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Λ-Nonlocality of Multipartite States and the Related Nonlocality Inequalities

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Abstract Correlations between subsystems of a composite quantum system include Bell nonlocality, steerability, entanglement and quantum discord. Bell nonlocality of a bipartite state is one of important quantum correlations demonstrated by some local quantum measurements. In this paper, we discuss nonlocality of a multipartite quantum system. The Λ-locality and Λ-nonlocality of multipartite states are firstly introduced, some related properties are discussed. Some related nonlocality inequalities are established for $\{1, 2; 3\}$-local, $\{1; 2, 3\}$-local, and Λ-local states, respectively. The violation of one of these inequalities gives a sufficient condition for Λ-nonlocal states. As application, genuinely nonlocality of a tripartite state is checked. Finally, a class of 2-separable nonlocal states are given, which shows that a 2-separable tripartite state is not necessarily local.

Keywords Λ-Locality · Λ-nonlocality · Nonlocality inequality · Multipartite state

1 Introduction

Correlations among the results of space-like separated measurements on composite quantum systems can be incompatible with a local model [1]. Such phenomenon, known as quantum nonlocality, is an intrinsic quantum feature and lies behind several applications in quantum information theory [2–6]. By performing local measurements on an n-partite entangled state one obtains outcomes that may be nonlocal, in the sense that they violate a Bell inequality [7]. Since the seminal work of Bell, nonlocality has been a central subject of study in

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the foundations of quantum theory and has been supported by many experiments [8, 9]. More recently, it has also been realized that it plays a key role in various quantum information applications [10, 11], where it represents a resource different from entanglement. For instance, the security of device independent quantum key distribution requires the existence of nonlocal correlations between the honest parties, very much in the spirit of Ekert's protocol [2, 3, 12, 13], and the only entanglement witnesses that do not rely on assumptions on the dimension of the Hilbert spaces are Bell inequalities, i.e. witnesses of nonlocality [3]. While nonlocality has been extensively studied in the bipartite ($n = 2$) and to a lesser extent in the tripartite ($n = 3$) case, the general n-partite case remains much unexplored, their characterization remains a general unsolved problem. The physics of many-particle systems, however, is well known to differ fundamentally from the one of a few particles and to give rise to new interesting phenomena, such as phase transitions or quantum computing. Entanglement theory, in particular, appears to have a much more complex and richer structure in the n-partite case than it has in the bipartite setting [14, 15]. This is reflected by the fact that multipartite entanglement is a very active field of research that has led to important insights into our understanding of many-particle physics [16, 17]. In this point of view, it seems worthy to investigate how nonlocality manifests itself in a multipartite scenario. Generalized Bell inequalities have been reported for n-particle systems which show that quantum mechanics violates local realism in these situations [18–20]. However such results are insufficient to show that all of the particles in a system are acting nonlocality, it is possible to imagine a nonlocal many-particle system as consisting of a finite number of nonlocal subsystems, but with only local correlations present between these subsystems. For example a state of three particles $|\psi\rangle_{123}$ which can be decomposed as $|\psi\rangle_1|\psi\rangle_{23}$ only exhibits nonlocal correlations between particles 2 and 3. Hence, it is necessary for us to extend the concept of locality of all particles in a multipartite system to the locality of groups of subsystems.

In this paper, we introduce Λ-locality and Λ-nonlocality of multipartite states and prove the related nonlocality inequalities. The remain of this paper are organized as follows. In Section 2, we introduce the Λ-locality of multipartite states and discuss the related properties. In Section 3, we establish some nonlocality inequalities, which are necessary conditions for a Λ-local state. In Section 4, we give a class of 2-separable nonlocal states, which shows that a 2-separable tripartite state is not necessarily local.

2 Λ-Nonlocality of Multipartite States

We consider the composite system $\mathcal{H}^{(n)} := \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n$, and use $D(\mathcal{H}^{(n)})$ to denote the set of all mixed states of the system $\mathcal{H}^{(n)}$ and I_k to denote the identity operator on $\mathcal{H}_k$.

To describe different local cases of the composite system $\mathcal{H}^{(n)}$, we use $\Omega = \{1, 2, \ldots, n\}$ to denote the set of all indices of the subsystems. And for a subset $\{i_1, i_2, \ldots, i_k\}$ of Ω with $i_0 + 1 = 1 \le i_1 < i_2 < \ldots < i_k = n$, put

$$\Lambda = \{i_0 + 1, \ldots, i_1; i_1 + 1, \ldots, i_2; \ldots; i_{k-1} + 1, \ldots, i_k\}, \tag{2.1}$$

called a *local pattern*.

For a local pattern (2.1), put

$$A_1 = \{i_0 + 1, \ldots, i_1\}, A_2 = \{i_1 + 1, \ldots, i_2\}, \ldots,$$
$$A_k = \{i_{k-1} + 1, \ldots, i_k\}.$$

We denote Λ simply by $\Lambda = (A_1, A_2, \ldots, A_k)$. Put $\mathcal{H}_{A_s} = \mathcal{H}_{i_{s-1}+1} \otimes \ldots \otimes \mathcal{H}_{i_s}$, then

$$\mathcal{H}^{(n)} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \ldots \otimes \mathcal{H}_{A_k}.$$

Thus, every n-partite state ρ of $\mathcal{H}^{(n)}$ can be viewed as a k-partite state. In this case, we use $\mathrm{tr}_{A_i}(\rho)$ to denote the reduced state of ρ with respect to ith subsystem $\mathcal{H}_{A_i}$, which is a state of $\mathcal{H}_{A_1} \otimes \ldots \otimes \mathcal{H}_{A_{i-1}} \otimes \mathcal{H}_{A_{i+1}} \otimes \ldots \otimes \mathcal{H}_{A_k}$.

Definition 2.1 Let Λ be a local pattern given by (2.1).

(1) A state $\rho \in D(\mathcal{H}^{(n)})$ is said to be Λ-*local* if for every measurement assemblage

$$\mathcal{M} = \{M^{x_1,x_2,\ldots,x_k} : x_j = 1, 2, \ldots, m_j (1 \le j \le k)\} \equiv \{M^{x_1,x_2,\ldots,x_k}\}_{x_1,x_2,\ldots,x_k}$$

of local POVMs:

$$M^{x_1,x_2,\ldots,x_k} = \{M^{x_1}_{b_1} \otimes M^{x_2}_{b_2} \otimes \ldots \otimes M^{x_k}_{b_k} : b_i \in N_i (1 \le i \le k)\}$$

on $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \ldots \otimes \mathcal{H}_{A_k}$, there exists a probability distribution $\Pi = \{\Pi_\lambda\}_{\lambda\in\Gamma}$ such that

$$\mathrm{tr}(M^{x_1}_{b_1} \otimes M^{x_2}_{b_2} \otimes \ldots \otimes M^{x_k}_{b_k})\rho = \sum_{\lambda\in\Gamma} \Pi_\lambda P_{A_1}(b_1|x_1, \lambda) P_{A_2}(b_2|x_2, \lambda) \ldots P_{A_k}(b_k|x_k, \lambda) \tag{2.2}$$

for all x_i, b_i,where $P_{A_i}(b_i|x_i, \lambda) \ge 0, \sum_{b_i} P_{A_i}(b_i|x_i, \lambda) = 1 (i = 1, 2, \ldots, k)$. Otherwise, ρ is said to be Λ-*nonlocal*.

(2) A mixed state $\rho \in D(\mathcal{H}^{(n)})$ is said to be *genuinely nonlocal* if it is Λ-nonlocal for every Λ.

Remark 2.1 By definition, ρ is Λ-local if and only if for every $\mathcal{M}$, there exists a PD Π such that (2.2) holds; ρ is Λ-nonlocal if and only if there exists an $\mathcal{M}$, the PD Π satisfying (2.2) does not exits.

Remark 2.2 By definition above, we see that when a state $\rho \in D(\mathcal{H}^{(n)})$ is Λ-local, then for every measurement assemblage $\mathcal{M}$, there exists a probability distribution $\Pi = \{\Pi_\lambda\}_{\lambda\in\Gamma}$ such that

$$\mathrm{tr}(M^{x_1}_{b_1} \otimes M^{x_2}_{b_2} \otimes \ldots \otimes M^{x_k}_{b_k})\rho = \sum_{\lambda\in\Gamma} \Pi_\lambda P_{A_1}(b_1|x_1, \lambda) P_{A_2}(b_2|x_2, \lambda) \ldots P_{A_k}(b_k|x_k, \lambda)$$

for all x_i, b_i. Finding the sums of two sides for b_j yields that

$$\begin{aligned}&\mathrm{tr}(M^{x_1}_{b_1} \otimes \ldots \otimes M^{x_{j-1}}_{b_{j-1}} \otimes I_{A_j} \otimes M^{x_{j+1}}_{b_{j+1}} \otimes \ldots \otimes M^{x_k}_{b_k})\rho \\ &= \sum_{\lambda\in\Gamma} \Pi_\lambda P_{A_1}(b_1|x_1, \lambda) \ldots P_{A_{j-1}}(b_{j-1}|x_{j-1}, \lambda) P_{A_{j+1}}(b_{j+1}|x_{j+1}, \lambda) \ldots P_{A_k}(b_k|x_k, \lambda).\end{aligned}$$

This shows that the measurement results of the other subsystems except the subsystem $\mathcal{H}_{A_j}$ are independent of the measurements of the subsystem $\mathcal{H}_{A_j}$.

Remark 2.3 When a state $\rho \in D(\mathcal{H}^{(n)})$ can be written as

$$\rho = \sum_{\lambda=1}^{m} p_\lambda \rho^\lambda_{1\ldots i_1} \otimes \rho^\lambda_{(i_1+1)\ldots i_2} \otimes \ldots \otimes \rho^\lambda_{(i_{k-1}+1)\ldots i_k},$$

where $\rho^{\lambda}_{(i_{j-1}+1)\ldots i_j} \in D(\mathcal{H}_{A_j})$, and $\{p_\lambda\}_{\lambda=1}^{m}$ is a probability distribution, for every measurement assemblage $\mathcal{M}$, we compute that (2.2) holds for

$$\Pi_\lambda = p_\lambda,\ P_{A_j}(b_j|x_j,\lambda) = \mathrm{tr}(M_{b_j}^{x_j}\rho^{\lambda}_{(i_{j-1}+1)\ldots i_j})(j = 1, 2, \ldots, k).$$

This shows that ρ is Λ-local.

Theorem 2.1 *Suppose that ρ is a $(A_1, A_2, \ldots, A_k)$-local state of $\mathcal{H}^{(n)}$, then $\rho_{12\ldots(k-1)} := tr_{A_k}(\rho)$ is a $(A_1, A_2, \ldots, A_{k-1})$-local state of $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \ldots \otimes \mathcal{H}_{A_{k-1}}$.*

Proof Suppose that ρ is $(A_1, A_2, \ldots, A_k)$-local. For every measurement assemblage

$$\mathcal{N} = \{M^{x_1,x_2,\ldots,x_{k-1}} : x_j = 1, 2, \ldots, m_j(j = 1, 2, \ldots, k-1)\}$$

of local POVMs:

$$M^{x_1,x_2,\ldots,x_{k-1}} = \{M_{b_1}^{x_1} \otimes M_{b_2}^{x_2} \otimes \ldots \otimes M_{b_{k-1}}^{x_{k-1}} : b_i \in N_i(1 \le i \le k-1)\}$$

on $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \ldots \otimes \mathcal{H}_{A_{k-1}}$, by letting $M_{b_k}^{x_k} = I_{A_k}(x_k = 1, b_k = 1)$, the identity operator on $\mathcal{H}_{A_k}$, we obtain a measurement assemblage $\mathcal{M} = \{M^{x_1,x_2,\ldots,x_k} : x_j = 1, 2, \ldots, m_j(j = 1, 2, \ldots, k)\}$ of local POVMs:

$$M^{x_1,x_2,\ldots,x_k} = \{M_{b_1}^{x_1} \otimes M_{b_2}^{x_2} \otimes \ldots \otimes M_{b_k}^{x_k} : b_i \in N_i(1 \le i \le k)\}$$

on $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \ldots \otimes \mathcal{H}_{A_k}$ with $m_k = 1$, $N_k = \{1\}$. By Definition 2.1, we know that there exists a probability distribution $\{\Pi_\lambda\}_{\lambda\in\Gamma}$ such that $\forall x_i, b_i$, it holds that

$$\begin{aligned}&\mathrm{tr}(M_{b_1}^{x_1} \otimes \ldots \otimes M_{b_{k-1}}^{x_{k-1}} \otimes M_{b_k}^{x_k})\rho \\ &= \sum_{\lambda\in\Gamma} \Pi_\lambda P_{A_1}(b_1|x_1,\lambda)\ldots P_{A_{k-1}}(b_{k-1}|x_{k-1},\lambda)P_{A_k}(b_k|x_k,\lambda),\end{aligned}$$

where $P_{A_i}(b_i|x_i,\lambda) \ge 0$, $\sum_{b_i} P_{A_i}(b_i|x_i,\lambda) = 1(i = 1, 2, \ldots, k)$. Since $x_k = 1, b_k = 1$ and $\sum_{b_k} P_{A_k}(b_k|x_k,\lambda) = 1$, we get that $P_{A_k}(b_k|x_k,\lambda) = 1$. Consequently,

$$\begin{aligned}&\mathrm{tr}(M_{b_1}^{x_1} \otimes M_{b_2}^{x_2} \otimes \ldots \otimes M_{b_{k-1}}^{x_{k-1}})\rho_{12\ldots(k-1)} \\ &= \mathrm{tr}(M_{b_1}^{x_1} \otimes M_{b_2}^{x_2} \otimes \ldots \otimes M_{b_{k-1}}^{x_{k-1}} \otimes M_{b_k}^{x_k})\rho \\ &= \sum_{\lambda\in\Gamma} \Pi_\lambda P_{A_1}(b_1|x_1,\lambda)\ldots P_{A_{k-1}}(b_{k-1}|x_{k-1},\lambda),\end{aligned}$$

for all $x_i = 1, 2, \ldots, m_i, b_i \in N_i(i = 1, 2, \ldots, k-1)$. By using Definition 2.1 again, we conclude that $\rho_{12\ldots(k-1)}$ is a $(A_1, A_2, \ldots, A_{k-1})$-local state of $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \ldots \otimes \mathcal{H}_{A_{k-1}}$. The proof is completed. □

3 Λ-Nonlocality Inequalities

Suppose that $\rho \in D(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$. If ρ is $\{1, 2; 3\}$-local, then the following three Clauser-Horne-Shimony-Holt Bell inequalities hold, which were mentioned in [21](5a-5c):

$$|\langle\sigma_z \otimes \sigma_z \otimes P\rangle_\rho + \langle\sigma_z \otimes \sigma_z \otimes Q\rangle_\rho + \langle\sigma_z \otimes \sigma_x \otimes P\rangle_\rho - \langle\sigma_z \otimes \sigma_x \otimes Q\rangle_\rho| \le 2, \quad (3.1)$$

$$|\langle\sigma_x \otimes \sigma_z \otimes P\rangle_\rho + \langle\sigma_x \otimes \sigma_z \otimes Q\rangle_\rho + \langle\sigma_x \otimes \sigma_x \otimes P\rangle_\rho - \langle\sigma_x \otimes \sigma_x \otimes Q\rangle_\rho| \le 2, \quad (3.2)$$

$$|\langle I \otimes \sigma_z \otimes P\rangle_\rho + \langle I \otimes \sigma_z \otimes Q\rangle_\rho + \langle I \otimes \sigma_x \otimes P\rangle_\rho - \langle I \otimes \sigma_x \otimes Q\rangle_\rho| \le 2, \quad (3.3)$$

where P, Q are ±1-valued observables. Next, we will generalize the above inequalities and obtain the following theorem.

Theorem 3.1 *Suppose that $\rho \in D(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3)$. If ρ is $\{1, 2; 3\}$-local, then the inequality*

$$|\langle CP\rangle_\rho + \langle CQ\rangle_\rho + \langle DP\rangle_\rho - \langle DQ\rangle_\rho| \leq 2 \tag{3.4}$$

holds for all ± 1-valued observables C, D on $\mathcal{H}_{A_1} := \mathcal{H}_1 \otimes \mathcal{H}_2$, and ± 1-valued observables P, Q on $\mathcal{H}_{A_2} := \mathcal{H}_3$, where $CP = (C \otimes I_3)(I_1 \otimes I_2 \otimes P)$ and so on.

Proof Because that C, D, P and Q are ± 1-valued observables, they have their spectral decompositions:

$$C = C^+ - C^-, D = D^+ - D^-, P = P^+ - P^-, Q = Q^+ - Q^-.$$

By taking

$$M_+^{x_1} = C^+, M_-^{x_1} = C^-, M_+^{x_2} = P^+, M_-^{x_2} = P^-;$$
$$M_+^{y_1} = D^+, M_-^{y_1} = D^-, M_+^{y_2} = Q^+, M_-^{y_2} = Q^-,$$

we obtain POVMs $M^{x_1} = \{M_+^{x_1}, M_-^{x_1}\}$ and $M^{y_1} = \{M_+^{y_1}, M_-^{y_1}\}$ on $\mathcal{H}_{A_1}$, and $M^{x_2} = \{M_+^{x_2}, M_-^{x_2}\}$ and $M^{y_2} = \{M_+^{y_2}, M_-^{y_2}\}$ on $\mathcal{H}_{A_2}$. Thus, we obtain a measurement assemblage:

$$\mathcal{M} = \{M^{x_1} \otimes M^{x_2},\ M^{x_1} \otimes M^{y_2},\ M^{y_1} \otimes M^{x_2},\ M^{y_1} \otimes M^{y_2}\}$$

of four local POVMs on $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$. Explicitly,

$$M^{x_1} \otimes M^{x_2} = \{C^+ \otimes P^+, C^+ \otimes P^-, C^- \otimes P^+, C^- \otimes P^-\},$$
$$M^{x_1} \otimes M^{y_2} = \{C^+ \otimes Q^+, C^+ \otimes Q^-, C^- \otimes Q^+, C^- \otimes Q^-\},$$
$$M^{y_1} \otimes M^{x_2} = \{D^+ \otimes P^+, D^+ \otimes P^-, D^- \otimes P^+, C^- \otimes P^-\},$$
$$M^{y_1} \otimes M^{y_2} = \{D^+ \otimes Q^+, D^+ \otimes Q^-, D^- \otimes Q^+, D^- \otimes Q^-\}.$$

Since ρ is $\{1, 2; 3\}$-local, by Definition 2.1, for this $\mathcal{M}$, there exists a probability distribution $\{\Pi_\lambda\}_{\lambda \in \Gamma}$ such that (2.2) holds. Hence,

$$\mathrm{tr}(M_+^{x_1} \otimes M_+^{x_2})\rho = \sum_{\lambda \in \Gamma} \Pi_\lambda P_{A_1}(+|x_1, \lambda) P_{A_2}(+|x_2, \lambda),$$

and so on. Thus,

$$\begin{aligned}
\langle CP\rangle_\rho &= \mathrm{tr}(C^+ \otimes P^+)\rho - \mathrm{tr}(C^+ \otimes P^-)\rho - \mathrm{tr}(C^- \otimes P^+)\rho + \mathrm{tr}(C^- \otimes P^-)\rho \\
&= \mathrm{tr}(M_+^{x_1} \otimes M_+^{x_2})\rho - \mathrm{tr}(M_+^{x_1} \otimes M_-^{x_2})\rho - \mathrm{tr}(M_-^{x_1} \otimes M_+^{x_2})\rho + \mathrm{tr}(M_-^{x_1} \otimes M_-^{x_2})\rho \\
&= \sum_{\lambda \in \Gamma} \Pi_\lambda \left\{P_{A_1}(+|x_1, \lambda)P_{A_2}(+|x_2, \lambda) - P_{A_1}(+|x_1, \lambda)P_{A_2}(-|x_2, \lambda)\right. \\
&\quad \left. - P_{A_1}(-|x_1, \lambda)P_{A_2}(+|x_2, \lambda) + P_{A_1}(-|x_1, \lambda)P_{A_2}(-|x_2, \lambda)\right\} \\
&= \sum_{\lambda \in \Gamma} \Pi_\lambda (P_{A_1}(+|x_1, \lambda) - P_{A_1}(-|x_1, \lambda))(P_{A_2}(+|x_2, \lambda) - P_{A_2}(-|x_2, \lambda)).
\end{aligned}$$

Similarly,

$$\langle CQ\rangle_\rho = \sum_{\lambda \in \Gamma} \Pi_\lambda (P_{A_1}(+|x_1, \lambda) - P_{A_1}(-|x_1, \lambda))(P_{A_2}(+|y_2, \lambda) - P_{A_2}(-|y_2, \lambda)),$$

$$\langle DQ\rangle_\rho = \sum_{\lambda \in \Gamma} \Pi_\lambda (P_{A_1}(+|y_1, \lambda) - P_{A_1}(-|y_1, \lambda))(P_{A_2}(+|y_2, \lambda) - P_{A_2}(-|y_2, \lambda)),$$

$$\langle DP\rangle_\rho = \sum_{\lambda \in \Gamma} \Pi_\lambda (P_{A_1}(+|y_1, \lambda) - P_{A_1}(-|y_1, \lambda))(P_{A_2}(+|x_2, \lambda) - P_{A_2}(-|x_2, \lambda)).$$

Therefore

$$\Delta := \langle CP\rangle_\rho + \langle CQ\rangle_\rho + \langle DP\rangle_\rho - \langle DQ\rangle_\rho = \sum_{\lambda \in \Gamma} \Pi_\lambda \widetilde{p_2}(\lambda),$$

where

$$\widetilde{p_2}(\lambda) = (\alpha_1 - \alpha_2)(a - b + x - y) + (\beta_1 - \beta_2)(a - b - x + y),$$

where

$$P_{A_1}(+|x_1, \lambda) = \alpha_1,\ P_{A_1}(-|x_1, \lambda)) = \alpha_2,\ P_{A_1}(+|y_1, \lambda) = \beta_1,\ P_{A_1}(-|y_1, \lambda) = \beta_2,$$

$$P_{A_2}(+|x_2, \lambda) = a,\ P_{A_2}(-|x_2, \lambda) = b,\ P_{A_2}(+|y_2, \lambda) = x,\ P_{A_2}(-|y_2, \lambda) = y.$$

Clearly, we get that

$$\alpha_1 + \alpha_2 = 1,\quad \alpha_1, \alpha_2 \geq 0;$$
$$\beta_1 + \beta_2 = 1,\quad \beta_1, \beta_2 \geq 0;$$
$$a + b = 1,\quad a, b \geq 0;$$
$$x + y = 1,\quad x, y \geq 0.$$

Put $a - b = m_1, x - y = m_2$. Clearly, $-1 \leq m_i \leq 1 (i = 1, 2)$ and

$$\widetilde{p_2}(\lambda) = (\alpha_1 - \alpha_2)(m_1 + m_2) + (\beta_1 - \beta_2)(m_1 - m_2).$$

Therefore

$$|\widetilde{p_2}(\lambda)|^2 \leq (|m_1 + m_2| + |m_1 - m_2|)^2 = 4 \max\{m_1^2, m_2^2\} \leq 4.$$

So, $|\widetilde{p_2}(\lambda)| \leq 2$ for all λ and thus $|\Delta| \leq 2$. i.e.

$$|\langle CP \rangle_\rho + \langle CQ \rangle_\rho + \langle DP \rangle_\rho - \langle DQ \rangle_\rho| \leq 2.$$

The proof is completed. □

Remark 3.1 From Theorem 3.1 we know that every $\{1, 2; 3\}$-local state satisfies inequality (3.4). Thus, if there exist C, D, P, Q such that the inequality (3.4) is not satisfied, then ρ is $\{1, 2; 3\}$-nonlocal.

Remark 3.2 It is easy to see that $\widetilde{p_2}(\lambda)$ is obtained by changing C to $P_{A_1}(+|x_1, \lambda) - P_{A_1}(-|x_1, \lambda)$, D to $P_{A_1}(+|y_1, \lambda) - P_{A_1}(-|y_1, \lambda)$, P to $P_{A_2}(+|x_2, \lambda) - P_{A_2}(-|x_2, \lambda)$, and Q to $P_{A_2}(+|y_2, \lambda) - P_{A_2}(-|y_2, \lambda)$ in the expression of $CP + CQ + DP - DQ$.

Specially, when $C = \sigma_z \otimes \sigma_z, D = \sigma_z \otimes \sigma_x$ and $C = \sigma_x \otimes \sigma_z, D = \sigma_x \otimes \sigma_x$ and $C = I \otimes \sigma_z, D = I \otimes \sigma_x$ in Theorem 3.1, respectively, we get (3.1)-(3.3).

Similar to the proof of Theorem 3.1, one can prove the following conclusion.

Theorem 3.2 *Suppose that* $\rho \in D(\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n)$. *If* ρ *is* (A_1, A_2)*-local, then the inequality*

$$|\langle CP \rangle_\rho + \langle CQ \rangle_\rho + \langle DP \rangle_\rho - \langle DQ \rangle_\rho| \leq 2 \tag{3.5}$$

holds for all ± 1*-valued observables* C, D *on* $\mathcal{H}_{A_1} := \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_j$, *and* ± 1*-valued observables* P, Q *on* $\mathcal{H}_{A_2} := \mathcal{H}_{j+1} \otimes \ldots \otimes \mathcal{H}_n$.

To describe more general locality ($k > 2$), we suppose that Λ is given by (2.1) and $\{A_{a_s}^{x_s}\} (s = 1, 2, \ldots, k)$ is ± 1-valued observables on $\mathcal{H}_{A_s}$. Let the two party Mermin polynomial [20] be

$$M_2 = \frac{1}{2}(A_{a_1}^0 A_{a_2}^0 + A_{a_1}^0 A_{a_2}^1 + A_{a_1}^1 A_{a_2}^0 - A_{a_1}^1 A_{a_2}^1), \tag{3.6}$$

$$M_2' = \frac{1}{2}(A_{a_1}^1 A_{a_2}^1 + A_{a_1}^1 A_{a_2}^0 + A_{a_1}^0 A_{a_2}^1 - A_{a_1}^0 A_{a_2}^0). \tag{3.6'}$$

Then M_m and M'_m are generated from M_{m-1} by recursion relation:

$$M_m = \frac{1}{2}[M_{m-1}(A^0_{a_m} + A^1_{a_m}) + M'_{m-1}(A^0_{a_m} - A^1_{a_m})]. \tag{3.7}$$

$$M'_m = \frac{1}{2}[M'_{m-1}(A^1_{a_m} + A^0_{a_m}) + M_{m-1}(A^1_{a_m} - A^0_{a_m})]. \tag{3.7$'$}$$

Following [22] we define the Svetlichny polynomials as

$$S_m = \begin{cases} M_m, & m = 2n; \\ \frac{1}{2}(M_m + M'_m), & m = 2n+1. \end{cases} \tag{3.8}$$

$$S'_m = \begin{cases} M'_m, & m = 2n; \\ \frac{1}{2}(M'_m + M_m), & m = 2n+1. \end{cases} \tag{3.8$'$}$$

By these definitions, we see that

$$M_m + M'_m = S_m + S'_m, \quad \forall m = 2, 3, \ldots, k.$$

Remark 3.3 Let ρ be $\Lambda = (A_1, A_2)$-local. Then we can get

$$|\langle S_2\rangle_\rho| = \frac{1}{2}\left|\langle A^0_{a_1} A^0_{a_2}\rangle_\rho + \langle A^0_{a_1} A^1_{a_2}\rangle_\rho + \langle A^1_{a_1} A^0_{a_2}\rangle_\rho - \langle A^1_{a_1} A^1_{a_2}\rangle_\rho\right| \le 1,$$

by using (3.5) for $C = A^0_{a_1}$, $D = A^1_{a_1}$, $P = A^0_{a_2}$ and $Q = A^1_{a_2}$.

Similarly,

$$|\langle S'_2\rangle_\rho| = \frac{1}{2}\left|\langle A^1_{a_1} A^1_{a_2}\rangle_\rho + \langle A^1_{a_1} A^0_{a_2}\rangle_\rho + \langle A^0_{a_1} A^1_{a_2}\rangle_\rho - \langle A^0_{a_1} A^0_{a_2}\rangle_\rho\right| \le 1.$$

Furthermore, since ρ is $\Lambda = (A_1, A_2)$-local, by Definition 2.1, there exists a probability distribution $\{\Pi_\lambda\}_{\lambda\in\Gamma}$ such that $\langle S_2\rangle_\rho = \sum_{\lambda\in\Gamma} \Pi_\lambda p_2(\lambda)$ and $\langle S'_2\rangle_\rho = \sum_{\lambda\in\Gamma} \Pi_\lambda p'_2(\lambda)$, where

$$\begin{aligned} p_2(\lambda) \;=\; \frac{1}{2}\big\{&(P_{A_1}(+|x_1,\lambda) - P_{A_1}(-|x_1,\lambda))(P_{A_2}(+|x_2,\lambda) - P_{A_2}(-|x_2,\lambda)) \\ &+(P_{A_1}(+|x_1,\lambda) - P_{A_1}(-|x_1,\lambda))(P_{A_2}(+|y_2,\lambda) - P_{A_2}(-|y_2,\lambda)) \\ &+(P_{A_1}(+|y_1,\lambda) - P_{A_1}(-|y_1,\lambda))(P_{A_2}(+|x_2,\lambda) - P_{A_2}(-|x_2,\lambda)) \\ &-(P_{A_1}(+|y_1,\lambda) - P_{A_1}(-|y_1,\lambda))(P_{A_2}(+|y_2,\lambda) - P_{A_2}(-|y_2,\lambda))\big\}, \end{aligned}$$

$$\begin{aligned} p'_2(\lambda) \;=\; \frac{1}{2}\big\{&(P_{A_1}(+|y_1,\lambda) - P_{A_1}(-|y_1,\lambda))(P_{A_2}(+|y_2,\lambda) - P_{A_2}(-|y_2,\lambda)) \\ &+(P_{A_1}(+|y_1,\lambda) - P_{A_1}(-|y_1,\lambda))(P_{A_2}(+|x_2,\lambda) - P_{A_2}(-|x_2,\lambda)) \\ &+(P_{A_1}(+|x_1,\lambda) - P_{A_1}(-|x_1,\lambda))(P_{A_2}(+|y_2,\lambda) - P_{A_2}(-|y_2,\lambda)) \\ &-(P_{A_1}(+|x_1,\lambda) - P_{A_1}(-|x_1,\lambda))(P_{A_2}(+|x_2,\lambda) - P_{A_2}(-|x_2,\lambda))\big\}. \end{aligned}$$

Clearly, $p_2(\lambda)$ and $p'_2(\lambda)$ are obtained by changing $A^0_{a_1}$ to $P_{A_1}(+|x_1,\lambda) - P_{A_1}(-|x_1,\lambda)$, $A^1_{a_1}$ to $P_{A_1}(+|y_1,\lambda) - P_{A_1}(-|y_1,\lambda)$, $A^0_{a_2}$ to $P_{A_2}(+|x_2,\lambda) - P_{A_2}(-|x_2,\lambda)$, and $A^1_{a_2}$ to $P_{A_2}(+|y_2,\lambda) - P_{A_2}(-|y_2,\lambda)$ in the definitions of S_2 and S'_2, respectively. It is easy to see that $p_2(\lambda) = \frac{1}{2}\widetilde{p}_2(\lambda)$. From the proof of Theorem 3.1, we can get $|p_2(\lambda)| \le 1$ for all λ. Similarly, we can prove $|p'_2(\lambda)| \le 1$ for all λ.

Theorem 3.3 *Suppose that ρ is $\Lambda = (A_1, A_2, A_3)$-local, then*

$$|\langle S_3\rangle_\rho| = |\langle S'_3\rangle_\rho| \le 1. \tag{3.9}$$

Proof Using (3.6)-(3.8) yield

$$
\begin{aligned}
S_3 &= \frac{1}{2}(M_3 + M_3') \\
&= \frac{1}{2}\left\{\frac{1}{2}[M_2(A^0_{a_3} + A^1_{a_3}) + M_2'(A^0_{a_3} - A^1_{a_3})]\right. \\
&\quad \left. + \frac{1}{2}[M_2'(A^1_{a_3} + A^0_{a_3}) + M_2(A^1_{a_3} - A^0_{a_3})]\right\} \\
&= \frac{1}{2}(M_2 A^1_{a_3} + M_2' A^0_{a_3}) \\
&= \frac{1}{4}\left\{(A^0_{a_1}A^0_{a_2} + A^0_{a_1}A^1_{a_2} + A^1_{a_1}A^0_{a_2} - A^1_{a_1}A^1_{a_2})A^1_{a_3}\right. \\
&\quad \left. + (A^1_{a_1}A^1_{a_2} + A^1_{a_1}A^0_{a_2} + A^0_{a_1}A^1_{a_2} - A^0_{a_1}A^0_{a_2})A^0_{a_3}\right\}.
\end{aligned}
$$

Because that $A^0_{a_1}, A^1_{a_1}, A^0_{a_2}, A^1_{a_2}, A^0_{a_3}, A^1_{a_3}$ are ± 1-valued observables, they have their spectral decompositions:

$$A^0_{a_i} = (A^0_{a_i})^+ - (A^0_{a_i})^-, A^1_{a_i} = (A^1_{a_i})^+ - (A^1_{a_i})^-, i = 1, 2, 3.$$

By taking

$$M^{x_1}_+ = (A^0_{a_1})^+, M^{x_1}_- = (A^0_{a_1})^-,$$

$$M^{x_2}_+ = (A^0_{a_2})^+, M^{x_2}_- = (A^0_{a_i})^-,$$

$$M^{x_3}_+ = (A^0_{a_3})^+, M^{x_3}_- = (A^0_{a_3})^-,$$

$$M^{y_1}_+ = (A^1_{a_1})^+, M^{y_1}_- = (A^1_{a_1})^-,$$

$$M^{y_2}_+ = (A^1_{a_2})^+, M^{y_2}_- = (A^1_{a_2})^-,$$

$$M^{y_3}_+ = (A^1_{a_3})^+, M^{y_3}_- = (A^1_{a_3})^-,$$

we obtain POVMs $M^{x_1} = \{M^{x_1}_+, M^{x_1}_-\}$ and $M^{y_1} = \{M^{y_1}_+, M^{y_1}_-\}$ on $\mathcal{H}_{A_1}$, and $M^{x_2} = \{M^{x_2}_+, M^{x_2}_-\}$ and $M^{y_2} = \{M^{y_2}_+, M^{y_2}_-\}$ on $\mathcal{H}_{A_2}$ and $M^{x_3} = \{M^{x_3}_+, M^{x_3}_-\}$ and $M^{y_3} = \{M^{y_3}_+, M^{y_3}_-\}$ on $\mathcal{H}_{A_3}$. Thus, we obtain a measurement assemblage $\mathcal{M}$ consisting of the following eight local POVMs on $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{A_3}$:

$$M^{x_1} \otimes M^{x_2} \otimes M^{y_3},\ M^{x_1} \otimes M^{y_2} \otimes M^{y_3},\ M^{y_1} \otimes M^{x_2} \otimes M^{y_3}, M^{y_1} \otimes M^{y_2} \otimes M^{y_3},$$

$$M^{y_1} \otimes M^{y_2} \otimes M^{x_3},\ M^{y_1} \otimes M^{x_2} \otimes M^{x_3},\ M^{x_1} \otimes M^{y_2} \otimes M^{x_3}, M^{x_1} \otimes M^{x_2} \otimes M^{x_3}.$$

Since ρ is (A_1, A_2, A_3)-local, by Definition 2.1, for this $\mathcal{M}$, there exists a probability distribution $\{\Pi_\lambda\}_{\lambda\in\Gamma}$ such that (2.2) holds. Thus, we have

$$\mathrm{tr}(M^{x_1}_+ \otimes M^{x_2}_+ \otimes M^{y_3}_+)\rho = \sum_{\lambda\in\Gamma} \Pi_\lambda P_{A_1}(+|x_1, \lambda) P_{A_2}(+|x_2, \lambda) P_{A_3}(+|y_3, \lambda),$$

and so on. Hence,

$$
\begin{aligned}
&\langle A^0_{a_1} A^0_{a_2} A^1_{a_3}\rangle_\rho \\
=\ & \mathrm{tr}((A^0_{a_1})^+ \otimes (A^0_{a_2})^+ \otimes (A^1_{a_3})^+)\rho - \mathrm{tr}((A^0_{a_1})^+ \otimes (A^0_{a_2})^- \otimes (A^1_{a_3})^+)\rho \\
& -\mathrm{tr}((A^0_{a_1})^- \otimes (A^0_{a_2})^+ \otimes (A^1_{a_3})^+)\rho + \mathrm{tr}((A^0_{a_1})^- \otimes (A^0_{a_2})^- \otimes (A^1_{a_3})^+)\rho \\
& -\mathrm{tr}((A^0_{a_1})^+ \otimes (A^0_{a_2})^+ \otimes (A^1_{a_3})^-)\rho + \mathrm{tr}((A^0_{a_1})^+ \otimes (A^0_{a_2})^- \otimes (A^1_{a_3})^-)\rho \\
& +\mathrm{tr}((A^0_{a_1})^- \otimes (A^0_{a_2})^+ \otimes (A^1_{a_3})^-)\rho - \mathrm{tr}((A^0_{a_1})^- \otimes (A^0_{a_2})^- \otimes (A^1_{a_3})^-)\rho \\
=\ & \mathrm{tr}(M^{x_1}_+ \otimes M^{x_2}_+ \otimes M^{y_3}_+)\rho - \mathrm{tr}(M^{x_1}_+ \otimes M^{x_2}_- \otimes M^{y_3}_+)\rho - \mathrm{tr}(M^{x_1}_- \otimes M^{x_2}_+ \otimes M^{y_3}_+)\rho \\
& +\mathrm{tr}(M^{x_1}_- \otimes M^{x_2}_- \otimes M^{y_3}_+)\rho - \mathrm{tr}(M^{x_1}_+ \otimes M^{x_2}_+ \otimes M^{y_3}_-)\rho + \mathrm{tr}(M^{x_1}_+ \otimes M^{x_2}_- \otimes M^{y_3}_-)\rho \\
& +\mathrm{tr}(M^{x_1}_- \otimes M^{x_2}_+ \otimes M^{y_3}_-)\rho - \mathrm{tr}(M^{x_1}_- \otimes M^{x_2}_- \otimes M^{y_3}_-)\rho \\
=\ & \sum_{\lambda\in\Gamma} \Pi_\lambda \big\{ P_{A_1}(+|x_1,\lambda) P_{A_2}(+|x_2,\lambda) P_{A_3}(+|y_3,\lambda) \\
& - P_{A_1}(+|x_1,\lambda) P_{A_2}(-|x_2,\lambda) P_{A_3}(+|y_3,\lambda) - P_{A_1}(-|x_1,\lambda) P_{A_2}(+|x_2,\lambda) P_{A_3}(+|y_3,\lambda) \\
& + P_{A_1}(-|x_1,\lambda) P_{A_2}(-|x_2,\lambda) P_{A_3}(+|y_3,\lambda) - P_{A_1}(+|x_1,\lambda) P_{A_2}(+|x_2,\lambda) P_{A_3}(-|y_3,\lambda) \\
& + P_{A_1}(+|x_1,\lambda) P_{A_2}(-|x_2,\lambda) P_{A_3}(-|y_3,\lambda) + P_{A_1}(-|x_1,\lambda) P_{A_2}(+|x_2,\lambda) P_{A_3}(-|y_3,\lambda) \\
& - P_{A_1}(-|x_1,\lambda) P_{A_2}(-|x_2,\lambda) P_{A_3}(-|y_3,\lambda) \big\} \\
=\ & \sum_{\lambda\in\Gamma} \Pi_\lambda (P_{A_1}(+|x_1,\lambda) - P_{A_1}(-|x_1,\lambda))(P_{A_2}(+|x_2,\lambda) - P_{A_2}(-|x_2,\lambda)) \\
& \times (P_{A_3}(+|y_3,\lambda) - P_{A_3}(-|y_3,\lambda)).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\langle A^1_{a_1} A^0_{a_2} A^1_{a_3}\rangle_\rho &= \sum_{\lambda\in\Gamma} \Pi_\lambda (P_{A_1}(+|y_1,\lambda) - P_{A_1}(-|y_1,\lambda))(P_{A_2}(+|x_2,\lambda) - P_{A_2}(-|x_2,\lambda)) \\
&\quad \times (P_{A_3}(+|y_3,\lambda) - P_{A_3}(-|y_3,\lambda)), \\
\langle A^0_{a_1} A^1_{a_2} A^1_{a_3}\rangle_\rho &= \sum_{\lambda\in\Gamma} \Pi_\lambda (P_{A_1}(+|x_1,\lambda) - P_{A_1}(-|x_1,\lambda))(P_{A_2}(+|y_2,\lambda) - P_{A_2}(-|y_2,\lambda)) \\
&\quad \times (P_{A_3}(+|y_3,\lambda) - P_{A_3}(-|y_3,\lambda)), \\
\langle A^1_{a_1} A^1_{a_2} A^1_{a_3}\rangle_\rho &= \sum_{\lambda\in\Gamma} \Pi_\lambda (P_{A_1}(+|y_1,\lambda) - P_{A_1}(-|y_1,\lambda))(P_{A_2}(+|y_2,\lambda) - P_{A_2}(-|y_2,\lambda)) \\
&\quad \times (P_{A_3}(+|y_3,\lambda) - P_{A_3}(-|y_3,\lambda)), \\
\langle A^1_{a_1} A^1_{a_2} A^0_{a_3}\rangle_\rho &= \sum_{\lambda\in\Gamma} \Pi_\lambda (P_{A_1}(+|y_1,\lambda) - P_{A_1}(-|y_1,\lambda))(P_{A_2}(+|y_2,\lambda) - P_{A_2}(-|y_2,\lambda)) \\
&\quad \times (P_{A_3}(+|x_3,\lambda) - P_{A_3}(-|x_3,\lambda)), \\
\langle A^0_{a_1} A^1_{a_2} A^0_{a_3}\rangle_\rho &= \sum_{\lambda\in\Gamma} \Pi_\lambda (P_{A_1}(+|x_1,\lambda) - P_{A_1}(-|x_1,\lambda))(P_{A_2}(+|y_2,\lambda) - P_{A_2}(-|y_2,\lambda)) \\
&\quad \times (P_{A_3}(+|x_3,\lambda) - P_{A_3}(-|x_3,\lambda)), \\
\langle A^1_{a_1} A^0_{a_2} A^0_{a_3}\rangle_\rho &= \sum_{\lambda\in\Gamma} \Pi_\lambda (P_{A_1}(+|y_1,\lambda) - P_{A_1}(-|y_1,\lambda))(P_{A_2}(+|x_2,\lambda) - P_{A_2}(-|x_2,\lambda)) \\
&\quad \times (P_{A_3}(+|x_3,\lambda) - P_{A_3}(-|x_3,\lambda))
\end{aligned}
$$

$$\langle A_{a_1}^0 A_{a_2}^0 A_{a_3}^0 \rangle_\rho = \sum_{\lambda\in\Gamma} \Pi_\lambda (P_{A_1}(+|x_1,\lambda) - P_{A_1}(-|x_1,\lambda))(P_{A_2}(+|x_2,\lambda) - P_{A_2}(-|x_2,\lambda)) \times (P_{A_3}(+|x_3,\lambda) - P_{A_3}(-|x_3,\lambda))$$

Therefore,

$$\begin{aligned}\langle S_3\rangle_\rho &= \frac{1}{4}\sum_{\lambda\in\Gamma}\Pi_\lambda \Big\{\{(P_{A_1}(+|x_1,\lambda) - P_{A_1}(-|x_1,\lambda))(P_{A_2}(+|x_2,\lambda) - P_{A_2}(-|x_2,\lambda))\\
&+(P_{A_1}(+|x_1,\lambda) - P_{A_1}(-|x_1,\lambda))(P_{A_2}(+|y_2,\lambda) - P_{A_2}(-|y_2,\lambda))\\
&+(P_{A_1}(+|y_1,\lambda) - P_{A_1}(-|y_1,\lambda))(P_{A_2}(+|x_2,\lambda) - P_{A_2}(-|x_2,\lambda))\\
&-(P_{A_1}(+|y_1,\lambda) - P_{A_1}(-|y_1,\lambda))(P_{A_2}(+|y_2,\lambda) - P_{A_2}(-|y_2,\lambda))\}\\
&\times(P_{A_3}(+|y_3,\lambda) - P_{A_3}(-|y_3,\lambda)) + \{(P_{A_1}(+|y_1,\lambda) - P_{A_1}(-|y_1,\lambda))\\
&\times(P_{A_2}(+|y_2,\lambda) - P_{A_2}(-|y_2,\lambda)) + (P_{A_1}(+|y_1,\lambda) - P_{A_1}(-|y_1,\lambda))\\
&\times(P_{A_2}(+|x_2,\lambda) - P_{A_2}(-|x_2,\lambda)) + (P_{A_1}(+|x_1,\lambda) - P_{A_1}(-|x_1,\lambda))\\
&\times(P_{A_2}(+|y_2,\lambda) - P_{A_2}(-|y_2,\lambda)) - (P_{A_1}(+|x_1,\lambda) - P_{A_1}(-|x_1,\lambda))\\
&\times(P_{A_2}(+|x_2,\lambda) - P_{A_2}(-|x_2,\lambda))\}(P_{A_3}(+|x_3,\lambda) - P_{A_3}(-|x_3,\lambda))\Big\}\\
&= \sum_{\lambda\in\Gamma}\Pi_\lambda p_3(\lambda),\end{aligned}$$

where

$$p_3(\lambda) = \frac{1}{2}\left\{p_2(\lambda)(P_{A_3}(+|y_3,\lambda) - P_{A_3}(-|y_3,\lambda)) + p_2'(\lambda)(P_{A_3}(+|x_3,\lambda) - P_{A_3}(-|x_3,\lambda))\right\},$$

and $p_2(\lambda)$ and $p_2'(\lambda)$ are defined in Remark 3.3. Since $|p_2(\lambda)| \le 1$ and $|p_2'(\lambda)| \le 1$, we obtain that

$$\begin{aligned}|p_3(\lambda)| &\le \frac{1}{2}\left\{|p_2(\lambda)|\cdot|P_{A_3}(+|y_3,\lambda) - P_{A_3}(-|y_3,\lambda)| + |p_2'(\lambda)|\cdot|P_{A_3}(+|x_3,\lambda)\right.\\
&\qquad \left.-P_{A_3}(-|x_3,\lambda)|\right\}\\
&\le \frac{1}{2}(|p_2(\lambda)| + |p_2'(\lambda)|)\\
&\le 1.\end{aligned}$$

Thus,

$$|\langle S_3\rangle_\rho| = \left|\sum_{\lambda\in\Gamma}\Pi_\lambda p_3(\lambda)\right| \le 1.$$

Since $S_3 = S_3'$, so $|\langle S_3'\rangle_\rho| = |\langle S_3\rangle_\rho| \le 1$. The proof is completed. □

Remark 3.4 Combining Theorem 2.1 with Remark 3.3, we can get that if ρ is $\Lambda = (A_1, A_2, A_3)$-local, then $|\langle S_2\rangle_{\rho_{12}}| \le 1$ and $|\langle S_2'\rangle_{\rho_{12}}| \le 1$.

Now let's generalize above results to the general case.

Theorem 3.4 *Suppose that ρ is $\Lambda = (A_1, A_2, \ldots, A_k)$-local, then*

$$|\langle S_k\rangle_\rho| \le 2^{k-\lceil\frac{k}{2}\rceil-1}, \tag{3.10}$$

$$|\langle S_k'\rangle_\rho| \le 2^{k-\lceil\frac{k}{2}\rceil-1}, \tag{3.11}$$

where $\lceil\frac{k}{2}\rceil$ indicates rounding up to the next nearest integer.

Proof Because that $A_{a_i}^0, A_{a_i}^1 (i = 1, 2, \ldots, k)$ are ± 1-valued observables, they have their spectral decompositions:

$$A_{a_i}^0 = (A_{a_i}^0)^+ - (A_{a_i}^0)^-, A_{a_i}^1 = (A_{a_i}^1)^+ - (A_{a_i}^1)^-, i = 1, 2, \ldots, k.$$

For every $i = 1, 2, \ldots, k$, by taking

$$M_+^{x_i} = (A_{a_i}^0)^+, M_-^{x_i} = (A_{a_i}^0)^-, M_+^{y_i} = (A_{a_i}^1)^+, M_-^{y_i} = (A_{a_i}^1)^-,$$

we obtain POVMs $M^{x_i} = \{M_+^{x_i}, M_-^{x_i}\}$ and $M^{y_i} = \{M_+^{y_i}, M_-^{y_i}\}$ on $\mathcal{H}_{A_i} (i = 1, 2, \ldots, k)$. Thus, we obtain a measurement assemblage

$$\mathcal{M} = \{M^{t_1} \otimes M^{t_2} \otimes \ldots \otimes M^{t_k} : t_i = x_i, y_i (i = 1, 2, \ldots, k)\}$$

of 2^k local POVMs on $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \ldots \otimes \mathcal{H}_{A_k}$.

Since ρ is $\Lambda = (A_1, A_2, \ldots, A_k)$-local, by Definition 2.1, for this $\mathcal{M}$, there exists a probability distribution $\{\Pi_\lambda\}_{\lambda \in \Gamma}$ such that (2.2) holds. Thus, similar to the proofs of Theorem 3.1 and Theorem 3.2, we obtain that

$$\langle S_k \rangle_\rho = \sum_{\lambda \in \Gamma} \Pi_\lambda p_k(\lambda), \quad \langle S_k' \rangle_\rho = \sum_{\lambda \in \Gamma} \Pi_\lambda p_k'(\lambda),$$

where $p_k(\lambda)$ and $p_k'(\lambda)$ are obtained by changing $A_{a_i}^0$ to $P_{A_i}(+|x_i, \lambda) - P_{A_i}(-|x_i, \lambda)$, $A_{a_i}^1$ to $P_{A_i}(+|y_i, \lambda) - P_{A_i}(-|y_i, \lambda) (i = 1, 2, \ldots, k)$ in the definitions of S_k and S_k', respectively.

Let $\lambda \in \Gamma$ be fixed. Next, we prove that $|p_{2m}(\lambda)| \le 2^{m-1}$ and $|p_{2m}'(\lambda)| \le 2^{m-1}$ hold for $1 \le m \le \lfloor \frac{k}{2} \rfloor$, where $\lfloor \frac{k}{2} \rfloor$ indicates rounding down to the next nearest integer.

When $m = 1$, by Remark 3.3, we know that $|p_2(\lambda)| \le 1$ and $|p_2'(\lambda)| \le 1$ since $\rho_{12} = \mathrm{tr}_{A_3 A_4 \ldots A_k}(\rho)$ is (A_1, A_2)-local(Theorem 2.1). Suppose that $|p_{2m}(\lambda)| \le 2^{m-1}$ and $|p_{2m}'(\lambda)| \le 2^{m-1}$ hold. Using (3.6)-(3.8) yield that

$$\begin{aligned}
S_{2m+2} = & \frac{1}{2}[M_{2m+1}(A_{a_{2m+2}}^0 + A_{a_{2m+2}}^1) + M_{2m+1}'(A_{a_{2m+2}}^0 - A_{a_{2m+2}}^1)] \\
= & \frac{1}{2}\left\{\frac{1}{2}[M_{2m}(A_{a_{2m+1}}^0 + A_{a_{2m+1}}^1) + M_{2m}'(A_{a_{2m+1}}^0 - A_{a_{2m+1}}^1)](A_{a_{2m+2}}^0 + A_{a_{2m+2}}^1)\right. \\
& \left. + \frac{1}{2}[M_{2m}'(A_{a_{2m+1}}^1 + A_{a_{2m+1}}^0) + M_{2m}(A_{a_{2m+1}}^1 - A_{a_{2m+1}}^0)](A_{a_{2m+2}}^0 - A_{a_{2m+2}}^1)\right\} \\
= & \frac{1}{2}\left\{M_{2m}(A_{a_{2m+1}}^0 A_{a_{2m+2}}^1 + A_{a_{2m+1}}^1 A_{a_{2m+2}}^0) + M_{2m}'(A_{a_{2m+1}}^0 A_{a_{2m+2}}^0 - A_{a_{2m+1}}^1 A_{a_{2m+2}}^1)\right\} \\
= & \frac{1}{2}\left\{S_{2m}(A_{a_{2m+1}}^0 A_{a_{2m+2}}^1 + A_{a_{2m+1}}^1 A_{a_{2m+2}}^0) + S_{2m}'(A_{a_{2m+1}}^0 A_{a_{2m+2}}^0 - A_{a_{2m+1}}^1 A_{a_{2m+2}}^1)\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
p_{2m+2}(\lambda) = & \frac{1}{2}\left\{p_{2m}(\lambda)\{(P_{A_{2m+1}}(+|x_{2m+1}, \lambda) - P_{A_{2m+1}}(-|x_{2m+1}, \lambda))\right. \\
& \times (P_{A_{2m+2}}(+|y_{2m+2}, \lambda) - P_{A_{2m+2}}(-|y_{2m+2}, \lambda)) \\
& + (P_{A_{2m+1}}(+|y_{2m+1}, \lambda) - P_{A_{2m+1}}(-|y_{2m+1}, \lambda)) \\
& \times (P_{A_{2m+2}}(+|x_{2m+2}, \lambda) - P_{A_{2m+2}}(-|x_{2m+2}, \lambda))\} \\
& + p_{2m}'(\lambda)\{(P_{A_{2m+1}}(+|x_{2m+1}, \lambda) - P_{A_{2m+1}}(-|x_{2m+1}, \lambda)) \\
& \times (P_{A_{2m+2}}(+|x_{2m+2}, \lambda) - P_{A_{2m+2}}(-|x_{2m+2}, \lambda)) \\
& - (P_{A_{2m+1}}(+|y_{2m+1}, \lambda) - P_{A_{2m+1}}(-|y_{2m+1}, \lambda)) \\
& \left. \times (P_{A_{2m+2}}(+|y_{2m+2}, \lambda) - P_{A_{2m+2}}(-|y_{2m+2}, \lambda))\}\right\},
\end{aligned}$$

therefore,

$$\begin{aligned}
|p_{2m+2}(\lambda)| \le\ & \frac{1}{2}\Big\{|p_{2m}(\lambda)|\cdot\{|P_{A_{2m+1}}(+|x_{2m+1},\lambda)-P_{A_{2m+1}}(-|x_{2m+1},\lambda)| \\
& \cdot|P_{A_{2m+2}}(+|y_{2m+2},\lambda)-P_{A_{2m+2}}(-|y_{2m+2},\lambda)| \\
& +|P_{A_{2m+1}}(+|y_{2m+1},\lambda)-P_{A_{2m+1}}(-|y_{2m+1},\lambda)| \\
& \cdot|P_{A_{2m+2}}(+|x_{2m+2},\lambda)-P_{A_{2m+2}}(-|x_{2m+2},\lambda)|\} \\
& +|p'_{2m}(\lambda)|\cdot\{|P_{A_{2m+1}}(+|x_{2m+1},\lambda)-P_{A_{2m+1}}(-|x_{2m+1},\lambda)| \\
& \cdot|P_{A_{2m+2}}(+|x_{2m+2},\lambda)-P_{A_{2m+2}}(-|x_{2m+2},\lambda)| \\
& +|P_{A_{2m+1}}(+|y_{2m+1},\lambda)-P_{A_{2m+1}}(-|y_{2m+1},\lambda)| \\
& \cdot|P_{A_{2m+2}}(+|y_{2m+2},\lambda)-P_{A_{2m+2}}(-|y_{2m+2},\lambda)|\}\Big\} \\
\le\ & |p_{2m}(\lambda)|+|p'_{2m}(\lambda)| \\
\le\ & 2^m.
\end{aligned}$$

S'_{2m+2} can be obtained from S_{2m+2}:

$$S'_{2m+2}=\frac{1}{2}[S'_{2m}(A^1_{a_{2m+1}}A^0_{a_{2m+2}}+A^0_{a_{2m+1}}A^1_{a_{2m+2}})+S_{2m}(A^1_{a_{2m+1}}A^1_{a_{2m+2}}-A^0_{a_{2m+1}}A^0_{a_{2m+2}})].$$

Similarly, we can get

$$|p'_{2m+2}(\lambda)|\le 2^m.$$

Therefore, by induction, we have $|p_{2m}(\lambda)| \le 2^{m-1}$ and $|p'_{2m}(\lambda)| \le 2^{m-1}$ hold for all $m = 1, 2, \ldots, \lfloor \frac{k}{2} \rfloor$ and all $\lambda \in \Gamma$.

Using (3.6)-(3.8) yield that

$$\begin{aligned}
S_{2m+1} &= \frac{1}{2}(M_{2m+1}+M'_{2m+1}) \\
&= \frac{1}{2}\left\{\frac{1}{2}[M_{2m}(A^0_{a_{2m+1}}+A^1_{a_{2m+1}})+M'_{2m}(A^0_{a_{2m+1}}-A^1_{a_{2m+1}})]\right. \\
&\quad \left.+\frac{1}{2}[M'_{2m}(A^1_{a_{2m+1}}+A^0_{a_{2m+1}})+M_{2m}(A^1_{a_{2m+1}}-A^0_{a_{2m+1}})]\right\} \\
&= \frac{1}{2}(M_{2m}A^1_{a_{2m+1}}+M'_{2m}A^0_{a_{2m+1}}) \\
&= \frac{1}{2}(S_{2m}A^1_{a_{2m+1}}+S'_{2m}A^0_{a_{2m+1}}),
\end{aligned}$$

then

$$\begin{aligned}
p_{2m+1}(\lambda) = \frac{1}{2}\Big\{& p_{2m}(\lambda)[P_{A_{2m+1}}(+|y_{2m+1},\lambda)-P_{A_{2m+1}}(-|y_{2m+1},\lambda)] \\
& +p'_{2m}(\lambda)[P_{A_{2m+1}}(+|x_{2m+1},\lambda)-P_{A_{2m+1}}(-|x_{2m+1},\lambda)]\Big\}.
\end{aligned}$$

By using the fact that $|p_{2m}(\lambda)| \le 2^{m-1}$ and $|p'_{2m}(\lambda)| \le 2^{m-1}$ for all λ, we get that

$$\begin{aligned}
|p_{2m+1}(\lambda)| \le\ & \frac{1}{2}\Big\{|p_{2m}(\lambda)|\cdot|P_{A_{2m+1}}(+|y_{2m+1},\lambda)-P_{A_{2m+1}}(-|y_{2m+1},\lambda)| \\
& +|p'_{2m}(\lambda)|\cdot|P_{A_{2m+1}}(+|x_{2m+1},\lambda)-P_{A_{2m+1}}(-|x_{2m+1},\lambda)|\Big\} \\
\le\ & \frac{1}{2}(|p_{2m}(\lambda)|+|p'_{2m}(\lambda)|) \\
\le\ & 2^{m-1}.
\end{aligned}$$

Since $S'_{2m+1} = \frac{1}{2}(S'_{2m}A^0_{a_{2m+1}} + S_{2m}A^1_{a_{2m+1}}) = S_{2m+1}$, we have

$$|p'_{2m+1}(\lambda)| = |p_{2m+1}(\lambda)| \leq 2^{m-1}.$$

Therefore, we have $|p_{2m+1}(\lambda)| \leq 2^{m-1}$ and $|p'_{2m+1}(\lambda)| \leq 2^{m-1}$ hold for all $m = 1, 2, \ldots, \lfloor\frac{k-1}{2}\rfloor$ and all $\lambda \in \Gamma$.

From what has been discussed above, we can get that

$$|p_k(\lambda)| \leq 2^{k-\lceil\frac{k}{2}\rceil-1},\ |p'_k(\lambda)| \leq 2^{k-\lceil\frac{k}{2}\rceil-1}.$$

Hence, we have

$$|\langle S_k\rangle_\rho| = \left|\sum_{\lambda\in\Gamma}\Pi_\lambda p_k(\lambda)\right| \leq 2^{k-\lceil\frac{k}{2}\rceil-1},\ |\langle S'_k\rangle_\rho| = \left|\sum_{\lambda\in\Gamma}\Pi_\lambda p'_k(\lambda)\right| \leq 2^{k-\lceil\frac{k}{2}\rceil-1}.$$

The proof is completed. □

Example 3.1 The 3-qubit state $|\psi\rangle = \frac{1}{\sqrt{2}}(|001\rangle - |110\rangle)$, i.e. $\rho = |\psi\rangle\langle\psi|$ is genuinely nonlocality, i.e. it is not Λ-local for every Λ.

Proof By computing, we get

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}(|001\rangle\langle 001| - |001\rangle\langle 110| - |110\rangle\langle 001| + |110\rangle\langle 110|).$$

Generally, for all real unit vectors:

$$\boldsymbol{a} = (a_x, a_y, a_z), \boldsymbol{b} = (b_x, b_y, b_z), \boldsymbol{c} = (c_x, c_y, c_z),$$

and the Pauli operator vector $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$,we have

$$\begin{aligned} &\boldsymbol{a}\cdot\boldsymbol{\sigma}\otimes\boldsymbol{b}\cdot\boldsymbol{\sigma}\otimes\boldsymbol{c}\cdot\boldsymbol{\sigma} \\ &= (a_x\sigma_x + a_y\sigma_y + a_z\sigma_z)\otimes(b_x\sigma_x + b_y\sigma_y + b_z\sigma_z)\otimes(c_x\sigma_x + c_y\sigma_y + c_z\sigma_z) \\ &= \sum_{i,j,k} a_i b_j c_k \sigma_i\otimes\sigma_j\otimes\sigma_k, \end{aligned}$$

with $\langle\sigma_i\otimes\sigma_j\otimes\sigma_k\rangle_\rho = 0$ for all $i, j, k \in \{x, y, z\}$ except for the following four cases: $\langle\sigma_x\otimes\sigma_x\otimes\sigma_x\rangle_\rho = -1$, $\langle\sigma_x\otimes\sigma_y\otimes\sigma_y\rangle_\rho = -1$, $\langle\sigma_y\otimes\sigma_x\otimes\sigma_y\rangle_\rho = -1$, $\langle\sigma_y\otimes\sigma_y\otimes\sigma_x\rangle_\rho = 1$. Thus

$$\langle\boldsymbol{a}\cdot\boldsymbol{\sigma}\otimes\boldsymbol{b}\cdot\boldsymbol{\sigma}\otimes\boldsymbol{c}\cdot\boldsymbol{\sigma}\rangle_\rho = -(a_xb_x - a_yb_y)c_x - (a_xb_y + a_yb_x)c_y. \tag{3.12}$$

Especially, (3.12) holds for

$$\begin{aligned} \boldsymbol{a} &= (a_x, a_y, a_z) = (\cos\alpha, \sin\alpha, 0), \\ \boldsymbol{b} &= (b_x, b_y, b_z) = (\cos\beta, \sin\beta, 0), \\ \boldsymbol{c} &= (c_x, c_y, c_z) = (\cos\gamma, \sin\gamma, 0), \end{aligned}$$

in this case, we have

$$\langle\boldsymbol{a}\cdot\boldsymbol{\sigma}\otimes\mathbf{b}\cdot\boldsymbol{\sigma}\otimes\boldsymbol{c}\cdot\boldsymbol{\sigma}\rangle_\rho = -\cos(\alpha+\beta-\gamma). \tag{3.13}$$

Firstly, we show that ρ is not $\{1; 2; 3\}$-local. Suppose that this is not the case, then Theorem 3.3 yields that $|\langle S_3\rangle_\rho| \leq 1$, where

$$\begin{aligned} S_3 = \frac{1}{4}\Big(&A^0_{a_1}A^0_{a_2}A^1_{a_3} + A^0_{a_1}A^1_{a_2}A^1_{a_3} + A^1_{a_1}A^0_{a_2}A^1_{a_3} - A^1_{a_1}A^1_{a_2}A^1_{a_3} \\ &+A^1_{a_1}A^1_{a_2}A^0_{a_3} + A^1_{a_1}A^0_{a_2}A^0_{a_3} + A^0_{a_1}A^1_{a_2}A^0_{a_3} - A^0_{a_1}A^0_{a_2}A^0_{a_3}\Big), \end{aligned}$$

and $A_{a_s}^{x_s}(s = 1, 2, 3)$ are any ±1-valued observables of $\mathcal{H}_{A_s}$. But for the ±1-valued observables

$$\begin{aligned}
A_{a_1}^0 &= \boldsymbol{a}_1^0 \cdot \boldsymbol{\sigma}, \ \ \boldsymbol{a}_1^0 = (\cos\alpha_0, \sin\alpha_0, 0),\\
A_{a_1}^1 &= \boldsymbol{a}_1^1 \cdot \boldsymbol{\sigma}, \ \ \boldsymbol{a}_1^1 = (\cos\alpha_1, \sin\alpha_1, 0),\\
A_{a_2}^0 &= \boldsymbol{a}_2^0 \cdot \boldsymbol{\sigma}, \ \ \boldsymbol{a}_2^0 = (\cos\beta_0, \sin\beta_0, 0),\\
A_{a_2}^1 &= \boldsymbol{a}_2^1 \cdot \boldsymbol{\sigma}, \ \ \boldsymbol{a}_2^1 = (\cos\beta_1, \sin\beta_1, 0),\\
A_{a_3}^0 &= \boldsymbol{a}_3^0 \cdot \boldsymbol{\sigma}, \ \ \boldsymbol{a}_3^0 = (\cos\gamma_0, \sin\gamma_0, 0),\\
A_{a_3}^1 &= \boldsymbol{a}_3^1 \cdot \boldsymbol{\sigma}, \ \ \boldsymbol{a}_3^1 = (\cos\gamma_1, \sin\gamma_1, 0),
\end{aligned}$$

Equation (3.13) implies that

$$\begin{aligned}
\langle S_3\rangle_\rho = \frac{1}{4}[&-\cos(\alpha_0+\beta_0-\gamma_1)-\cos(\alpha_0+\beta_1-\gamma_1)\\
&-\cos(\alpha_1+\beta_0-\gamma_1)+\cos(\alpha_1+\beta_1-\gamma_1)\\
&-\cos(\alpha_1+\beta_1-\gamma_0)-\cos(\alpha_1+\beta_0-\gamma_0)\\
&-\cos(\alpha_0+\beta_1-\gamma_0)+\cos(\alpha_0+\beta_0-\gamma_0)].
\end{aligned}$$

Especially, we take

$$\alpha_0 = 0, \alpha_1 = -\frac{\pi}{2}, \beta_0 = \frac{3\pi}{4}, \beta_1 = \frac{\pi}{4}, \gamma_0 = 0, \gamma_1 = \frac{\pi}{2},$$

and get

$$\begin{aligned}
\cos(\alpha_0+\beta_0-\gamma_1) &= \frac{\sqrt{2}}{2}, \cos(\alpha_0+\beta_1-\gamma_1) = \frac{\sqrt{2}}{2},\\
\cos(\alpha_1+\beta_0-\gamma_1) &= \frac{\sqrt{2}}{2}, \cos(\alpha_1+\beta_1-\gamma_1) = -\frac{\sqrt{2}}{2},\\
\cos(\alpha_1+\beta_1-\gamma_0) &= \frac{\sqrt{2}}{2}, \cos(\alpha_1+\beta_0-\gamma_0) = \frac{\sqrt{2}}{2},\\
\cos(\alpha_0+\beta_1-\gamma_0) &= \frac{\sqrt{2}}{2}, \cos(\alpha_0+\beta_0-\gamma_0) = -\frac{\sqrt{2}}{2}.
\end{aligned}$$

Thus,

$$|\langle S_3\rangle_\rho| = \frac{1}{4}\times 8\times\frac{\sqrt{2}}{2} = \sqrt{2} > 1,$$

a contradiction. This shows that ρ is not {1; 2; 3}-local.

Similarly, by taking ±1-valued observables

$$A_{a_1}^0 = \sigma_x\otimes\frac{\sigma_x+\sigma_y}{\sqrt{2}},\ A_{a_1}^1 = -\sigma_y\otimes\frac{\sigma_x+\sigma_y}{\sqrt{2}},\ A_{a_2}^0 = \sigma_x,\ A_{a_2}^1 = \sigma_y$$

and using (3.12), we get

$$\begin{aligned}
\langle S_2\rangle_\rho &= \frac{1}{2}\Big\langle\sigma_x\otimes\frac{\sigma_x+\sigma_y}{\sqrt{2}}\otimes\sigma_x+\sigma_x\otimes\frac{\sigma_x+\sigma_y}{\sqrt{2}}\otimes\sigma_y\\
&\quad -\sigma_y\otimes\frac{\sigma_x+\sigma_y}{\sqrt{2}}\otimes\sigma_x+\sigma_y\otimes\frac{\sigma_x+\sigma_y}{\sqrt{2}}\otimes\sigma_y\Big\rangle_\rho\\
&= \frac{1}{2}\times(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}})\\
&= -\sqrt{2}.
\end{aligned}$$

This shows that $|\langle S_2\rangle_\rho| > 1$ and then ρ is not {1, 2; 3}-local (Remark 3.3).

Finally, by taking

$$A_{a_1}^0 = -\sigma_y,\ A_{a_1}^1 = \sigma_x,\ A_{a_2}^0 = \frac{-\sigma_x + \sigma_y}{\sqrt{2}} \otimes \sigma_y,\ A_{a_2}^1 = \frac{-\sigma_x + \sigma_y}{\sqrt{2}} \otimes \sigma_x$$

and using (3.12), we get

$$\begin{aligned}\langle S_2\rangle_\rho &= \frac{1}{2}\Big\langle -\sigma_y \otimes \frac{-\sigma_x + \sigma_y}{\sqrt{2}} \otimes \sigma_y - \sigma_y \otimes \frac{-\sigma_x + \sigma_y}{\sqrt{2}} \otimes \sigma_x \\ &\quad +\sigma_x \otimes \frac{-\sigma_x + \sigma_y}{\sqrt{2}} \otimes \sigma_y - \sigma_x \otimes \frac{-\sigma_x + \sigma_y}{\sqrt{2}} \otimes \sigma_x \Big\rangle_\rho \\ &= \frac{1}{2} \times (-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}) \\ &= -\sqrt{2}.\end{aligned}$$

This shows that $|\langle S_2\rangle_\rho| > 1$ and therefore ρ is not $\{1; 2, 3\}$-local (Remark 3.3). □

As a conclusion, ρ is Λ-nonlocal for every Λ. The proof is completed.

4 A Class of 2-separable Nonlocal States

According to Refs. [23, 24], a 2-partition $A_1|A_2$ of the index set $\{1, 2, \dots, n\}$ consists of a pairwise disjoint subsets

$$A_1 = \{m_1, m_2, \dots, m_j\},\ A_2 = \{m_{j+1}, \dots, m_n\},$$

such that $A_1 \cup A_2 = \{1, 2, \dots, n\}$. An n-partite pure state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_n$ was called *2-separable* in [23, 24] means that there is a 2-partition $A_1|A_2$, and two states $|\psi_1\rangle_{A_1}$ of $\mathcal{H}_{m_1} \otimes \mathcal{H}_{m_2} \otimes \dots \otimes \mathcal{H}_{m_j}$ and $|\psi_2\rangle_{A_2}$ of $\mathcal{H}_{m_{j+1}} \otimes \mathcal{H}_{m_{j+2}} \otimes \dots \otimes \mathcal{H}_{m_n}$ such that $|\psi\rangle = |\psi_1\rangle_{A_1} \otimes |\psi_2\rangle_{A_2}$. An n-partite mixed state ρ was called *2-separable* [23, 24] if it can be written as a convex combination of 2-separable pure states: $\rho = \sum_{m=1}^d p_m|\varphi_m\rangle\langle\varphi_m|$,where $|\varphi_m\rangle(m = 1, 2, \dots, d)$ are 2-separable, but may be with respect to different 2-partitions.

It is well-known that a separable bipartite state must be Bell-local. For multipartite case, k-separability [23, 24] does not imply locality, because that there exists a class of 2-separable states which are nonlocal, see Example 4.1.

Example 4.1 Consider the following pure states of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}|0\rangle_1 \otimes (|00\rangle + |11\rangle)_{23},$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{12} \otimes |0\rangle_3,$$

being 2-separable with respect to 2-partitions $\{1\}|\{2, 3\}$ and $\{1, 2\}|\{3\}$, respectively. Thus, the mixed state

$$\rho = p|\psi_1\rangle\langle\psi_1| + (1-p)|\psi_2\rangle\langle\psi_2|$$

is 2-separable for all $0 \le p \le 1$ according to the definition in [23, 24]. Next, we show that ρ is neither $\{1, 2; 3\}$-local when $\sqrt{2}-1 < p \le 1$ nor $\{1; 2, 3\}$-local when $0 \le p < 2-\sqrt{2}$. In other words, ρ has no any locality.

In fact, we compte that

$$\rho = \frac{p}{2}|000\rangle\langle 000| + \frac{p}{2}|000\rangle\langle 011| + \frac{p}{2}|011\rangle\langle 000| + \frac{p}{2}|011\rangle\langle 011| + \frac{1-p}{2}|000\rangle\langle 000| + \frac{1-p}{2}|000\rangle\langle 110| + \frac{1-p}{2}|110\rangle\langle 000| + \frac{1-p}{2}|110\rangle\langle 110|.$$

Let P, Q be as in (3.1). Then

$$\begin{aligned}\langle \sigma_z \otimes \sigma_z \otimes P\rangle_\rho &= \mathrm{tr}(\sigma_z \otimes \sigma_z \otimes P)\rho \\ &= \frac{p}{2}\mathrm{tr}(P|0\rangle\langle 0|) - \frac{p}{2}\mathrm{tr}(P|1\rangle\langle 1|) \\ &\quad + \frac{1-p}{2}\mathrm{tr}(P|0\rangle\langle 0|) \\ &\quad + \frac{1-p}{2}\mathrm{tr}(P|0\rangle\langle 0|) \\ &= \frac{2-p}{2}\langle 0|P|0\rangle - \frac{p}{2}\langle 1|P|1\rangle,\end{aligned}$$

$$\begin{aligned}\langle \sigma_z \otimes \sigma_x \otimes P\rangle_\rho &= \mathrm{tr}(\sigma_z \otimes \sigma_x \otimes P)\rho \\ &= \frac{p}{2}\mathrm{tr}(P|0\rangle\langle 1|) + \frac{p}{2}\mathrm{tr}(P|1\rangle\langle 0|) \\ &= \frac{p}{2}\langle 1|P|0\rangle + \frac{p}{2}\langle 0|P|1\rangle.\end{aligned}$$

Similarly,

$$\begin{aligned}\langle \sigma_z \otimes \sigma_z \otimes Q\rangle_\rho &= \frac{2-p}{2}\langle 0|Q|0\rangle - \frac{p}{2}\langle 1|Q|1\rangle, \\ \langle \sigma_z \otimes \sigma_x \otimes Q\rangle_\rho &= \frac{p}{2}\langle 1|Q|0\rangle + \frac{p}{2}\langle 0|Q_|1\rangle.\end{aligned}$$

So,

$$\begin{aligned}\Delta &:= |\langle \sigma_z \otimes \sigma_z \otimes P\rangle_\rho + \langle \sigma_z \otimes \sigma_z \otimes Q\rangle_\rho + \langle \sigma_z \otimes \sigma_x \otimes P\rangle_\rho - \langle \sigma_z \otimes \sigma_x \otimes Q\rangle_\rho| \\ &= \left|\frac{2-p}{2}\langle 0|P|0\rangle - \frac{p}{2}\langle 1|P|1\rangle + \frac{2-p}{2}\langle 0|Q|0\rangle - \frac{p}{2}\langle 1|Q|1\rangle \right. \\ &\quad \left. + \frac{p}{2}\langle 1|P|0\rangle + \frac{p}{2}\langle 0|P|1\rangle - \frac{p}{2}\langle 1|Q|0\rangle - \frac{p}{2}\langle 0|Q|1\rangle\right| \\ &= \left|\frac{2-p}{2}\langle 0|P+Q|0\rangle - \frac{p}{2}\langle 1|P+Q|1\rangle + \frac{p}{2}\langle 1|P-Q|0\rangle + \frac{p}{2}\langle 0|P-Q|1\rangle\right|.\end{aligned}$$

Especially, when

$$P = \frac{\sigma_z + \sigma_x}{\sqrt{2}}, \ Q = \frac{\sigma_z - \sigma_x}{\sqrt{2}},$$

have

$$P + Q = \sqrt{2}\sigma_z = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix}, \ P - Q = \sqrt{2}\sigma_x = \begin{pmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}.$$

Thus

$$\Delta = \frac{\sqrt{2}}{2}(2-p) + \frac{\sqrt{2}}{2}p + \frac{\sqrt{2}}{2}p + \frac{\sqrt{2}}{2}p = \sqrt{2} + \sqrt{2}p.$$

Clearly,

$$\Delta > 2 \Leftrightarrow p > \sqrt{2} - 1.$$

Therefore, we have if $1 \geq p > \sqrt{2} - 1$, then ρ is $\{1, 2; 3\}$-nonlocal.

Similarly, we can prove that ρ is $\{1; 2, 3\}$-nonlocal when $0 \leq p < 2 - \sqrt{2}$.

5 Conclusions

In this paper, we have introduced Λ-locality and Λ-nonlocality of n-partite states, discussed some related properties and established some related nonlocality inequalities for $\{1, 2; 3\}$-local, $\{1; 2, 3\}$-local, and Λ-local states, respectively. The violation of one of these inequalities exhibits Λ-nonlocality. As application, we have checked genuinely nonlocality of a tripartite state by a violation of nonlocality inequality. Finally, we have given a class of 2-separable nonlocal states, which shows that a 2-separable tripartite state is not necessarily local. When $n = 2, k = 2, A_1 = \{1\}, A_2 = \{2\}$, Λ-locality is equivalent to Bell-locality. This implies that Λ-locality generalizes the Bell-locality of bipartite states and Λ-nonlocality inequalities are generalization of the usual Bell-inequalities.

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