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Generalized Steering Robustness of Bipartite Quantum States

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Abstract EPR steering is a kind of quantum correlation that is intermediate between entanglement and Bell nonlocality. In this paper, by recalling the definitions of unsteerability and steerability, some properties of them are given, e.g, it is proved that a local quantum channel transforms every unsteerable state into an unsteerable state. Second, a way of quantifying quantum steering, which we called the generalized steering robustness (GSR), is introduced and some interesting properties are established, including: (1) GSR of a state vanishes if and only if the state is unsteerable; (2) a local quantum channel does not increase GSR of any state; (3) GSR is invariant under each local unitary operation; (4) as a function on the state space, GSR is convex and lower-semi continuous. Lastly, by using the majorization between the reduced states of two pure states, GSR of the two pure states are compared, and it is proved that every maximally entangled state has the maximal GSR.

Keywords Quantum steering · Unsteerability · Steerability · Generalized steering robustness

1 Introduction

The nonlocality of entangled states, a key feature of quantum mechanics, was first pointed out in 1935 by Einstein, Podolsky, and Rosen (EPR) [1]. The EPR paper provoked an interesting response from Schrödinger [2], who introduced the term steering for Alice's ability to affect Bob's state through her choice of measurement basis. That is, by measuring her subsystem, Alice can remotely change the state of Bob's subsystem in such a way that would be impossible if their systems were only classically correlated.

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EPR steering was recently given an operational interpretation [3]: Alice wants to convince Bob, who does not trust her, that they share an entangled state. Bob, in order to be convinced, asks Alice to remotely prepare a collection of states of his subsystem. Alice performs her measurements (which are unknown to Bob) and communicates the results to him. By looking at the conditional states prepared by Alice, Bob is able to certify if they must have come from measurements on an entangled state. Thus, it follows that steerability is stronger than nonseparability and weaker than Bell nonlocality.

Since then, although our understanding of EPR steering has advanced greatly recently, two fundamental questions remains open: Given that a quantum state, (i) how to judge it steerable or not? (ii) how to quantify it? There exist many works which demonstrate steering through the violation of various kinds of steering inequalities [4–14]. Apart from the fundamental interest in steering, there is also an applied motivation for studying and implementing [15, 16]. Skrzypczyk et al. introduced an operationally motivated method by semidefinite programming to quantify EPR steering of arbitrary finite-dimensional bipartite quantum states [17, 18].

Usually, robustness describes the endurance of some property of an object with respect to disturbance. There exist some works on robustness of quantum features [13, 19–23], where Piani and Watrous introduced steering robustness of a state and applied it to prove that every steerable state is useful in subchannel discrimination with one-way LOCC measurements[13], Vidal and Tarrach discussed robustness of entanglement by investigating the effect of mixing certain entangled state with any separable state and investigated the minimal amount of entanglement-free mixing needed to wipe out all entanglement[19], Steiner introduced the generalized robustness of entanglement by mixing a state with any state (not necessarily separable)[20], Meng introduced and discussed in [21, 22] the robustness and generalized robustness of contextuality and Guo possed the robustness of quantum correlation[23]. Based on these ideas, our aim is to introduce generalized steering robustness, which can quantify steerability of a quantum state and describe steering endurance against disturbance.

The reminder parts of this paper are organized as follows. In Section 2, we recall the definitions of unsteerability and steerability and discuss some related properties. In Section 3, we introduce the generalized steering robustness (GSR) of a state and discuss its properties. In Section 4, by using the majorization between the reduced states of two pure states, we compare GSR of the two states. The last section is devoted to a summary and the conclusions of this paper.

2 Unsteerable and Steerable Quantum States

Consider a bipartite system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, denote by $\mathcal{D}_{AB}$ and $\mathcal{P}_{AB}$ the sets of all quantum states and pure states on $\mathcal{H}_{AB}$, respectively. Alice and Bob share a quantum state ρ. On her subsystem, Alice makes m_A positive operator-valued measurements (POVMs): $M_x = \{M_{a|x}\}_{a=1}^{o_A} (x = 1, 2, \ldots, m_A)$. Put

$$\mathcal{M}_A = \{M_1, M_2, \ldots, M_{m_A}\} \equiv \{M_{a|x}\}_{a,x},$$

called a *measurement assemblage* of Alice, in which x denotes the measurement choice of Alice and a is the corresponding output. When Alice chooses a measurement x, i.e. M_x, from her measurement assemblage $\mathcal{M}_A$ and get output a, the possible states of Bob are as follows:

$$\rho_{a|x} := \text{tr}_A((M_{a|x} \otimes I_B)\rho). \tag{1}$$

We call the collection $\{\rho_{a|x}\}_{a,x}$ of un-normalized density operators $\rho_{a|x}$ a *state assemblage* of Bob induced by a state ρ and a measurement assemblage $\mathcal{M}_A$.

Definition 2.1 [3] Let $\rho \in \mathcal{D}_{AB}$. A state ρ of AB is said to be *unsteerable by Alice*, if for every measurement assemblage $\mathcal{M}_A$ of A, there exists a probability distribution $\{\pi(\lambda)\}_{\lambda\in\Lambda}$ and a family of quantum states $\{\sigma(\lambda)\}_{\lambda\in\Lambda}$ in $\mathcal{D}_B$ such that

$$\rho_{a|x} = \sum_{\lambda\in\Lambda} \pi(\lambda) p_A(a|x,\lambda)\sigma(\lambda), \quad \forall x, a, \tag{2}$$

where $p_A(a|x,\lambda) \geq 0$ for all a, x and all $\lambda \in \Lambda$, satisfying $\sum_{a=1}^{o_A} p_A(a|x,\lambda) = 1$ for all x, λ. In this case, we call the system of (2) a *local hidden state (LHS) model* of ρ with respect to $\mathcal{M}_A$, or an LHS model of the state assemblage $\{\rho_{a|x}\}_{a,x}$.

A state ρ of AB is said to be *steerable by Alice* if it is not unsteerable. Explicitly, ρ is steerable by Alice means that Alice has a measurement assemblage $\mathcal{M}_A$ such that the state assemblage $\{\rho_{a|x}\}_{a,x}$ has no an LHS model (2).

Unsteerability (steerability) is from [3], the definition given above is more mathematical. Here is a physical explanation. According to [3], quantum steerability means the possibility of remotely generating ensembles that could not be produced by an LHS model. When a state ρ is unsteerable by Alice, for every chosen measurement assemblage $\mathcal{M}_A$, an LHS model (2) exists, which can refer to the situation where a source sends a classical message λ to Alice with a probability $\pi(\lambda)$, and a corresponding quantum state $\sigma(\lambda)$ to Bob. When Alice decides to apply a POVM $M_x = \{M_{a|x}\}_{a=1}^{o_A}$, the variable λ instructs Alice's measurement device to output the result a with probability $p_A(a|x,\lambda)$. Bob does not have access to the classical variable λ, the final assemblage he observed is composed by the elements $\pi(\lambda)$, $p_A(a|x,\lambda)$ and $\sigma(\lambda)$ according to (2). Moreover, (2) implies that $\sum_{\lambda\in\Lambda} \pi(\lambda)\sigma(\lambda) = \rho_B$.

Next let us introduce unsteerability and steerability of a state by Alice with a given measurement assemblage $\mathcal{M}_A$.

Definition 2.2 Let $\rho \in \mathcal{D}_{AB}$ and let

$$\mathcal{M}_A = \{\{M_{a|x}\}_{a=1}^{o_A} : x = 1, 2, \ldots, m_A\}$$

be a measurement assemblage of A. If there exists a probability distribution $\{\pi(\lambda)\}_{\lambda\in\Lambda}$ and a family of quantum states $\{\sigma(\lambda)\}_{\lambda\in\Lambda}$ in $\mathcal{D}_B$ such that (2) holds, then ρ is said to be *unsteerable by Alice with* $\mathcal{M}_A$. Otherwise, we say that ρ is *steerable by Alice with* $\mathcal{M}_A$.

By definition, a state ρ is unsteerable by Alice if and only if for every $\mathcal{M}_A$, it is unsteerable by Alice with $\mathcal{M}_A$; ρ is steerable by Alice if and only if there exists an $\mathcal{M}_A$ such that ρ is steerable by Alice with $\mathcal{M}_A$.

In what follows, we denote by $\mathcal{US}_A$ and $\mathcal{QS}_A$ the sets of all states of $\mathcal{H}_{AB}$ that are unsteerable and steerable by Alice, respectively. If we use $\mathcal{US}_A(\mathcal{M}_A)$ (resp. $\mathcal{S}_A(\mathcal{M}_A)$) to denote the set of all states which are unsteerable (resp. steerable) by Alice with $\mathcal{M}_A$, then we have

$$\mathcal{US}_A = \bigcap_{\mathcal{M}_A} \mathcal{US}_A(\mathcal{M}_A), \mathcal{QS}_A = \bigcup_{\mathcal{M}_A} \mathcal{S}_A(\mathcal{M}_A), \tag{3}$$

where the intersection and the union were taken over all measurement assemblages $\mathcal{M}_A$ of Alice. These expressions can help us to prove that the set $\mathcal{US}_A$ is a convex compact subset of $\mathcal{D}_{AB}$ and so $\mathcal{QS}_A$ is an open subset of $\mathcal{D}_{AB}$, see [30] for the proofs.

Example 2.1 Every separable state is unsteerable by Alice.

Indeed, when ρ is separable, we write $\rho = \sum_{k=1}^{d} \pi_k \rho_k^A \otimes \rho_k^B$ and get that for any $\mathcal{M}_A$, it holds that

$$\mathrm{tr}_A[(M_{a|x} \otimes I_B)\rho] = \sum_{k=1}^{d} \pi_k p_A(a|x,k)\rho_k^B,$$

where $p_A(a|x,k) = \mathrm{tr}(M_{a|x}\rho_k^A)$. Thus, ρ is unsteerable by Alice with any $\mathcal{M}_A$. Thus, ρ is unsteerable by Alice.

Example 2.2 [30] Let $|\psi\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |\varepsilon_i\rangle|\varepsilon_i\rangle$ be a maximally entangled state induced by a real orthonormal basis $\{|\varepsilon_i\rangle\}_{i=1}^n$ for $\mathbb{C}^n$. Then $\rho = |\psi\rangle\langle\psi|$ is steerable by Alice.

Proof Notice that

$$\mathrm{tr}_A\left((|x^*\rangle\langle x^*| \otimes I_B)\rho\right) = \frac{1}{n}|x\rangle\langle x|, \quad \forall |x\rangle \in \mathbb{C}^n, \tag{4}$$

where $|x^*\rangle$ denotes the conjugation of $|x\rangle$. Choose an orthonormal basis $e = \{|e_i\rangle\}_{i=1}^n$ for $H_A = \mathbb{C}^n$ and an $n \times n$ unitary matrix $U = [u_{ij}]$ such that $\sum_i \mathrm{sgn}|u_{ij}| \geq 2$ for each j, that is, each row of U has at least two nonzero entries. Then $f = Ue = \{\sum_{i=1}^n u_{ij}|e_i\rangle\}_{j=1}^n$ is an orthonormal basis for $\mathbb{C}^n$ with

$$(\mathbb{R}|e_i\rangle) \cap (\mathbb{R}|f_j\rangle) = \{0\}, \quad \forall i, j. \tag{5}$$

Define

$$P = \{|e_i^*\rangle\langle e_i^*| : i = 1, 2, \ldots, n\}, \; Q = \{|f_i^*\rangle\langle f_i^*| : i = 1, 2, \ldots, n\},$$

then $\mathcal{M}_A := \{P, Q\}$ is a measurement assemblage of Alice.

Suppose that $\rho = |\psi\rangle\langle\psi|$ is unsteerable by Alice. Then for $\mathcal{M}_A$, there exists a probability distribution $\{\pi(\lambda)\}_{\lambda\in\Lambda}$ and a family of quantum states $\{\sigma(\lambda)\}_{\lambda\in\Lambda}$ in $\mathcal{D}_B$ such that

$$\mathrm{tr}_A((|e_i^*\rangle\langle e_i^*| \otimes I_B)\rho^{AB}) = \sum_{\lambda\in\Lambda} \pi(\lambda)p_A(i|P,\lambda)\sigma(\lambda), \quad \forall i,$$
$$\mathrm{tr}_A((|f_i^*\rangle\langle f_i^*| \otimes I_B)\rho^{AB}) = \sum_{\lambda\in\Lambda} \pi(\lambda)p_A(i|Q,\lambda)\sigma(\lambda), \quad \forall i,$$

where $p_A(i|x,\lambda) \geq 0$ for all $i = 1, 2, \ldots, n$, $x = P, Q$ and all $\lambda \in \Lambda$, satisfying $\sum_{i=1}^n p_A(i|x,\lambda) = 1$ for all $x = P, Q, \lambda \in \Lambda$. By using (4), we get that

$$\sum_{\lambda\in\Lambda} \pi(\lambda)p_A(i|P,\lambda)\sigma(\lambda) = \frac{1}{n}|e_i\rangle\langle e_i| (i = 1, 2, \ldots, n),$$

$$\sum_{\lambda\in\Lambda} \pi(\lambda)p_A(i|Q,\lambda)\sigma(\lambda) = \frac{1}{n}|f_i\rangle\langle f_i| (i = 1, 2, \ldots, n).$$

Thus, for each $\lambda \in \Lambda$ and $i = 1, 2, \ldots, n$, there exist $a_{i,\lambda}, b_{i,\lambda} \in [0, 1]$ such that

$$\pi(\lambda)p_A(i|P,\lambda)\sigma(\lambda) = \frac{1}{n}a_{i,\lambda}|e_i\rangle\langle e_i|,$$

$$\pi(\lambda)p_A(i|Q,\lambda)\sigma(\lambda) = \frac{1}{n}b_{i,\lambda}|f_i\rangle\langle f_i|.$$

Because that $\sum_{i=1}^n p_A(i|P,\lambda) = 1$ for each $\lambda \in \Lambda$, we conclude that for each $\lambda \in \Lambda$, there exists an i_λ such that $p_A(i_\lambda|P,\lambda) \neq 0$ and so

$$\pi(\lambda)\sigma(\lambda) = \frac{a_{i_\lambda,\lambda}}{np_A(i_\lambda|P,\lambda)}|e_{i_\lambda}\rangle\langle e_{i_\lambda}|.$$

This shows that

$$\{\pi(\lambda)\sigma(\lambda) : \lambda \in \Lambda\} \subset \bigcup_{i=1}^{n}(\mathbb{R}|e_i\rangle\langle e_i|) := S_P.$$

Similarly,

$$\{\pi(\lambda)\sigma(\lambda) : \lambda \in \Lambda\} \subset \bigcup_{i=1}^{n}(\mathbb{R}|f_i\rangle\langle f_i|) := S_Q.$$

Thus, $\{\pi(\lambda)\sigma(\lambda) : \lambda \in \Lambda\} \subset S_P \bigcap S_Q$. From (5), we see that $S_P \bigcap S_Q = \{0\}$. Thus, $\pi(\lambda)\sigma(\lambda) = 0$ for all $\lambda \in \Lambda$. This contradicts that fact that

$$\sum_{\lambda\in\Lambda} \pi(\lambda)\sigma(\lambda) = \rho_B.$$

□

The following theorem gives a characterization of unsteerable states.

Theorem 2.1 *Let $\rho \in \mathcal{D}_{AB}$. Then ρ is unsteerable by Alice if and only if for every measurement assemblage $\mathcal{M}_A = \{\{M_{a|x}\}_{a=1}^{o_A} : x = 1, 2, \ldots, m_A\}$ of Alice, there exists a family $\{\sigma'(\lambda)\}_{\lambda\in\Lambda}$ of positive operators on $\mathcal{H}_B$ such that*

$$\rho_{a|x} = \sum_{\lambda\in\Lambda} p_A(a|x, \lambda)\sigma'(\lambda), \quad \forall x, a, \tag{6}$$

where $p_A(a|x, \lambda) \geq 0$ for all x, a and all λ, satisfying $\sum_{a=1}^{o_A} p_A(a|x, \lambda) = 1$ for all x and all λ.

Proof The necessity is proved by taking $\sigma'(\lambda) = \pi(\lambda)\sigma(\lambda)$. To prove the sufficiency, we assume that for every $\mathcal{M}_A$, a family $\{\sigma'(\lambda)\}_{\lambda\in\Lambda}$ satisfying (6) does exist. Then we have

$$\sum_{\lambda\in\Lambda} \mathrm{tr}(\sigma'(\lambda)) = \mathrm{tr}\left(\sum_{a} \rho_{a|x}\right) = \mathrm{tr}(\rho) = 1.$$

Put $\pi(\lambda) = \mathrm{tr}(\sigma'(\lambda))$ and define $\sigma(\lambda) = \frac{1}{\pi(\lambda)}\sigma'(\lambda)$ if $\pi(\lambda) \neq 0$; $\sigma(\lambda) = \rho_B$ (the reduced state of ρ on B) if $\pi(\lambda) = 0$. Then $\{\sigma(\lambda)\}_{\lambda\in\Lambda} \subset \mathcal{D}_B$ and $\sum_{\lambda\in\Lambda} \pi(\lambda) = 1$, (6) becomes (2). This shows that ρ is unsteerable by Alice. □

Corollary 2.1 *Let $\rho \in \mathcal{US}_A$ and $T \geq 0$ such that $\rho' := (I_A \otimes T)\rho(I_A \otimes T^\dagger) \neq 0$. Then $\rho_T := \frac{1}{tr(\rho')}\rho' \in \mathcal{US}_A$.*

Proof Since $\rho \in \mathcal{US}_A$, it follows from Theorem 2.1 that for every $\mathcal{M}_A$, there exists a family $\{\sigma'(\lambda)\}_{\lambda\in\Lambda}$ of positive operators on $\mathcal{H}_B$ such that (6) holds. Hence, for each x and each a, we have

$$\mathrm{tr}_A[(M_{a|x} \otimes I_B)\rho_T] = \frac{1}{\mathrm{tr}(\rho')} T\,\mathrm{tr}_A[(M_{a|x} \otimes I_B)\rho]T^\dagger = \sum_{\lambda\in\Lambda} p_A(a|x, \lambda)\sigma(\lambda),$$

where $\sigma(\lambda) = \frac{1}{\mathrm{tr}(\rho')} T\sigma'(\lambda)T^\dagger$ are positive operators on $\mathcal{H}_B$. Consequently, Theorem 2.1 yields that $\rho_T \in \mathcal{US}_A$. □

At the end of this section, we discuss the influence of a local quantum channel on unsteerability of a state. To do this, we let Φ be a quantum channel of a quantum system described by a Hilbert space $\mathcal{H}$. It was proved by Choi in [24] that Φ has the form of

$$\Phi(X) = \sum_{i=1}^{m} E_i X E_i^\dagger, \quad \forall X \in B(\mathcal{H}),$$

where $E_1, E_2, \ldots, E_m$ (called Kraus operators of Φ) are some bounded linear operators on $\mathcal{H}$ satisfying $\sum_{i=1}^{m} E_i^\dagger E_i = I$. Define the dual channel $\Phi^\dagger$ of Φ as

$$\Phi^\dagger(X) = \sum_i E_i^\dagger X E_i, \quad \forall X \in B(\mathcal{H}).$$

Then $\Phi^\dagger(I) = I$ and

$$\mathrm{tr}(X^\dagger \Phi(Y)) = \mathrm{tr}((\Phi^\dagger(X))^\dagger Y), \quad \forall X, Y \in B(\mathcal{H}).$$

Theorem 2.2 *Let $\rho \in \mathcal{US}_A$ and $\Phi = \Phi_A \otimes \Phi_B$ be a local quantum channel. Then $\Phi(\rho)$ is unsteerable by Alice.*

Proof Since $\rho \in \mathcal{US}_A$, we see by definition that there exists a probability distribution $\{\pi(\lambda)\}_{\lambda\in\Lambda}$ and a family of quantum states $\{\sigma(\lambda)\}_{\lambda\in\Lambda}$ in $\mathcal{D}_B$ such that for any POVM $M_x = \{M_{a|x}\}_{a=1}^{o_A}$ of $\mathcal{H}_A$, (2) holds, where $p_A(a|x,\lambda) \geq 0$ for all a, x and all $\lambda \in \Lambda$, satisfying $\sum_{a=1}^{o_A} p_A(a|x,\lambda) = 1 (\lambda \in \Lambda)$. For any POVM $M_x = \{M_{a|x}\}_{a=1}^{o_A}$ on $\mathcal{H}_A$, we see that $M_{x'} := \{\Phi_A^\dagger(M_{a|x})\}_{a=1}^{o_A} = \{M_{a|x'}\}_{a=1}^{o_A}$ is a POVM of $\mathcal{H}_A$ and so (2) yields that

$$\begin{aligned}\mathrm{tr}_A[(M_{a|x} \otimes I_B)\Phi(\rho)] &= \Phi_B\left(\mathrm{tr}_A[\Phi_A^\dagger(M_{a|x}) \otimes I_B)\rho]\right) \\ &= \sum_{\lambda\in\Lambda} \pi(\lambda) p_A(a|x',\lambda)\Phi_B(\sigma(\lambda)).\end{aligned}$$

This shows that $\Phi(\rho)$ is unsteerable by Alice. □

3 Generalized Steering Robustness

In order to quantify the steerability of quantum states, the steering robustness of states was introduced by Piani and Watrous [13], the definition is as follows.

$$R_{\mathrm{steer}}^{A\to B}(\rho) = \sup_{\mathcal{M}_A} R(\mathcal{A}), \tag{7}$$

where the supremum was taken over all measurement assemblages $\mathcal{M}_A = \{M_{a|x}\}$ on A, $\mathcal{A}$ is the assemblage induced by a state ρ and $\mathcal{M}_A$, and $R(\mathcal{A})$ is the steering robustness of $\mathcal{A}$, defined essentially by

$$R(\mathcal{A}) = \min\left\{t \in [0, +\infty) : \exists\sigma \in \mathcal{D}_{AB} \text{ s.t. } \tau_{\rho,\sigma}(t) \in \mathcal{US}_A(\mathcal{M}_A)\right\}, \tag{8}$$

where

$$\tau_{\rho,\sigma}(t) = \frac{1}{1+t}\rho + \frac{t}{1+t}\sigma, \quad t \in [0, +\infty). \tag{9}$$

In this section, we introduce the generalized steering robustness in a way similar to the generalized robustness of entanglement [20] and the generalized robustness of contextuality [22], and explore the related properties.

For all $\rho, \sigma \in \mathcal{D}_{AB}$, we define

$$\Delta_{\rho,\sigma} = \{t \in [0, +\infty) : \tau_{\rho,\sigma}(t) \in \mathcal{US}_A\}.$$

It was proved [19, Theorem 1] that any state $\rho \in \mathcal{D}_{AB}$ can be expressed as

$$\rho = (1 + t_0)\rho^+ - t_0\rho^-,$$

where ρ^+ is a separable state, $\rho^- = \frac{1}{d_{AB}} I_{AB}$ and t_0 is a non-negative finite real number. Thus, $\tau_{\rho,\rho^-}(t_0) = \rho^+$, which is unsteerable by Alice. Hence, $t_0 \in \Delta_{\rho,\rho^-}$. Generally, $\Delta_{\rho,\sigma}$ may be empty. When $\Delta_{\rho,\sigma} \neq \emptyset$, it must be closed since $\mathcal{US}_A$ is closed and has a lower bound 0. So, it has a minimal element. This leads to the following definition.

Definition 3.1 For every $\rho, \sigma \in \mathcal{D}_{AB}$, when $\Delta_{\rho,\sigma} \neq \emptyset$, define $\mathcal{R}_s^A(\rho\|\sigma) = \min \Delta_{\rho,\sigma}$; when $\Delta_{\rho,\sigma} = \emptyset$, define $\mathcal{R}_s^A(\rho\|\sigma) = +\infty$. We call

$$\mathcal{R}_s^A(\rho) := \inf\{\mathcal{R}_s^A(\rho\|\sigma) : \ \sigma \in \mathcal{D}_{AB}\} \tag{10}$$

the *generalized steering robustness* (GSR) of ρ.

By definition, we see that if $\rho = (1 + t)\rho_1 - t\sigma$ for a nonnegative real number t, an unsteerable state ρ_1 by Alice, and a state σ, then $\mathcal{R}_s^A(\rho) \leq \mathcal{R}_s^A(\rho||\sigma) \leq t$. Moreover, if there are nonnegative real numbers a, b, an unsteerable state ρ_1 by Alice and a state σ such that $\rho = b\rho_1 - a\sigma$, we have $1 = b - a$ and so $\mathcal{R}_s^A(\rho) \leq \mathcal{R}_s^A(\rho||\sigma) \leq a$.

Since $\mathcal{R}_s^A(\rho) \leq \mathcal{R}_s\left(\rho\|\rho^-\right) \leq t_0 < \infty$, we see that $0 \leq \mathcal{R}_s^A(\rho) < +\infty$ for every state $\rho \in \mathcal{D}_{AB}$. Also, we have

$$\mathcal{R}_s^A(\rho) = \inf\{\mathcal{R}_s^A(\rho\|\sigma) : \ \sigma \in \mathcal{D}_{AB} \text{ with } \Delta_{\rho,\sigma} \neq \emptyset\} \tag{11}$$

for all state ρ. Thus, for each state ρ, there is a sequence $\{\sigma_n\} \subset \mathcal{D}_{AB}$ with $\Delta_{\rho,\sigma_n} \neq \emptyset$ for all n such that

$$t_n := \mathcal{R}_s^A(\rho\|\sigma_n) \to t := \mathcal{R}_s^A(\rho)$$

as $n \to +\infty$. Since $\mathcal{D}_{AB}$ is a compact set, $\{\sigma_n\}$ has a convergent subsequence $\{\sigma_{n_k}\}$, let $\sigma = \lim_{k\to+\infty} \sigma_{n_k}$. Thus, as $k \to +\infty$,

$$\tau_{\rho,\sigma_{n_k}}(t_{n_k}) = \frac{1}{1 + t_{n_k}}\rho + \frac{t_{n_k}}{1 + t_{n_k}}\sigma_{n_k} \to \frac{1}{1 + t}\rho + \frac{t}{1 + t}\sigma = \tau_{\rho,\sigma}(t).$$

Since $\mathcal{US}_A$ is convex and closed ([29], [30]), we conclude that $\tau_{\rho,\sigma}(t) \in \mathcal{US}_A$. This shows that $\mathcal{R}_s^A(\rho\|\sigma) \leq t = \mathcal{R}_s^A(\rho) \leq \mathcal{R}_s^A(\rho\|\sigma)$. Hence, $t = \mathcal{R}_s^A(\rho) = \mathcal{R}_s^A(\rho\|\sigma)$. Consequently, for any state ρ, it holds that

$$\mathcal{R}_s^A(\rho) = \min\{\mathcal{R}_s^A(\rho\|\sigma) : \ \sigma \in \mathcal{D}_{AB} \text{ with } \Delta_{\rho,\sigma} \neq \emptyset\} \tag{12}$$

$$= \min\{\mathcal{R}_s^A(\rho\|\sigma) : \ \sigma \in \mathcal{D}_{AB}\}. \tag{13}$$

From this formula, one can check that

$$\mathcal{R}_s^A(\rho) = \min\{t \in [0, +\infty) : \exists\sigma \in \mathcal{D}_{AB} \text{ s.t. } \tau_{\rho,\sigma}(t) \in \mathcal{US}_A\} \tag{14}$$

for every state ρ. Combining (7), (8) with (14), we obtain that for every state ρ,

$$\begin{aligned}
\mathcal{R}_s^A(\rho) &= \min\{t \in [0, +\infty) : \exists\sigma \in \mathcal{D}_{AB} \text{ s.t. } \tau_{\rho,\sigma}(t) \in \mathcal{US}_A\} \\
&\geq \min \bigcap_{\mathcal{M}_A} \{t \in [0, +\infty) : \exists\sigma \in \mathcal{D}_{AB} \text{ s.t. } \tau_{\rho,\sigma}(t) \in \mathcal{US}_A(\mathcal{M}_A)\} \\
&= \sup_{\mathcal{M}_A} \min\left\{t \in [0, +\infty) : \exists\sigma \in \mathcal{D}_{AB} \text{ s.t. } \tau_{\rho,\sigma}(t) \in \mathcal{US}_A(\mathcal{M}_A)\right\} \\
&= R_{\text{steer}}^{A\to B}(\rho).
\end{aligned}$$

This gives a relationship between the possed steering robustness $\mathcal{R}_s^A(\rho)$ and the previous one $R_{\text{steer}}^{A\to B}(\rho)$.

Moreover, from the definitions of the robustness of quantum correlation (RoQC)[23], the robustness of entanglement (RoE) [19], the generalized robustness of entanglement (GRoE)[20], one can check that

RoQC: $\mathcal{R}_c(\rho) = \min\{t \in [0, +\infty) : \exists\sigma \in \mathcal{CC}_{AB} \text{ s.t. } \tau_{\rho,\sigma}(t) \in \mathcal{CC}_{AB}\}$, where $\mathcal{CC}_{AB}$ is the set of all classically correlated states of AB [23, 25–27];

RoE: $\mathcal{R}_e(\rho) = \min\{t \in [0, +\infty) : \exists\sigma \in Sep_{AB} \text{ s.t. } \tau_{\rho,\sigma}(t) \in Sep_{AB}\}$, where Sep_{AB} is the set of all separable states of AB [19];

GRoE: $\mathcal{R}_{ge}(\rho) = \min\{t \in [0, +\infty) : \exists\sigma \in \mathcal{D}_{AB} \text{ s.t. } \tau_{\rho,\sigma}(t) \in Sep_{AB}\}$ [20].

Combining these formulas with (14), we obtain the following relations:

$$\mathcal{R}_c(\rho) \geq \mathcal{R}_e(\rho) \geq \mathcal{R}_{ge}(\rho) \geq \mathcal{R}_s^A(\rho) \geq R_{\text{steer}}^{A\to B}(\rho),\ \ \forall\rho \in \mathcal{D}_{AB}. \tag{15}$$

Some more properties of GSR are given by the following Theorems 3.1–3.3.

Theorem 3.1 *The generalized steering robustness function $\mathcal{R}_s$ has the following properties.*

(1) $\mathcal{R}_s^A(\rho) = 0$ *if and only if* $\rho \in \mathcal{US}_A$.

(2) *For each state ρ and a local quantum channel $\Phi = \Phi_A \otimes \Phi_B$, it holds that* $\mathcal{R}_s^A(\Phi(\rho)) \leq \mathcal{R}_s^A(\rho)$.

(3) *$\mathcal{R}_s$ is invariant under each local unitary operator $U = U_A \otimes U_B$:* $\mathcal{R}_s^A(\rho) = \mathcal{R}_s^A(U\rho U^\dagger)$.

(4) *$\mathcal{R}_s^A(\rho)$ is convex for ρ, that is,* $\mathcal{R}_s^A(\rho) \leq \sum_{k=1}^m p_k\mathcal{R}_s^A(\rho_k)$ *provided that* $\rho = \sum_{k=1}^m p_k\rho_k$, $p_k \geq 0$, $\sum_{k=1}^m p_k = 1$ *and* $\rho_k \in \mathcal{D}_{AB}(k = 1, 2, \ldots, m)$.

(5) *$\mathcal{R}_s : \mathcal{D}_{AB} \to \mathbb{R}$ is lower-semi continuous, i.e. when $\rho_n \in \mathcal{D}_{AB}(n = 1, 2, \ldots)$ with $\lim\limits_{n\to\infty}\rho_n = \rho$, it holds that*

$$\mathcal{R}_s^A(\rho) = \mathcal{R}_s^A(\lim_{n\to\infty}\rho_n) \leq \varliminf_{n\to\infty}\mathcal{R}_s^A(\rho_n).$$

Proof (1) Suppose that $\mathcal{R}_s^A(\rho) = 0$, then (13) implies that there exists a state σ such that $\mathcal{R}_s^A(\rho\|\sigma) = 0$. We have $\rho = \tau_{\rho,\sigma}(0) \in \mathcal{US}_A$. Conversely, if $\rho \in \mathcal{US}_A$, then for any state σ, we have $\tau_{\rho,\sigma}(0) = \rho \in \mathcal{US}_A$, so $\mathcal{R}_s^A(\rho) = 0$.

(2) By (13), there exists a state σ such that $\mathcal{R}_s^A(\rho) = \mathcal{R}_s^A(\rho\|\sigma)$, and

$$\tau_{\rho,\sigma}(\mathcal{R}_s^A(\rho)) = \frac{1}{1+\mathcal{R}_s^A(\rho)}\rho + \frac{\mathcal{R}_s^A(\rho)}{1+\mathcal{R}_s^A(\rho)}\sigma \in \mathcal{US}_A.$$

By Theorem 2.2, we know that

$$\frac{1}{1+\mathcal{R}_s^A(\rho)}\Phi(\rho) + \frac{\mathcal{R}_s^A(\rho)}{1+\mathcal{R}_s^A(\rho)}\Phi(\sigma) = \Phi\left(\tau_{\rho,\sigma}(\mathcal{R}_s^A(\rho))\right) \in \mathcal{US}_A.$$

This shows that

$$\mathcal{R}_s^A(\Phi(\rho)) \leq \mathcal{R}_s^A(\rho||\Phi(\sigma)) \leq \mathcal{R}_s^A(\rho).$$

(3) By using (2) for $\Phi(X) = (U_A\otimes U_B)X(U_A\otimes U_B)^\dagger$, we see that $\mathcal{R}_s^A(U\rho U^\dagger) \leq \mathcal{R}_s^A(\rho)$ for all $\rho \in \mathcal{D}_{AB}$ and all unitary operators U_A and U_B on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. By using (2) for $\Phi(X) = (U_A \otimes U_B)^\dagger X(U_A \otimes U_B)$ and the state $U\rho U^\dagger$, we obtain that

$$\mathcal{R}_s^A(\rho) = \mathcal{R}_s^A(U^\dagger U\rho U^\dagger U) \leq \mathcal{R}_s^A(U\rho U^\dagger).$$

Thus, $\mathcal{R}_s^A(\rho) = \mathcal{R}_s^A(U\rho U^\dagger)$.

(4) By (13), for any k, there exists a state σ_k such that $\tau_{\rho_k,\sigma_k}(\mathcal{R}_s^A(\rho_k)) \in \mathcal{US}_A$ and

$$\rho_k = (1 + \mathcal{R}_s^A(\rho_k))\tau_{\rho_k,\sigma_k}(\mathcal{R}_s^A(\rho_k)) - \mathcal{R}_s^A(\rho_k)\sigma_k.$$

So

$$\rho = \sum_k p_k \rho_k = \xi_1 - \xi_2,$$

where

$$\xi_1 = \sum_k p_k(1 + \mathcal{R}_s^A(\rho_k))\tau_{\rho_k,\sigma_k}(\mathcal{R}_s^A(\rho_k)),\ \xi_2 = \sum_k p_k \mathcal{R}_s^A(\rho_k)\sigma_k.$$

When $\xi_2 = 0$, we have $\sum_{k=1}^m p_k \mathcal{R}_s^A(\rho_k) = \mathrm{tr}(\xi_2) = 0$, and so

$$\rho = \sum_{k=1}^m p_k \tau_{\rho_k,\sigma_k}(\mathcal{R}_s^A(\rho_k)) \in \mathcal{US}_A$$

since $\mathcal{US}_A$ is convex ([29, 30]). Hence, conclusion (1) yields that $\mathcal{R}_s^A(\rho) = 0 \leq \sum_{k=1}^m p_k \mathcal{R}_s^A(\rho_k)$.

When $\xi_2 \neq 0$, we have $\frac{\xi_2}{\mathrm{tr}(\xi_2)} \in \mathcal{D}_{AB}$ and $\xi_1 \neq 0$. Thus,

$$\frac{\xi_1}{\mathrm{tr}(\xi_1)} = \sum_{k=1}^m \frac{p_k(1 + \mathcal{R}_s^A(\rho_k))}{\mathrm{tr}(\xi_1)} \tau_{\rho_k,\sigma_k}(\mathcal{R}_s^A(\rho_k)) \in \mathcal{US}_A$$

by the convexity of $\mathcal{US}_A$. Since $\rho = \xi_1 - \xi_2$, we get $\mathrm{tr}(\xi_1) = \mathrm{tr}(\xi_2) + 1$ and consequently,

$$\rho = (1 + \mathrm{tr}(\xi_2)) \frac{\xi_1}{\mathrm{tr}(\xi_1)} - \mathrm{tr}(\xi_2) \frac{\xi_2}{\mathrm{tr}(\xi_2)}.$$

Therefore,

$$\mathcal{R}_s^A(\rho) \leq \mathrm{tr}(\xi_2) = \sum_{k=1}^m p_k \mathcal{R}_s^A(\rho_k).$$

(5) Let $\{\rho_n\} \subset \mathcal{D}_{AB}$ with $\lim\limits_{n\to\infty} \rho_n = \rho$, and put $t = \varliminf\limits_{n\to\infty} \mathcal{R}_s^A(\rho_n)$. Then $\{\rho_n\}_{n=1}^\infty$ has a subsequence $\{\rho_{n_k}\}_{k=1}^\infty$ such that

$$t_k := \mathcal{R}_s^A(\rho_{n_k}) \to t\ (k \to \infty).$$

From the definition of $\mathcal{R}_s^A(\rho)$, we can choose $\sigma_k \in \mathcal{US}_A$ such that

$$t_k = \mathcal{R}_s^A(\rho_{n_k}) = \mathcal{R}_s^A(\rho_{n_k} \| \sigma_k)(k = 1, 2, \ldots).$$

Since $\mathcal{D}_{AB}$ is compact, $\{\sigma_k\}$ has a convergent subsequence, say $\{\sigma_{k_j}\}_{j=1}^\infty$. Put $\sigma = \lim\limits_{j\to\infty} \sigma_{k_j}$. Then

$$\frac{1}{1 + t_{k_j}} \rho_{n_{k_j}} + \frac{t_{k_j}}{1 + t_{k_j}} \sigma_{k_j} \to \frac{1}{1 + t} \rho + \frac{t}{1 + t} \sigma,$$

as $j \to \infty$. Note that

$$\frac{1}{1+t_{k_j}}\rho_{n_{k_j}} + \frac{t_{k_j}}{1+t_{k_j}}\sigma_{k_j} \in \mathcal{US}_A (j = 1, 2, \ldots)$$

we conclude that $\frac{1}{1+t}\rho + \frac{t}{1+t}\sigma \in \mathcal{US}_A$(Theorem 2.2). Thus, $\mathcal{R}_s^A(\rho) \le \mathcal{R}_s^A(\rho\|\sigma) \le t$. This shows that

$$\mathcal{R}_s^A(\lim_{n\to\infty} \rho_n) \le \varliminf_{n\to\infty} \mathcal{R}_s^A(\rho_n).$$

□

Theorem 3.2 *Let $|\Phi\rangle, |\Psi\rangle \in \mathcal{P}_{AB}$ with the same Schmidt coefficients. Then*

$$\mathcal{R}_s^A(|\Psi\rangle\langle\Psi|) = \mathcal{R}_s^A(|\Phi\rangle\langle\Phi|).$$

Proof Denote Schmidt decompositions of $|\Phi\rangle$ and $|\Psi\rangle$ as follows, respectively,

$$|\Psi\rangle = \sum_{i=1}^{m} a_i |\gamma_i^A\rangle \otimes |\gamma_i^B\rangle, \ |\Phi\rangle = \sum_{i=1}^{m} a_i |f_i^A\rangle \otimes |f_i^B\rangle,$$

where $a_i > 0$ for all i with $\sum\limits_{i=1}^{m} a_i^2 = 1$, $\{\gamma_i^A\}$ and $\{f_i^A\}$ are two orthonormal sets in $\mathcal{H}_A$, $\{\gamma_i^B\}$ and $\{f_i^B\}$ are two orthonormal sets in $\mathcal{H}_B$. Choose unitary operators U_A on $\mathcal{H}_A$ and U_B on $\mathcal{H}_B$ such that $U_A|f_i^A\rangle = |\gamma_i^A\rangle$ and $U_B|f_i^B\rangle = |\gamma_i^B\rangle$ for all i. Then

$$|\Psi\rangle\langle\Psi| = (U_A \otimes U_B)|\Phi\rangle\langle\Phi|(U_A \otimes U_B)^\dagger.$$

It follows from Theorem 3.1(3) that $\mathcal{R}_s^A(|\Psi\rangle\langle\Psi|) = \mathcal{R}_s^A(|\Phi\rangle\langle\Phi|)$. □

Theorem 3.3 *Let $\rho \in \mathcal{D}_{AB}$ and $\{M_k\}_{k=1}^m$ be a quantum measurement on B, i.e. $\sum_{k=1}^m M_k^\dagger M_k = I_B$. Denote*

$$\rho_k = \frac{1}{p_k(\rho)}(I_A \otimes M_k)\rho(I_A \otimes M_k)^\dagger, \tag{16}$$

where $p_k(\rho) = tr((I_A \otimes M_k)\rho(I_A \otimes M_k)^\dagger) \neq 0$. Then

$$\mathcal{R}_s^A(\rho) \ge \sum_{k=1}^{m} p_k(\rho)\mathcal{R}_s^A(\rho_k).$$

Proof By definition, there exists a state σ such that $\tau_{\rho,\sigma}(\mathcal{R}_s^A(\rho)) \in \mathcal{US}_A$ and

$$\rho = (1 + \mathcal{R}_s^A(\rho))\tau_{\rho,\sigma}(\mathcal{R}_s^A(\rho)) - \mathcal{R}_s^A(\rho)\sigma. \tag{17}$$

Put $q_k = p_k(\tau_{\rho,\sigma}(\mathcal{R}_s^A(\rho)))$, and

$$\rho_k' = \frac{(I_A \otimes M_k)\tau_{\rho,\sigma}(\mathcal{R}_s^A(\rho))(I_A \otimes M_k)^\dagger}{q_k}(q_k > 0), \quad \rho_k' = \frac{1}{d_A}I_A \otimes \frac{1}{d_B}I_B(q_k = 0),$$

$$\sigma_k = \frac{(I_A \otimes M_k)\sigma(I_A \otimes M_k)^\dagger}{p_k(\sigma)}(p_k(\sigma) > 0), \quad \sigma_k = \frac{1}{d_A}I_A \otimes \frac{1}{d_B}I_B(p_k(\sigma) = 0).$$

By (17), we have

$$\begin{aligned}\rho_k &= \frac{1}{p_k(\rho)}\Big[(1+\mathcal{R}_s^A(\rho))(I_A\otimes M_k)\tau_{\rho,\sigma}(\mathcal{R}_s^A(\rho))(I_A\otimes M_k)^\dagger \\ &\quad - \mathcal{R}_s^A(\rho)(I_A\otimes M_k)\sigma(I_A\otimes M_k)^\dagger\Big] \\ &= (1+\mathcal{R}_s^A(\rho))\frac{q_k}{p_k(\rho)}\cdot\rho_k' - \mathcal{R}_s^A(\rho)\frac{p_k(\sigma)}{p_k(\rho)}\cdot\sigma_k \\ &= \left(1+\mathcal{R}_s^A(\rho)\frac{p_k(\sigma)}{p_k(\rho)}\right)\rho_k' - \mathcal{R}_s^A(\rho)\frac{p_k(\sigma)}{p_k(\rho)}\sigma_k.\end{aligned}$$

Corollary 2.1 implies that $\rho_k' \in \mathcal{US}_A$ for all k. Also, $\sigma_k \in \mathcal{D}_{AB}$. By definition, we get

$$\mathcal{R}_s^A(\rho_k) \le \mathcal{R}_s^A(\rho)\frac{p_k(\sigma)}{p_k(\rho)}(k=1,2,\ldots,m).$$

Furthermore,

$$\sum_{k=1}^m p_k(\rho)\mathcal{R}_s^A(\rho_k) \le \sum_{k=1}^m \mathcal{R}_s^A(\rho)p_k(\sigma) = \mathcal{R}_s^A(\rho).$$

□

4 Comparison of GSR of Different States

We first introduce the concept of majorization between two vectors. Let $x = (x_1, x_2, \cdots, x_d)$ and $y = (y_1, y_2, \cdots, y_d)$ be two d-dimensional real vectors. We say that x is *majorizated* by y, written as $x \prec y$, if $x_1 \ge x_2 \ge \ldots \ge x_d,\ y_1 \ge y_2 \ge \ldots \ge y_d$ and

$$\sum_{j=1}^k x_j \le \sum_{j=1}^k y_j (k=1,2,\cdots,d-1),\quad \sum_{j=1}^d x_j = \sum_{j=1}^d y_j.$$

For every Hermitian operator X on $\mathbb{C}^d$, we use $\lambda(X)$ to denote the vector consisting d eigenvalues $c_1, c_2, \ldots, c_d$ of X in decreasing order, i.e. $\lambda(X) = (c_1, c_2, \ldots, c_d)$ with $c_1 \ge c_2 \ge \ldots \ge c_d$. Let ρ and σ be Hermitian operators. We say that ρ is *majorizated* by σ written as $\rho \prec \sigma$ if $\lambda(\rho) \prec \lambda(\sigma)$. Note that the relation $\prec$ is not a total ordering, generally, unless $\dim(\mathcal{H}) = 2$. The following example shows that the maximally mixed state $\frac{1}{n}I_n$ is a minimal element of $(\mathcal{D}(\mathbb{C}^n), \prec)$.

Example 4.1 For any state ρ of $\mathbb{C}^n$, we have $\sigma := \frac{1}{n}I_n \prec \rho$.

Indeed, let $\lambda(\rho) = (x_1, x_2, \ldots, x_n)$ with

$$x_1 \ge x_2 \ge \cdots \ge x_m \ge \frac{1}{n} > x_{m+1} \ge \cdots \ge x_n,$$

and let $\lambda(\sigma) = (y_1, y_2, \ldots, y_n)$ with $y_k = \frac{1}{n}$ for all k. Then

$$\sum_{j=1}^k x_j \ge \frac{k}{n} = \sum_{j=1}^k y_j (k=1,2,\ldots,m).$$

When $n > k > m$, we see from $x_1 + x_2 + \ldots + x_n = 1$ and $\frac{1}{n} > x_j (j = k+1, \ldots, n)$ that

$$\sum_{j=1}^{k} x_j = 1 - \sum_{j=k+1}^{n} x_j > 1 - \frac{n-k}{n} = \frac{k}{n} = \sum_{j=1}^{k} y_j.$$

Also, $\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j = 1$. This shows that $\sigma \prec \rho$.

In what follows, we will discuss the GSR of states by using majorization.

Lemma 4.1 [28] *Let ρ and σ be Hermitian operators on a finite dimensional Hilbert space. Then $\rho \prec \sigma$ if and only if there exists a probability distribution $\{p_j\}_{j=1}^m$ and a set $\{U_j\}_{j=1}^m$ of unitary matrices such that*

$$\rho = \sum_{j=1}^{m} p_j U_j \sigma U_j^{\dagger}.$$

Theorem 4.1 *Let $|\phi\rangle, |\psi\rangle \in \mathcal{P}_{AB}$, $\rho_\phi := \mathrm{tr}_A\,(|\phi\rangle\langle\phi|) \prec \rho_\psi := \mathrm{tr}_A\,(|\psi\rangle\langle\psi|)$. Then*

$$\mathcal{R}_s^A(|\psi\rangle\langle\psi|) \leq \mathcal{R}_s^A(|\phi\rangle\langle\phi|).$$

Proof Since $\rho_\phi \prec \rho_\psi$, we see from Lemma 4.1 that there exists a probability distribution $\{p_j\}_{j=1}^m$ and a set $\{U_j\}_{j=1}^m$ of unitary matrices $\{U_j\}_{j=1}^m$ such that

$$\rho_\phi = \sum_{j=1}^{m} p_j U_j \rho_\psi U_j^{\dagger}. \tag{18}$$

First, ρ_ϕ is represented as a 2×2 operator matrix

$$\rho_\phi = \begin{pmatrix} \rho_1 & 0 \\ 0 & 0 \end{pmatrix}$$

relative to the space decomposition $\mathcal{H}_{AB} = \ker(\rho_\phi)^{\perp} \bigoplus \ker(\rho_\phi)$, where ρ_1 is an invertible positive operator on $\ker(\rho_\phi)^{\perp}$. From (4.1), we have

$$\ker(\rho_\phi) \subseteq \ker(U_j \rho_\psi U_j^{\dagger}),\ j = 1, 2, \ldots, m$$

and so

$$U_j \rho_\psi U_j^{\dagger} = \begin{pmatrix} \sigma_j & 0 \\ 0 & 0 \end{pmatrix},$$

where σ_j is a state on $\ker(\rho_\phi)^{\perp}$ with $\rho_1 = \sum_j p_j \sigma_j$. Choose

$$M_j = U_j^{\dagger} \begin{pmatrix} \sqrt{p_j}\,\sigma_j^{\frac{1}{2}} \rho_1^{-\frac{1}{2}} & 0 \\ 0 & \frac{1}{\sqrt{m}} I_{\ker(\rho_\phi)} \end{pmatrix} (j = 1, 2, \ldots, m).$$

Then

$$\sum_{j=1}^{m} M_j^{\dagger} M_j = I_B,$$

$$M_j \rho_\phi M_j^{\dagger} = p_j \rho_\psi (j = 1, 2, \ldots, m).$$

It is clear that $\{M_j\}_{j=1}^m$ is a POVM. When quantum state $|\phi\rangle$ is measured by $\{M_j\}_{j=1}^m$, the corresponding state is

$$|\phi_j\rangle = \frac{(I_A \otimes M_j)|\phi\rangle}{\sqrt{\langle\phi|(I_A \otimes M_j^{\dagger} M_j)|\phi\rangle}},$$

satisfying

$$\begin{aligned}
\mathrm{tr}_A\left(|\phi_j\rangle\langle\phi_j|\right) &= \frac{1}{\mathrm{tr}\left((I_A\otimes M_j^\dagger M_j)|\phi\rangle\langle\phi|\right)}\left(M_j \mathrm{tr}_A\left(|\phi\rangle\langle\phi|\right) M_j^\dagger\right)\\
&= \frac{1}{\mathrm{tr}\left((I_A\otimes M_j^\dagger M_j)|\phi\rangle\langle\phi|\right)}\left(M_j\rho_\phi M_j^\dagger\right)\\
&= \frac{p_j}{\mathrm{tr}\left((I_A\otimes M_j^\dagger M_j)|\phi\rangle\langle\phi|\right)}\rho_\psi\\
&= \frac{p_j}{\mathrm{tr}\left((I_A\otimes M_j^\dagger M_j)|\phi\rangle\langle\phi|\right)}\mathrm{tr}_A\left(|\psi\rangle\langle\psi|\right)\\
&= \mathrm{tr}_A\left(|\psi\rangle\langle\psi|\right).
\end{aligned}$$

Thus

$$\mathrm{tr}_A\left(|\phi_j\rangle\langle\phi_j|\right) = \mathrm{tr}_A\left(|\psi\rangle\langle\psi|\right), \forall j.$$

Therefore, $|\phi_j\rangle(\forall j)$ and $|\psi\rangle$ have the same Schmidt coefficients. It follows from Theorem 3.2 that

$$\mathcal{R}_s^A(|\psi\rangle\langle\psi|) = \mathcal{R}_s^A\left(|\phi_j\rangle\langle\phi_j|\right)(\forall j).$$

Using Theorem 3.3 again, we get that

$$\mathcal{R}_s^A(|\psi\rangle\langle\psi|) = \sum_j p_j\mathcal{R}_s^A(|\psi\rangle\langle\psi|) = \sum_j p_j\mathcal{R}_s^A(|\phi_j\rangle\langle\phi_j|) \le \mathcal{R}_s^A(|\phi\rangle\langle\phi|).$$

□

Note that there exists the majorization relation between any two qubit states, we see from Theorem 4.1 that for any two pure states $|\phi\rangle$ and $|\psi\rangle$ for a two-qubit system, their reduced states ρ_ϕ and ρ_ψ are comparable with respect to the majorization and so their GSR can be compared. However, for the mixed state case, comparison of GSR of two states is very hard. However, the following Corollary 4.1 shows that GSR of any state ρ is always less than or equal to GSR of any maximally mixed state.

Corollary 4.1 *Let $\{|\varepsilon_i\rangle\}_{i=1}^n$ be an orthonormal basis for $H_A = H_B = \mathbb{C}^n$, $|\phi\rangle = \frac{1}{\sqrt{n}}\sum_{i=1}^n|\varepsilon_i\rangle|\varepsilon_i\rangle$ be a maximally entangled state. Then for every state $\rho \in \mathcal{D}_{AB}$, it holds that*

$$\mathcal{R}_s^A(\rho) \le \mathcal{R}_s^A(|\phi\rangle\langle\phi|) \le \frac{n^2}{2}. \tag{19}$$

Proof It is easy to check that for every pure state $|\psi\rangle \in \mathcal{P}_{AB}$, the reduced state ρ_ϕ of ρ^{AB} on system A is the maximally mixed state $\frac{1}{n}I_n$. From Example 4.1, we know that $\rho_\phi \prec \rho_\psi$ and so Theorem 4.1 implies that $\mathcal{R}_s^A(|\psi\rangle\langle\psi|) \le \mathcal{R}_s^A(|\phi\rangle\langle\phi|)$. Furthermore, for every mixed state $\rho \in \mathcal{D}_{AB}$, it has its spectral decomposition $\rho = \sum_{i=1}^n p_i|\psi_i\rangle\langle\psi_i|$. From the convexity of $\mathcal{R}_s$ (Theorem 3.1(4)), we have

$$\mathcal{R}_s^A(\rho) \le \sum_{i=1}^n p_i\mathcal{R}_s^A(|\psi_i\rangle\langle\psi_i|) \le \sum_{i=1}^n p_i\mathcal{R}_s^A(|\phi\rangle\langle\phi|) = \mathcal{R}_s^A(|\phi\rangle\langle\phi|).$$

The last inequality is from Eq. (3.9) and [19, Appendix C] □

By using Theorem 4.1, we have the following.

Corollary 4.2 *Let* $|\phi\rangle, |\varphi\rangle \in \mathcal{P}_{AB}$, $\rho_\phi = \mathrm{tr}_A(|\phi\rangle\langle\phi|)$, $\sigma_\varphi = \mathrm{tr}_A(|\varphi\rangle\langle\varphi|)$. *If* $\rho_\phi \prec \sigma_\varphi$ *and* $\sigma_\varphi \prec \rho_\phi$, *then* $\mathcal{R}_s^A(|\varphi\rangle\langle\varphi|) = \mathcal{R}_s^A(|\phi\rangle\langle\phi|)$.

Example 4.2 Consider any pure state $|\phi_{x,y}\rangle = x|00\rangle + \mathrm{e}^{\mathrm{i}\theta}y|11\rangle$ of $\mathbb{C}^2 \otimes \mathbb{C}^2$ with $x, y \in [0, 1]$, $x^2 + y^2 = 1$ and $\theta \in \mathbb{R}$. Then

$$\begin{aligned}|\phi_{x,y}\rangle\langle\phi_{x,y}| = \;& x^2|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \mathrm{e}^{-\mathrm{i}\theta}xy|0\rangle\langle 1| \otimes |0\rangle\langle 1| \\ & + y^2|1\rangle\langle 1| \otimes |1\rangle\langle 1| + \mathrm{e}^{\mathrm{i}\theta}xy|1\rangle\langle 0| \otimes |1\rangle\langle 0|.\end{aligned}$$

We can compute that

$$\rho_{xy} := \mathrm{tr}_A\left(|\phi_{x,y}\rangle\langle\phi_{x,y}|\right) = x^2|0\rangle\langle 0| + y^2|1\rangle\langle 1| = \begin{pmatrix} x^2 & 0 \\ 0 & y^2 \end{pmatrix}.$$

Thus, $\rho_{yx} = \begin{pmatrix} y^2 & 0 \\ 0 & x^2 \end{pmatrix}$. It is clear that $\lambda(\rho_{xy}) = \lambda(\rho_{yx})$ and so $\rho_{xy} \prec \rho_{yx}$ while $\rho_{yx} \prec \rho_{xy}$. We conclude from Theorem 4.1 that

$$\mathcal{R}_s^A(|\phi_{x,y}\rangle\langle\phi_{x,y}|) = \mathcal{R}_s^A(|\phi_{y,x}\rangle\langle\phi_{y,x}|).$$

Furthermore, by Corollary 4.1, we know that

$$\mathcal{R}_s^A(|\phi_{x,y}\rangle\langle\phi_{x,y}|) \leq \mathcal{R}_s^A(|\phi_{x_0,y_0}\rangle\langle\phi_{x_0,y_0}|), \tag{20}$$

where $x_0 = y_0 = \frac{1}{\sqrt{2}}$, and $|\phi_{x_0,y_0}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + \mathrm{e}^{\mathrm{i}\theta}|11\rangle)$ is a maximally entangled state. Next, we will prove (20) by simply using Theorem 4.1. To do so, we let

$$f(x) = \mathcal{R}_s^A(|\phi_{x,y(x)}\rangle\langle\phi_{x,y(x)}|),\ \ y(x) = \sqrt{1-x^2}(0 \leq x \leq 1),$$

then when $\frac{1}{\sqrt{2}} \leq x_1 < x_2 \leq 1$, $\rho_{x_1 y(x_1)} \prec \rho_{y(x_2)x_2}$ and Theorem 4.1 yields that $f(x_1) \geq f(x_2)$. This shows that $f(x)$ is decreasing on $\left[\frac{1}{\sqrt{2}}, 1\right]$. Similarly, $f(x)$ is increasing on $\left[0, \frac{1}{\sqrt{2}}\right]$ and so $f\left(\frac{1}{\sqrt{2}}\right) = \max_{0\leq x\leq 1} f(x)$. This implies that (20) holds.

5 Conclusions

In summary, we have obtained a characterization of an unsteerable state. We have introduced a new method, called the generalized steering robustness (GSR), to quantify the steering power, which can describe steering endurance of a state against disturbance. Our discussion shows that GSR has many good properties, such as (1) GSR of a state vanishes if and only if the state is unsteerable; (2) a local quantum channel does not increase GSR of any state; (3) GSR is invariant under each local unitary operation; (4) as a function on the state space, GSR is convex and lower-semi continuous. Also, GSR of two pure states can be compared by the majorization of the their reduced states. In a quantitative way, we have proved that maximally entangled states have the maximal GSR and then are maximally steerable.

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