisfies the d'Alembert equation onto the flat Lorentzian part spanned by $\{(z,t) \in \mathbf{R} \times \mathbf{R}\}$.

This mini-theorem has a series of immediate consequences on the calculations and onto the available exact solutions of the Einstein–Nambu–Goldstone system. For instance, if one considers the complementary two cases of either a static or a purely coherent Nambu–Goldstone field, then, the "generated" spacetimes come into light on the spot, as being respectively described by the metrics

$$ds^2_{st} = |Kz + G_0|\delta_{AB}\, dx^A dx^B + (dz)^2 - (dt)^2\ ,$$

and

$$ds^2_{ch} = |\Omega t + G_0|\delta_{AB}\, dx^A dx^B + (dz)^2 - (dt)^2\ .$$

The corresponding Einstein Equations fall to just one, namely

$$(f,_3)^2 = \frac{\kappa_0}{2}\,(\phi,_3)^2 \ \Rightarrow\ \phi(z) = \phi_0 \pm \frac{1}{\sqrt{2\kappa_0}} \log|Kz + G_0|\ \ ,$$

and respectively

$$(f,_4)^2 = \frac{\kappa_0}{2}\,(\phi,_4)^2 \ \Rightarrow\ \phi(t) = \phi_0 \pm \frac{1}{\sqrt{2\kappa_0}} \log|\Omega t + G_0|\ \ .$$

In spite of their simple appearance, each of these spacetimes is highly pathological and represents, in fact, a very good exactly solvable model for studying naked singularities, non-trivial embeddings and geodesical (in)completeness.

At the end of this section, we are going to manage, mainly algebraically, the equations in (14) in such a way to get them obviously completely integrable. During this process, one would be able to notice the key role played by the "radiative" equation $G_{34} = \kappa_0 T_{34}$ in the system diagonalization. Using the geometry governing Eq. (15), the first Einstein equation in (14) becomes

$$(f,_4)^2 - (f,_3)^2 = \frac{\kappa_0}{2}\left[(\phi,_4)^2 - (\phi,_3)^2\right] \tag{16}$$

and is going to be important a little bit later.

For now, from the first Einstein Equation in (14) and from the sum of the ones corresponding to G_{33} and G_{44}, we get the following relations

$$f,_{33} + (f,_3)^2 = -\frac{\kappa_0}{2}\,(\phi,_3)^2\ \ ;\ \ f,_{44} + (f,_4)^2 = -\frac{\kappa_0}{2}\,(\phi,_4)^2\ \ ,$$

which bring the initial Einstein system in (14) to the much clearer form

$$\begin{aligned}
&(a)\ \ f,_{44} - f,_{33} + 2\left[(f,_4)^2 - (f,_3)^2\right] = 0;\\
&(b)\ \ (f,_4)^2 - (f,_3)^2 = (\psi,_4)^2 - (\psi,_3)^2\,;\\
&(c)\ \ f,_{33} + (f,_3)^2 = -\,(\psi,_3)^2\,;\\
&(d)\ \ f,_{44} + (f,_4)^2 = -\,(\psi,_4)^2\,;\\
&(e)\ \ f,_{34} + f,_3\, f,_4 = -\,\psi,_3\,\psi,_4\ \ ,
\end{aligned} \tag{17}$$

where

$$\psi = \sqrt{\frac{\kappa_0}{2}}\,\phi$$

denotes the physically dimensionless and gravitationally normalized Nambu–Goldstone field. Next, we have to form the linear combinations $(c) + (d) + 2(e)$ and $(c) + (d) - 2(e)$ and to express everything in terms of the null-coordinates $\{u, v\}$, yielding

$$\begin{aligned}
&(1)\ \ f,_{uv} + 2f,_u f,_v = 0\\
&(2)\ \ f,_u f,_v = \psi,_u \psi,_v\\
&(3)\ \ f,_{uv} + f,_u f,_v = -\psi,_u \psi,_v\\
&(4)\ \ f,_{vv} + (f,_v)^2 = -(\psi,_v)^2\\
&(5)\ \ f,_{uu} + (f,_u)^2 = -(\psi,_u)^2\\
&(6)\ \ \psi,_{uv} + f,_v \psi,_u + f,_u \psi,_v = 0\,,
\end{aligned} \tag{18}$$

where we have added the corresponding form of the massless Gordon equation for completeness.

4 The Progressive Essential Einstein Equation and Some Exact Solutions

With respect to the Einstein–Nambu–Goldstone system written in terms of null-coordinates, once we look for progressive exact solutions, $\psi,_v = 0 = f,_v$, it automatically breaks down to just one equation, namely

$$\frac{d^2 f}{du^2} + \left(\frac{df}{du}\right)^2 = -\left(\frac{d\psi}{du}\right)^2 \;\Rightarrow\; \frac{d^2 F}{du^2} + \left(\frac{d\psi}{du}\right)^2 F = 0\,, \tag{19}$$

where $F(u) = e^{f(u)}$, which we term here as the Essential Equation. That is because it establishes a two-way connection between the spacetime geometry, controlled by the metric function $F(u)$ and the excitation matter-field of Nambu–Goldstone nature, with the essential component of the energy momentum tensor

$$T_{44} = T_{33} = \frac{1}{\kappa_0}\left(\frac{d\psi}{du}\right)^2 = -T_{34}\,.$$

In some respect, this fact is quite intriguing since if one had started with two unknown functions (ϕ for physics and f for the geometry), one would have expected to get a pair of nonlinearly coupled differential equations, for the two fields and not just one, as it actually happens. That comes from the fact that, for either progressive or regressive matter-excitations preserving the isometries of the Euclidean plane, the corresponding geometrical fundamental objects experience an algebraically exceptional relaxation, as it can be noticed below, for the case of progressive propagation, namely

$$R_{AB} \equiv 0\,,\ \ R_{A3} = 0 = R_{A4}\,,\ \ R_{33} = R_{44} = -\left[\frac{d^2 f}{du^2} + \left(\frac{df}{du}\right)^2\right] = -R_{34}\,,\ \ R \equiv 0\,.$$

Thus, one ends up with the case for *pure radiation*, in terms of the dual null-tetrads $\omega^3 = du$, $\omega^4 = dv$, where the whole Einstein tensor goes down to just one component and the same is true for the energy–momentum tensor, i.e.

$$G_{\hat{a}\hat{b}} = -2\left[\frac{d^2 f}{du^2} + \left(\frac{df}{du}\right)^2\right]\delta^3_{\hat{a}}\delta^3_{\hat{b}}\,, \quad T_{\hat{a}\hat{b}} = \frac{2}{\kappa_0}\left(\frac{d\psi}{du}\right)^2\delta^3_{\hat{a}}\delta^3_{\hat{b}}\,,$$

and therefore, the $\mathbf{G} = \kappa_0\mathbf{T}$ equations do only come to the essential one.

In addition, it can be proven that the second Bianchi identity and its contractions as well as the four-divergence of the energy–momentum tensor vanish (identically), so that they do not involve any other fundamental equation. Thence, unlike the usual situation in General Relativity, this sort of planar symmetric configurations with pure radiation seems to allow the observer to prepare the excitation field in a certain progressive distribution or, more fundamental, in a certain quantum state.

4.1 The Exact Solution for a Nambu–Goldstone Field of Constant Intensity

Once we deal with a pure-radiation distribution of Umov vector components

$$\Pi^\alpha = c\, w[\phi]\,\delta^\alpha_3 \;\Rightarrow\; \Pi^3 = \frac{c}{\kappa_0}\left(\frac{d\psi}{du}\right)^2\,, \tag{20}$$

where $c = 1$ in natural units, the elementary power transported along the normal to the elementary surface

$$d\Sigma_\alpha = \frac{1}{2}\varepsilon_{\alpha\beta\gamma}\, d\Omega^\beta \wedge d\Omega^\gamma\,,$$

will be given by the expression

$$d\mathcal{P} = \Pi^\alpha\, d\Sigma_\alpha = \frac{c}{\kappa_0}\left(\frac{d\psi}{du}\right)^2 \Omega^1 \wedge \Omega^2 = \frac{c}{\kappa_0}\, F^2(u)\left(\frac{d\psi}{du}\right)^2 d^2x\,,$$

so that the corresponding radiation intensity, measured in W/m^2, is going to read

$$\Pi_0 = \frac{d\mathcal{P}}{d^2x} = \frac{1}{\kappa_0}\, F^2(u)\left(\frac{d\psi}{du}\right)^2 .$$

Based on the finite energy criterion, it is normal to ask for a radiation field of constant intensity, i.e.

$$\left(\frac{d\psi}{du}\right)^2 = \frac{\kappa_0\Pi_0}{F^2(u)}\,, \tag{21}$$

which turns the essential Eq. (19) into the seemingly very simple form

$$F\frac{d^2F}{du^2} = -\kappa_0\,\Pi_0\,.$$

In reality, things are a lot more complicated since, with respect to the null-coordinate u, the F, ψ solutions get (highly) transcendent. One can notice this from the first-integral of the above equation, namely

$$\left(\frac{dF}{du}\right)^2 = C - 2\kappa_0\Pi_0\log\left(\frac{F}{F_0}\right),\ \text{with } C \in \mathbf{R}. \tag{22}$$

Besides the details regarding the reality constraint, one is getting in trouble with the solution to the differential equation coming from the above relation for it cannot be performed by quadratures. Fortunately, expressing $d\psi/du$ with respect to $d\psi/dF$, it yields

$$d\psi = \pm\sqrt{\kappa_0\Pi_0}\left[C - 2\kappa_0\Pi_0 \log\left|\frac{F}{F_0}\right|\right]^{-1/2}\frac{dF}{F}\,,$$

which leads to the workable relation between the two functions,

$$\psi = \frac{\mp 1}{\sqrt{\kappa_0\Pi_0}}\left[C - 2\kappa_0\Pi_0 \log\left|\frac{F}{F_0}\right|\right]^{1/2} \;\Leftrightarrow\; \psi(u) = \frac{sgn}{\sqrt{\kappa_0\Pi_0}}\left(\frac{dF}{du}\right).$$

Thus, it is possible to express the solutions with respect to ψ itself as

$$F(\psi) = \exp\left[-\frac{1}{2}\psi^2\right],$$

where, with no loss of generality, the constants F_0 and C have been set to 1 and 0, respectively. Also, from the initial Eq. (21), i.e.

$$\frac{d\psi}{du} = (sgn)\,(\kappa_0\Pi_0)^{1/2}\,F^{-1}\,,$$

it yields the important row of relations

$$\frac{d\psi}{du} = \pm\sqrt{\kappa_0\Pi_0}\exp\left[\frac{1}{2}\psi^2\right]\,,\quad \frac{du}{d\psi} = \frac{sgn}{\sqrt{\kappa_0\Pi_0}}\exp\left[-\frac{1}{2}\psi^2\right]\,,$$
$$u(\psi) = \frac{sgn}{\sqrt{\kappa_0\Pi_0}}\int \exp\left[-\frac{1}{2}\psi^2\right]d\psi\,. \tag{23}$$

The last relation gives the concrete expression of the null-coordinate u with respect to the (classical) rescaled field ψ, i.e.

$$u = \pm\frac{\sqrt{\pi/2}}{\sqrt{\kappa_0\Pi_0}}\,\mathrm{Erf}\left(\frac{\psi}{\sqrt{2}}\right)\,, \tag{24}$$

where we have used the definition of the error function [13]

$$\int_0^{\psi}\exp\left[-\frac{1}{2}\sigma^2\right]d\sigma = \sqrt{\frac{\pi}{2}}\,\mathrm{Erf}\left(\frac{\psi}{\sqrt{2}}\right)\,,\ \text{with } \psi\in\mathbf{R}\,.$$

Thus, there is no need to carry the signs any longer because the function in (24) is bijective on $\mathbf{R}$, admitting the very useful inverse

$$\psi(u) = \sqrt{2}\,\mathrm{Fre}\left[\frac{\sqrt{\kappa_0\Pi_0}}{\sqrt{\pi/2}}\,u\right], \tag{25}$$

where "Fre" is a short notation for the Inverse Error Function, $\mathrm{Fre} = \mathrm{Erf}^{-1}$.

The second relation in (23), together with the metric function $F(\psi)$, give the $\psi-$representation of the metric, i.e.

$$ds^2 = e^{-\psi^2}\left[\delta_{AB}\,dx^A dx^B - 2\frac{e^{\frac{1}{2}\psi^2}}{\sqrt{\kappa_0\Pi_o}}\,d\psi dv\right],$$

while the first relation (in (23)) leads to the concrete expressions of the non-vanishing curvature and energy–momentum tensor components

$$R_{A3B3} = R_{A4B4} = -R_{A3B4} = \frac{1}{2}\,(\kappa_0 \Pi_0)\, e^{\psi^2}\, \delta_{AB}\,,$$
$$T_{44} = T_{33} = -T_{34} = \Pi_0\, e^{\psi^2}\,.$$

4.2 The Progressive Plane Wave Solution

We have finally arrived at the case we had in mind when we had commenced the paper: what the planar symmetric spacetime would look like if it was generated and geometro-dynamically sustained by a somewhat realistic wave, of Nambu–Goldstone bosons, of amplitude ϕ_0 and pulsation ω, sent vertically straight-away.

Since, roughly speaking, any physically acceptable function of variable $\chi = ct - z$, i.e. $\chi = \sqrt{2}u$, can be represented by the Fourier transformation

$$\Phi(\chi) = \int_{\mathbf{R}_+} \left[C_+(k)\cos(k\chi) + C_-(k)\sin(k\chi)\right] dk\,,$$

we consider, for simplicity — as in the studies of alternative currents — just the sine-harmonic channel of pulsation $\omega = ck$, so that the simplest wave-form reads

$$\phi(\chi) = \phi_0 \sin(k\chi)\,,$$

where, in international units, k is in $(m)^{-1}$ and the amplitude ϕ_0 is measured in $[J/m]^{1/2}$. In realistic situations, talking about macroscopic waves of particles, the "single" amplitude ϕ_0 will not be measured directly, for experimentally (coming from quantum physics) it is not an observable, unlike the energy, the momentum or even the averaged Umov–Poynting intensity. Therefore, some care is needed when one matches ϕ_0^2 to the actual intensity $\Pi_3[\phi]$, in order to get the measurable value $\Pi_3(0)$ of the power transported through the unit area of the $\{z = 0\}$−plane, at the initial moment $t = 0$. Concretely, with the expression of the energy density

$$w = \frac{1}{2}\left[(\phi_{,4})^2 + (\phi_{,3})^2\right] = (k\phi_0)^2\cos^2(k\chi)\,,$$

the locally measured intensity (not the integrated one) reads

$$\Pi_3 = cw = c\,(k\phi_0)^2\cos^2(k\chi)\,,$$

from where it yields the exact relation we were talking about, i.e.

$$\phi_0 = \frac{1}{\omega}\sqrt{c\,\Pi_3(0)}\,. \tag{26}$$

As it can be checked, all these relations are consistent with respect to the System of International Units. It is now that we may measurably couple the gravity, by the Einstein's constant $\kappa_0 = (8\pi G_N)/c^4$, in m/J, into the concrete form of the Essential Equation

$$\frac{d^2F}{d\chi^2} + \frac{\kappa_0}{2}\,(k\phi_0)^2\cos^2(k\chi)F = 0\,.$$

Using (26), it does simply become the particular Mathieu Equation [14]

$$\frac{d^2F}{d\chi^2} + \frac{\kappa_0\Pi_3(0)}{2c}\cos^2\left(\frac{\omega}{c}\chi\right)F = 0\,, \tag{27}$$

wherein, contrary to the naive expectation of a $k^2 = \omega^2/c^2$ dependence of the Mathieu's coefficient, one does only have the spectral flat coefficient

$$K^2 = \frac{\kappa_0 \Pi_3(0)}{2c} = \frac{4\pi G_N}{c^5}\, \Pi_3(0)\,, \tag{28}$$

measured in $(m)^{-2}$. As it is known from the physics of gravitational collapse, the quantity $P_N = 2c^5/G_N$ has the concrete dimension of power and does actually represent (classically) the maximum gravitationally fully radiated power in a perfect collapse. Moreover, since at the Planck level one can define the corresponding power as the energy radiated by a Planck-mass particle once it vanishes in just a Planck-time duration, it yields the relation

$$P_* = \frac{M_P\, c^2}{\ell_P/c} = \frac{M_P^2\, c^4}{\hbar} = \frac{c^5}{G_N}\,.$$

Thus, albeit the factor of "2" which is related to the Schwarzschild term $2G_N M_P/c^2$ in the metric, in order to get a Planck-sized Black-Hole, the tremendous $P_N \sim 10^{52}(W)$ value has a surprisingly strong quantum support.

In terms of the dimensionless variable $\alpha = \omega\chi/c$ and parameters

$$q = \frac{\pi c^2}{\omega^2}\frac{\Pi_3(0)}{P_*} \equiv \frac{\pi c}{P_*}\phi_0^2\,, \quad a = 2q\,, \tag{29}$$

the Eq. (27) turns into the canonical form of the Mathieu equation [13]

$$\frac{d^2F}{d\alpha^2} + 2q\,[1+\cos(2\alpha)]\,F = 0\,, \tag{30}$$

whose solutions,

$$F = \{MathieuC\,[a,q,\alpha]\,, MathieuS\,[a,q,\alpha]\}\,,$$

are of the form $F(\alpha) \sim e^{i\gamma\alpha}u(\alpha)$, where $u(\alpha)$ is a periodic function, while the the Mathieu Characteristic Exponent (MCE), γ, may be real or imaginary, depending on the values of the model parameters [15]. In particular, looking for the solution as the complex Fourier series

$$F(\alpha) = \sum_{n=-\infty}^{\infty} F_n\, e^{i(n+\gamma)\alpha}\,,$$

we get the following recurrent equation for the Fourier amplitudes,

$$F_n + \zeta_n \left[F_{n+2} + F_{n-2}\right] = 0\,,$$

where

$$\zeta_n = \frac{q}{2q-(\gamma+n)^2}\,.$$

For $n = 1$, we impose the vanishing of the determinant associated to the homogeneous system for the unknowns $\{F_1,\ F_0\,,\ F_{-1}\}$, i.e.

$$\begin{vmatrix} 1 & 0 & \zeta_1 \\ 0 & 1 & 0 \\ \zeta_{-1} & 0 & 1 \end{vmatrix} = 0$$

which leads to the following relation between the parameter q and the MCE γ,

$$q^2 = \left[2q-(\gamma-1)^2\right]\left[2q-(\gamma+1)^2\right].$$

Denoting $\gamma^2 = \sigma$, we get a second degree equation, with the solutions

$$\sigma = 2q + 1 \pm \sqrt{q(q+8)}\,.$$

One may notice that, regarding the lower branch, for

$$2q + 1 < \sqrt{q(q+8)} \text{ i.e. } (q-1)\left(q - \frac{1}{3}\right) < 0\,,$$

the MCE, γ, gets purely imaginary, pointing out the first range of exponentially growing instabilities

$$\frac{1}{3} < q < 1\,. \tag{31}$$

On the other hand, the upper branch, i.e.

$$\gamma^2 = 2q + 1 + \sqrt{q(q+8)}\,,$$

is characterizing the stable periodic solutions, since q is always positive.

5 Conclusions

The present paper deals with the Einstein–Nambu–Goldstone (ENG) system of equations in the $SO(3,1)$-invariant formulation. Besides explicit calculations on the geometro-dynamics of the system, some connections between geometry and physics are also pointed out. The main attention has been given to the progressive essential equation coming from the ENG system written in terms of null-coordinates and some of its exact solutions were derived.

It has turned out that, if the effective intensity Π_0, measured in W/m^2 into the actually generated (curved) spacetime, is *constant* – for the Nambu–Goldstone radiation speeding upwards – then the required dimensionless field distribution, with respect to the null-coordinates "u", should be (25), i.e. the effective Nambu–Goldstone field, measured in $(J/m)^{1/2}$, is

$$\phi(z,t) = \frac{2}{\sqrt{\kappa_0}} \mathrm{Fre}\left[\sqrt{\frac{\kappa_0 \Pi_0}{\pi}}(ct - z)\right],$$

leading, as an exact solution, to the metric function

$$F(z,t) = \exp\left\{-\left[\mathrm{Fre}\left(\sqrt{\frac{\kappa_0 \Pi_0}{\pi}}(ct - z)\right)\right]^2\right\}$$

and obviously to the respective spacetime metric written in terms of the usual Minkowski coordinates

$$ds^2 = \exp\left\{-2\,\mathrm{Fre}^2\left[\sqrt{\frac{\kappa_0 \Pi_0}{\pi}}(ct - z)\right]\right\} \delta_{AB}\, dx^A dx^B + (dz)^2 - c^2(dt)^2\,.$$

Last but not least, the progressive plane wave solution has been expressed in terms of the Mathieu's functions of parameters (29). As stated in the general theory, for $q \ll 1$, the characteristic values a_n, coming from the resonance condition $a_n \approx n^2$, yield periodic solutions and separate the regions of stability. In the case under consideration, one has the relation $q = a/2$ and the stable regions, in the stability chart, situated between the characteristic curves $a_n(q)$, become more and more narrow as q, i.e. $\Pi_3(0)$, is increasing. In the first band of instability, (31), the imaginary part of the MCE comes into play, leading to the exponentially growing behaviour of the oscillatory metric function F, solution to (27). Finally, in the first stability region, $0 < q < 1/3$, at $q = 1/8$, which points out the gravitational resonant pulsation

$$\omega_G = \frac{2}{c}\sqrt{\frac{2\pi G_N \Pi_3(0)}{c}} = 2.5 \times 10^{-10}\sqrt{\Pi_3(0)}\ (years)^{-1}\,,$$

there is a strongly amplified mode of the scale function F.

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