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Varieties of Orthocomplemented Lattices Induced by Łukasiewicz-Groupoid-Valued Mappings

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Abstract In the logico-algebraic approach to the foundation of quantum mechanics we sometimes identify the set of events of the quantum experiment with an orthomodular lattice ("quantum logic"). The states are then usually associated with (normalized) finitely additive measures ("states"). The conditions imposed on states then define classes of orthomodular lattices that are sometimes found to be universal-algebraic varieties. In this paper we adopt a conceptually different approach, we relax orthomodular to orthocomplemented and we replace the states with certain subadditive mappings that range in the Łukasiewicz groupoid. We then show that when we require a type of "fulness" of these mappings, we obtain varieties of orthocomplemented lattices. Some of these varieties contain the projection lattice in a Hilbert space so there is a link to quantum logic theories. Besides, on the purely algebraic side, we present a characterization of orthomodular lattices among the orthocomplemented ones. - The intention of our approach is twofold. First, we recover some of the Mayet varieties in a principally different way (indeed, we also obtain many other new varieties). Second, by introducing an interplay of the lattice, measure-theoretic and fuzzy-set notions we intend to add to the concepts of quantum axiomatics.

Keywords Quantum logic · Orthocomplemented (orthomodular) lattice · Łukasiewicz statoid · Boolean algebra · Variety of algebras

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1 Introduction and Preliminaries

The orthomodular lattices (OMLs) have been introduced in theoretical physics in the quest for "quantum logic" ([1, 5, 12, 19], etc.). The study of OMLs then intensely continued for a longer time as seen from the series of monographs ([8, 9, 11, 21], etc.). From the point of view of (universal) algebra, a particulary interesting line presented the study of the state conditions and related classes of OMLs. Let us recall the initial work by R. Mayet ([16, 17]), but there have been some other related attempts ([2, 6, 7, 18, 20], etc.). Applying another approach, we arrive to the notion of (Łukasiewicz) statoid on orthocomplemented lattices (OCLs). Then we obtain classes of OCLs (= classes of orthocomplemented lattices) that came into existence by imposing certain term conditions on the statoids. Our main result says that we always obtain a "nice" class of algebras - we obtain a variety (Theorem 4.4). We also show how some of the known varieties could be obtained in the framework of our approach.

Let us introduce basic notions as we shall use them in the paper. Suppose that $\mathcal{L}_0 = \{\wedge, \vee, ^\perp, 0, 1\}$, where $\wedge, \vee$ are binary operational symbols, $^\perp$ is a unary operational symbol and 0,1 are nullary operational symbols. Let us recall that by an *orthocomplemented lattice* we mean an $\mathcal{L}_0$-algebra L such that L is a $\{0, 1\}$-lattice such that the following conditions are fulfiled ($x, y \in X$): (1) $\inf\{x, x^\perp\} = 0$, $\sup\{x, x^\perp\} = 1$, (2)$(x^\perp)^\perp = x$, (3)$x \leq y \Rightarrow y^\perp \leq x^\perp$. If L is an OCL, we shall sometimes denote its least (resp. greatest) element by 0_L (resp. 1_L). We write $x \perp y$ provided $x \leq y^\perp$. If L satisfies the orthomodular law, $x \leq y \Rightarrow y = x \vee (y \wedge x^\perp)$ for any $x, y \in L$, then L is said to be an *orthomodular lattice* (see e.g. [11] and [21]).

Let us introduce our basic definition. For any $x, y \in [0, 1]$, let us set $x \oplus y = 1 - |1 - (x + y)|$. It means that $x \oplus y = x + y$ provided $x + y \leq 1$ and $x \oplus y = 2 - (x + y)$ provided $x + y > 1$. We see that $\oplus : [0, 1]^2 \to [0, 1]$. Viewing this operation with the fuzzy-set eye, we take the liberty to call the couple $([0, 1], \oplus)$ the *Łukasiewicz groupoid*.

If L is an OCL then the symbol $\mathcal{F}(L)$ denotes the set of all functions $L \to [0, 1]$.

Definition 1.1 Let L be an OCL and let $s \in \mathcal{F}(L)$. Then s is called a (*Łukasiewicz*) *statoid* on L if (s_1) $s(1_L) = 1$, and
(s_2) if $x, y \in L$ and $x \perp y$, then $s(x \vee y) \leq s(x) \oplus s(y)$.

A certain justification for the notion of statoid could be given by the circumstance that if we consider a Łukasiewicz state on L ([15]), then the condition (s_2) becomes an equality.

Let us denote by $\mathcal{S}(L)$ the set of all statoids on L. Before launching on the study of $\mathcal{S}(L)$, let us observe the following simple fact. Let L be an OCL with $\mathrm{card}(L) \geq 2$. Let us define a function $s : L \to [0, 1]$ such that $s(0_L) = 0$, $s(1_L) = 1$ and $s(x) = \frac{1}{2}$ otherwise. Then s is a statoid on L. Thus, $\mathcal{S}(L) \neq \emptyset$. The following facts will be used in the sequel.

Proposition 1.2 *Let L be an OCL, let s be a statoid on L and let* $x, y \in L$.

(1) *If* $x \perp y$ *and* $s(x \vee y) = 1$, *then* $s(x) + s(y) = 1$.
(2) $s(x) + s(x^\perp) = 1$.
(3) *If* $x \leq y$ *then* $s(x \vee (y \wedge x^\perp)) = s(y)$.
(4) *If* $x \perp y$ *and* $s(x) = 0$, *then* $s(x \vee y) = s(y)$.
(5) *If* $x \perp y$ *and both* $s(x)$ *and* $s(y)$ *belong to* $\{0, 1\}$, *then* $s(x \vee y) = s(x) \oplus s(y)$.

Proof (1) Let us suppose that $x \perp y$ and $s(x \vee y) = 1$. According to the condition (s_2) of the Definition 1.1, we have $1 = s(x \vee y) \leq s(x) \oplus s(y) \leq 1$. Hence $s(x) \oplus s(y) = 1$. But we also have $s(x) \oplus s(y) = 1 - |1 - (s(x) + s(y))|$. It implies that $|1 - (s(x) + s(y))| = 0$, and therefore $s(x) + s(y) = 1$.
(2) Using the equality $x \vee x^{\perp} = 1_L$, we obtain $s(x \vee x^{\perp}) = s(1_L) = 1$. Since $x \perp x^{\perp}$, the previous consideration gives us $s(x) + s(x^{\perp}) = 1$.
(3) Since $x \leq y$, we also have $x \vee (y \wedge x^{\perp}) \leq y$ and therefore $x \vee (y \wedge x^{\perp}) \perp y^{\perp}$. Moreover, $(x \vee (y \wedge x^{\perp})) \vee y^{\perp} = (x \vee y^{\perp}) \vee (y \wedge x^{\perp}) = (x^{\perp} \wedge y)^{\perp} \vee (y \wedge x^{\perp}) = 1_L$. According to the condition (1), we have $s(x \vee (y \wedge x^{\perp})) + s(y^{\perp}) = 1$. This means that $s(x \vee (y \wedge x^{\perp})) = 1 - s(y^{\perp}) = s(y)$.
(4) Let us suppose that $x \perp y$ and $s(x) = 0$. Since $y \leq x^{\perp}$, we see by the condition (3) that $s(y \vee (x^{\perp} \wedge y^{\perp})) = s(x^{\perp}) = 1 - s(x) = 1$. Further, as $y \perp x^{\perp} \wedge y^{\perp}$, the condition (1) implies that $s(y) + s(x^{\perp} \wedge y^{\perp}) = 1$. Thus, $s(x \vee y) = 1 - s((x \vee y)^{\perp}) = 1 - s(x^{\perp} \wedge y^{\perp}) = s(y)$.
(5) Let us suppose that $x \perp y$ and $s(x), s(y) \in \{0, 1\}$. If either of the values $s(x)$ and $s(y)$ is equal to 0, it is sufficient to apply the condition (4). Suppose therefore that $s(x) = s(y) = 1$. Then $s(x) \oplus s(y) = 0$. According to the condition (s_2) of Definition 1.1, we have $0 \leq s(x \vee y) \leq s(x) \oplus s(y) = 0$. Hence $s(x \vee y) = 0$. □

It is a fact of a certain separate interest that the OMLs can be characterized, among OCLs, in terms of statoids. Suppose that L is an OCL. Let us recall that $F \subseteq L$ is said to be a *filter* in L if the following three conditions are fulfiled: (1) $1_L \in F$, (2) if $x \in F$ and $x \leq y$, then $y \in F$, (3) if $x, y \in F$ then $x \wedge y \in F$. Moreover, a filter F is said to be a *proper filter* if $0_L \notin F$. The notion dual to a filter is called an *ideal*.

Proposition 1.3 *Let L be an OML and card$(L) \geq 2$. Let F be a proper filter in L. Then there is a statoid $s \in \mathcal{S}(L)$ such that $F = \{u \in L; s(u) = 1\}$ and $s(L) \subseteq \{0, \frac{1}{2}, 1\}$.*

Proof Let us define a mapping $s : L \to [0, 1]$ such that $s(z) = 1$ for each $z \in F$, $s(z) = 0$ for each $z \in F^{\perp}$ and $s(z) = \frac{1}{2}$ otherwise (standardly, $F^{\perp} = \{a^{\perp};\ a \in F\}$ and thus $F^{\perp}$ is an ideal). Since $0 \notin F$, it follows that $F \cap F^{\perp} = \emptyset$ and therefore the definition of the mapping s is correct. Obviously, the mapping s fulfils the condition (s_1) of Definition 1.2. We are going to verify the condition (s_2). Let us choose elements $x, y \in L$ with $x \perp y$. We have to prove that $s(x \vee y) \leq s(x) \oplus s(y)$. Let us first suppose that $x \in F$. Since $x \leq y^{\perp}$, we have $y^{\perp} \in F$ and therefore $y \in F^{\perp}$. As a result, $s(x) = 1$ and $s(y) = 0$. Further, since $x \vee y \in F$ we see that $s(x \vee y) = 1$. In other words, in this case we actually have the equality $s(x \vee y) = s(x) \oplus s(y)$. The case of $y \in F$ argues similarly. What we have to discuss are the following four possibilities.

(1) If $x, y \in L \setminus (F \cup F^{\perp})$, then $s(x) \oplus s(y) = \frac{1}{2} \oplus \frac{1}{2} = 1$ and the required inequality is obviously true.
(2) If $x \in L \setminus (F \cup F^{\perp})$ and $y \in F^{\perp}$, then $s(x) = \frac{1}{2}$ and $s(y) = 0$. Let us show that $x \vee y \in L \setminus (F \cup F^{\perp})$. Looking for a contradiction let us suppose that $x \vee y \in F$. Making use of the assumption $y \in F^{\perp}$, we infer that $y^{\perp} \in F$. Since F is a filter, we also have $y^{\perp} \wedge (x \vee y) \in F$. Further, since $y \leq x^{\perp}$ the orthomodular law gives us $x^{\perp} = y \vee (x^{\perp} \wedge y^{\perp})$ and therefore $x = y^{\perp} \wedge (x \vee y)$. But $y^{\perp} \wedge (x \vee y) \in F$. We have shown that $x \in F$, which is a contradiction. Thus, $x \vee y \notin F$. It remains to show that $x \vee y \notin F^{\perp}$. Arguing again by contradiction, suppose that $x \vee y \in F^{\perp}$. Since $F^{\perp}$ is an ideal, we see that $x \in F^{\perp}$ and this gives us a contradiction. In summary, we

have verified that $x \vee y \in L \setminus (F \cup F^{\perp})$. This means that $s(x \vee y) = \frac{1}{2}$ and we finally obtain the equality $s(x \vee y) = s(x) \oplus s(y)$.

(3) If $x \in F^{\perp}$ and $y \in L \setminus (F \cup F^{\perp})$, then we proceed in an analogy to the case (2).

(4) If $x, y \in F^{\perp}$, then $x \vee y \in F^{\perp}$ and therefore $s(x) = s(y) = s(x \vee y) = 0$. □

The property formulated in Proposition 1.3 in fact characterizes the OMLs among the OCLs.

Proposition 1.4 *Let L be an OCL such that for any proper filter F in L there is a statoid $s \in \mathcal{S}(L)$ with $F = \{u \in L; s(u) = 1\}$. Then L is an OML.*

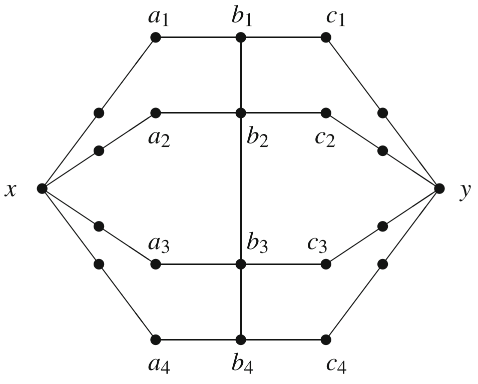
Proof Let $x, y \in L$ and $x \leq y$. We are to show that $y = x \vee (y \wedge x^{\perp})$. If $y = 0$ then there is nothing to prove. Suppose therefore that $y > 0$ and hence the set $[y, 1]$ is a proper filter in L. So there is a statoid $s \in \mathcal{S}(L)$ such that $[y, 1] = \{u \in L; s(u) = 1\}$. By Proposition 1.2 (3), we have $s(x \vee (y \wedge x^{\perp})) = s(y) = 1$. By our assumption, we obtain $x \vee (y \wedge x^{\perp}) \in [y, 1]$. This means that $y \leq x \vee (y \wedge x^{\perp})$. The reverse inequality is obvious. □

Definition 1.5 Let L be an OCL. Then L is said to be *statoid-rich* if for any $a, b \in L$ the following condition holds true: If $a \not\leq b$, then there exists a statoid s on L such that $s(a_1) = 1$ for any $a_1 \in [a, 1]$ and $s(b_1) = 0$ for any $b_1 \in [0, b]$.

Proposition 1.6 *Let L be a statoid-rich OCL. Then L is an OML.*

Proof Let us suppose that $x, y \in L$ and $x \leq y$. We are to show that $y \leq x \vee (y \wedge x^{\perp})$. Reasoning by contradiction, suppose that $y \not\leq x \vee (y \wedge x^{\perp})$. Then there exists an $s \in \mathcal{S}(L)$ such that $s([y, 1_L]) = \{1\}$ and $s([0_L, x \vee (y \wedge x^{\perp})]) = \{0\}$. In particular, $s(y) = 1$ and $s(x \vee (y \wedge x^{\perp})) = 0$. But this contradicts Proposition 1.2, (3). □

The following Greechie diagram [7] shows that there are OMLs that are not statoid-rich.



In order to see that this OML is not statoid-rich, let us consider the elements x and $y^{\perp}$. Let us suppose that there is a statoid s such that $s([x, 1]) = \{1\}$ and $s([0, y^{\perp}]) = \{0\}$. Choose an index $i \in \{1, 2, 3, 4\}$. Then $a_i^{\perp} \in [x, 1]$ and $c_i \in [0, y^{\perp}]$. The definition of s gives us that $s(a_i^{\perp}) = 1$ (and therefore $s(a_i) = 0$) and $s(c_i) = 0$. Since $a_i \perp c_i$, we can make use of Proposition 1.2 (5), to obtain $s(a_i \vee c_i) = 0$. But $a_i \vee c_i = b_i^{\perp}$. Thus,

$s(b_i) = 1 - s(b_i^\perp) = 1$ for any $i \in \{1, 2, 3, 4\}$. By Proposition 1.2 (5), we easily see that $s(b_1 \vee b_2 \vee b_3 \vee b_4) = s(b_1) \oplus s(b_2) \oplus s(b_3) \oplus s(b_4) = 1 \oplus 1 \oplus 1 \oplus 1 = 0$. But $b_1 \vee b_2 \vee b_3 \vee b_4 = 1$. So we have arrived to a contradiction in view of the requirement $s(1) = 1$.

In the next text we subject the definition of richness to additional conditions. We first express the conditions by certain logical formulas required for the statoids. Then we are concerned with the values of $s([0, b])$. We do not require that $s([0, b]) = \{0\}$ but we ask that $s([0, b]) \subseteq H$, where $H \subseteq [0, 1]$ is a preassigned set. We show that upon a suitable choice of these conditions we will obtain varieties of OCLs.

2 Q-Statoids

Through the paper, let us agree to adopt the following convention (universal-algebraic notions and results, though fairly standard, could be found in [3]).

Convention 2.1 (1) $\mathcal{L}$ is an arbitrary (but fixed) set of operational symbols with $\mathcal{L}_0 \subseteq \mathcal{L}$,
(2) $\mathcal{W}$ is an arbitrary (but fixed) variety of $\mathcal{L}$-algebras such that if $L \in \mathcal{W}$ and $L_{\mathcal{L}_0}$ is the restriction of the algebra L to the language $\mathcal{L}_0$, then $L_{\mathcal{L}_0}$ is an OCL,
(3) If $L \in \mathcal{W}$ then all notions related to OCLs will be adopted for L in the way that they will be considered in $L_{\mathcal{L}_0}$.

The following definitions prepare the stage for our basic definition to be introduced later. They may seem less intuitive and harder to digest. However, when a reader combines the reading of these definitions with Example 2.5, the matters become much clearer.

Definition 2.2 (1) By a *conjunction of atomic formulas* (abbr., by a *ca-formula*) we mean any formula $\phi(x_1, \dots, x_n)$ of the form $(p_1 = q_1)$ & ... & $(p_m = q_m)$, where $p_1, q_1, \dots, p_m, q_m$ are $\mathcal{L}$-terms with variables from the set $\{x_1, \dots, x_n\}$.
(2) If ϕ is a ca-formula of the previous form, $L \in \mathcal{W}$ and $a_1, \dots, a_n \in L$, we will write $L \models \phi[a_1, \dots, a_n]$ provided $p_1(a_1, \dots, a_n) = q_1(a_1, \dots, a_n), \dots, p_m(a_1, \dots, a_n) = q_m(a_1, \dots, a_n)$.
(3) Let $\Phi_\exists$ be the set of all ca-formulas $\phi(x_1, \dots, x_n)$ $(n \geq 1)$ with the following property: If $K, L \in \mathcal{W}$ and $f : K \to L$ is a surjective homomorphism, then for any $b_1, \dots, b_n \in L$ with $L \models \phi[b_1, \dots, b_n]$, there exist elements $a_1, \dots, a_n \in K$ such that $f(a_1) = b_1, \dots, f(a_n) = b_n$ and $K \models \phi[a_1, \dots, a_n]$.

Definition 2.3 By a *ca-condition* we mean any couple (ϕ, M), where $\phi(x_1, \dots, x_n)$ $(n \geq 1)$ is a ca-formula from the set $\Phi_\exists$ and M is a closed subset of $[0, 1]^n$. By a *condition* we mean any set of ca-conditions.

Definition 2.4 Suppose that $L \in \mathcal{W}$ and $s \in \mathcal{F}(L)$.

(1) Let $(\phi(x_1, \dots, x_n), M)$ be a ca-condition. Let us say that the mapping *s fulfils* (ϕ, M) if $(s(a_1), \dots, s(a_n)) \in M$ for any $a_1, \dots, a_n \in L$ such that $L \models \phi[a_1, \dots, a_n]$.
(2) Let Q be a condition. Let us say that the mapping *s fulfils* (*the condition*) Q, if s fulfils any ca-condition of Q.

Let us denote by $\mathcal{F}_Q(L)$ the set of all $s \in \mathcal{F}(L)$ that fulfil the condition Q. In the following example we will exhibit some very natural ca-conditions and their fulfilling by statoids.

Example 2.5 (1) Let us denote by $\phi_{(1)}(x)$ the ca-formula $x = 1$. If $L \in \mathcal{W}$ and $a \in L$ then we have $L \models \phi_{(1)}[a]$ provided $a = 1_L$. Obviously, $\phi_{(1)} \in \Phi_\exists$.
Let us set $M_{(1)} = \{1\}$. Then $M_{(1)}$ is a closed subset of $[0, 1]$. If $L \in \mathcal{W}$, $s \in \mathcal{F}(L)$, then s fulfils the ca-condition $(\phi_{(1)}, M_{(1)})$ exactly when $s(1_L) = 1$.
(2) Let us denote by $\phi^1_=(x)$ the ca-formula $x = x$. If $L \in \mathcal{W}$ and $a \in L$ then we automatically have $L \models \phi^1_=[a]$. Obviously, $\phi^1_= \in \Phi_\exists$.
Let us set $M_{0,1} = \{0, 1\}$. Then $M_{0,1}$ is a closed subset of $[0, 1]$. If $L \in \mathcal{W}$, $s \in \mathcal{F}(L)$, then s fulfils the ca-condition $(\phi^1_=, M_{0,1})$ exactly when $s(a) \in \{0, 1\}$ for any $a \in L$.
(3) Let us denote by $\phi_\leq(x, y)$ the ca-formula $x = x \wedge y$. If $L \in \mathcal{W}$ and $a, b \in L$ then we have $L \models \phi_\leq[a, b]$ provided $a \leq b$. We will show that $\phi_\leq \in \Phi_\exists$. Suppose that $K, L \in \mathcal{W}$, $f : K \to L$ is a surjective homomorphism and b_1, b_2 be elements of L with $b_1 \leq b_2$. Since f is surjective there are elements $c_1, c_2 \in K$ such that $f(c_1) = b_1$, $f(c_2) = b_2$. Let us set $a_1 = c_1, a_2 = c_1 \vee c_2$. Then $f(a_1) = f(c_1) = b_1$, $f(a_2) = f(c_1 \vee c_2) = f(c_1) \vee f(c_2) = b_1 \vee b_2 = b_2$. Moreover, $a_1 = c_1 \leq c_1 \vee c_2 = a_2$ and the verification is complete.
Let us set $M_\leq = \{(r_1, r_2) \in [0, 1]^2;\ r_1 \leq r_2\}$. Then $M_\leq$ is a closed subset of $[0, 1]^2$. If $L \in \mathcal{W}$, $s \in \mathcal{F}(L)$, then s fulfils the ca-condition $(\phi_\leq, M_\leq)$ exactly when $s(a) \leq s(b)$ for any elements $a, b \in L$ with $a \leq b$.
(4) Let us denote by $\phi_\perp(x, y, z)$ the ca-formula $x = x \wedge y^\perp \ \&\ z = x \vee y$. If $L \in \mathcal{W}$ and $a, b, c \in L$ then we have $L \models \phi_\perp[a, b, c]$ provided $a \perp b$ and $c = a \vee b$. In the analogy to the previous case (3) we have $\phi_\perp \in \Phi_\exists$.
(4a) Let us set $M_+ = \{(r_1, r_2, r_3) \in [0, 1]^3;\ r_3 = r_1 + r_2\}$. Then M_+ is a closed subset of $[0, 1]^3$. If $L \in \mathcal{W}$, $s \in \mathcal{F}(L)$, then s fulfils the ca-condition $(\phi_\perp, M_+)$ exactly when $s(a \vee b) = s(a) + s(b)$ for any elements $a, b \in L$ with $a \perp b$.
(4b) Let us set $M_\oplus = \{(r_1, r_2, r_3) \in [0, 1]^3;\ r_3 = r_1 \oplus r_2\}$. Then $M_\oplus$ is a closed subset of $[0, 1]^3$. If $L \in \mathcal{W}$, $s \in \mathcal{F}(L)$, then s fulfils the ca-condition $(\phi_\perp, M_\oplus)$ exactly when $s(a \vee b) = s(a) \oplus s(b)$ for any elements $a, b \in L$ with $a \perp b$.
(4c) Let us set $M^\leq_\oplus = \{(r_1, r_2, r_3) \in [0, 1]^3;\ r_3 \leq r_1 \oplus r_2\}$. Then $M^\leq_\oplus$ is a closed subset of $[0, 1]^3$. If $L \in \mathcal{W}$, $s \in \mathcal{F}(L)$, then s fulfils the ca-condition $(\phi_\perp, M^\leq_\oplus)$ exactly when $s(a \vee b) \leq s(a) \oplus s(b)$ for any elements $a, b \in L$ with $a \perp b$.
(5) Let us denote by $\phi_\vee(x, y, z)$ the ca-formula $z = x \vee y$. If $L \in \mathcal{W}$ and $a, b, c \in L$ then we have $L \models \phi_\vee[a, b, c]$ provided $c = a \vee b$. Obviously, $\phi_\vee \in \Phi_\exists$.
Let us set $M^\leq_+ = \{(r_1, r_2, r_3) \in [0, 1]^3;\ r_3 \leq r_1 + r_2\}$. Then $M^\leq_+$ is a closed subset of $[0, 1]^3$. If $L \in \mathcal{W}$ and $s \in \mathcal{F}(L)$, then s fulfils the ca-condition $(\phi_\vee, M^\leq_+)$ exactly when $s(a \vee b) \leq s(a) + s(b)$ for any elements $a, b \in L$.

The following two observations shed light on the interplay of our notions.

Proposition 2.6 *There exists a condition Q_0 such that $\mathcal{S}(L) = \mathcal{F}_{Q_0}(L)$ for any $L \in \mathcal{W}$.*

Proof It is sufficient to set $Q_0 = \{(\phi_{(1)}, M_{(1)}), (\phi_\perp, M^\leq_\oplus)\}$. □

Remark 2.7 If we take for Q the set $\{(\phi_{(1)}, M_{(1)}), (\phi_\perp, M_\oplus)\}$ we arrive to the Łukasiewicz states as investigated in [15]. If we take for Q the set $\{(\phi_{(1)}, M_{(1)}), (\phi_\perp, M_+)\}$, we arrive to the (standard) states.

In conclusion of this section, let us show how the fulfilling of the conditions of Definition 2.4 transfers on the mappings that are induced by natural algebraic constructions.

Proposition 2.8 *Let $K, L \in \mathcal{W}$ and let Q be a condition.*

(1) *Suppose that K is a sub-algebra of L and $s \in \mathcal{F}_Q(L)$ Then $s_1 \in \mathcal{F}_Q(K)$, where s_1 denotes the restriction of s to the algebra K.*
(2) *Suppose that $f : K \to L$ is a surjective homomorphism. Suppose further that $s_1 \in \mathcal{F}(K)$ and $s_2 \in \mathcal{F}(L)$. If $s_1 = f \circ s_2$, then s_1 fulfils Q if and only if s_2 fulfils Q.*

Proof We may suppose that Q contains only one ca-condition, say $(\phi(x_1, \ldots, x_n), M)$.

(1) Let us take $a_1, \ldots, a_n \in K$ with $K \models \phi[a_1, \ldots, a_n]$. Since K is a sub-algebra of L, we have $a_1, \ldots, a_n \in L$ and $L \models \phi[a_1, \ldots, a_n]$. Since s fulfils (ϕ, M), we infer that $(s(a_1), \ldots, s(a_n)) \in M$. Further, the mapping s_1 is the restriction of s to K, we obtain that $s_1(a_1) = s(a_1), \ldots, s_1(a_n) = s(a_1)$. Hence $(s_1(a_1), \ldots, s_1(a_n)) \in M$.
(2) Firstly, suppose that s_2 fulfils (ϕ, M). Suppose that $a_1, \ldots, a_n \in K$ with $K \models \phi[a_1, \ldots, a_n]$. As f is a homomorphism, we see that $L \models \phi[f(a_1), \ldots, f(a_n)]$. Since s_2 fulfils (ϕ, M), we obtain $(s_2(f(a_1)), \ldots, s_2(f(a_n))) \in M$. But $(s_2(f(a_1)), \ldots, s_2(f(a_n))) = (s_1(a_1), \ldots, s_1(a_n))$ and thus we have verified the first implication. Secondly, let s_1 fulfils (ϕ, M). Suppose that $b_1, \ldots, b_n \in L$ with $L \models \phi[b_1, \ldots, b_n]$. The definition of a ca-condition implies that there are elements $a_1, \ldots, a_n \in K$ such that $f(a_1) = b_1, \ldots, f(a_n) = b_n$ and $K \models \phi[a_1, \ldots, a_n]$. Since s_1 fulfils (ϕ, M), we have $(s_1(a_1), \ldots, s_1(a_n)) \in M$. As a result, $(s_2(f(a_1)), \ldots, s_2(f(a_n))) \in M$ and this means that $(s_2(b_1), \ldots, s_2(b_n)) \in M$. □

In case we restrict ourselves to statoids, we can even formulate a statement on the existence of statoids on the homomorphic images that fulfil the condition Q.

Definition 2.9 Let $L \in \mathcal{W}$, $s \in \mathcal{S}(L)$ and let Q be a condition. Then s is said to be a *Q-statoid* if the mapping s fulfils the condition Q.

Let us denote by $\mathcal{S}_Q(L)$ the set of all Q-statoids on L (thus, $\mathcal{S}_Q(L) = \mathcal{S}(L) \cap \mathcal{F}_Q(L)$).

Proposition 2.10 *Let $K, L \in \mathcal{W}$ and let $f : K \to L$ be a surjective homomorphism. Let $s \in \mathcal{S}_Q(K)$. Then there exists $\widetilde{s} \in \mathcal{S}_Q(L)$ such that $s = f \circ \widetilde{s}$ if and only if $s(x) = 1$ for any $x \in f^{-1}(1_L)$.*

Proof Let us suppose that $s = f \circ \widetilde{s}$ for some Q-statoid $\widetilde{s} \in \mathcal{S}_Q(L)$. Let us choose $x \in f^{-1}(1_L)$. Then $s(x) = (f \circ \widetilde{s})(x) = \widetilde{s}(f(x)) = \widetilde{s}(1_L) = 1$.

Conversely, let $s(x) = 1$ for any $x \in f^{-1}(1_L)$. If $y \in L$ with $y = f(x)$, then we can set $\widetilde{s}(y) = s(x)$. Let us show that this definition of $\widetilde{s}$ is correct. To this end, suppose that $y = f(x_1) = f(x_2)$. We have $f(x_1^\perp \vee (x_1 \wedge x_2)) = f(x_1^\perp) \vee (f(x_1) \wedge f(x_2)) = y^\perp \vee (y \wedge y) = 1_L$. Hence $x_1^\perp \vee (x_1 \wedge x_2) \in f^{-1}(1_L)$. By our assumption, this implies that $s(x_1^\perp \vee (x_1 \wedge x_2)) = 1$. Since, $x_1^\perp \perp (x_1 \wedge x_2)$, Proposition 1.3 implies that $s(x_1^\perp) + s(x_1 \wedge x_2) = 1$. This gives us $s(x_1 \wedge x_2) = 1 - s(x_1^\perp) = 1 - (1 - s(x_1)) = s(x_1)$. Analogously, $s(x_2) = s(x_1 \wedge x_2)$. This means that $s(x_1) = s(x_2)$. We have verified the correctness of $\widetilde{s}$. The definition of $\widetilde{s}$ gives us that $s = f \circ \widetilde{s}$. Finally, Proposition 2.8 implies that the mapping $\widetilde{s}$ is a Q-statoid on L. □

3 Topological Considerations and Stone's Lemma

In this section we will prove a generalized Stone's lemma to be applied in our main result. Let $L \in \mathcal{W}$. Let us consider the standard topology on the set $[0, 1]$ and let us consider the topological product $[0, 1]^L$. According to the Tychonoff theorem, $[0, 1]^L$ is a compact topological space. For any $a_1, \ldots, a_k \in L$ ($k \geq 1$), let us denote by $\pi_{a_1,\ldots,a_k}$ the projection mapping $\mathcal{F}(L) \to [0, 1]^k$ defined as follows: For any $s \in \mathcal{F}(L)$, $\pi_{a_1,\ldots,a_k}(s) = (s(a_1), \ldots, s(a_k))$. Obviously, the mappings $\pi_{a_1,\ldots,a_k}$ are continuous.

Proposition 3.1 *Let Q be a condition. Then $\mathcal{F}_Q(L)$ is a closed subset in* $[0, 1]^L$.

Proof If $Q = \emptyset$, then $\mathcal{F}_Q(L) = [0, 1]^L$ and the statement is obvious. Let us suppose that $Q \neq \emptyset$. Then $\mathcal{F}_Q(L) = \bigcap_{(\phi,M)\in Q} \mathcal{F}_{\{(\phi,M)\}}(L)$. It is sufficient to show the closedness of the set $\mathcal{F}_{\{(\phi,M)\}}(L)$ for a ca-condition (ϕ, M). Let $\phi = \phi(x_1, \ldots, x_n)$ and let $M \subseteq [0, 1]^n$ be closed. Write $L_\phi = \{(a_1, \ldots, a_n) \in L^n;\ L \models \phi[a_1, \ldots, a_n]\}$. Then $\mathcal{F}_{\{(\phi,M)\}}(L) = \{s \in \mathcal{F}(L);\ s \text{ fulfils } (\phi, M)\} = \bigcap_{(a_1,\ldots,a_n)\in L_\phi} \{s \in \mathcal{F}(L);\ (s(a_1), \ldots, s(a_n)) \in M\} = \bigcap_{(a_1,\ldots,a_n)\in L_\phi} \pi^{-1}_{a_1,\ldots,a_n}(M)$.

The continuity of $\pi_{a_1,\ldots,a_n}$ and the closedness of M imply that the sets $\pi^{-1}_{a_1,\ldots,a_n}(M)$ are closed. Hence, $\mathcal{F}_{\{(\phi,M)\}}(L)$ is also closed in $[0, 1]^L$ as an intersection of closed sets. □

Proposition 3.2 *Let Q be a condition. Then $\mathcal{S}_Q(L)$ is a closed subset in* $[0, 1]^L$.

Proof We have $\mathcal{S}_Q(L) = \mathcal{F}_{Q'}(L)$, where $Q' = Q \cup Q_0$ and Q_0 is the condition of Proposition 2.6. The set $\mathcal{F}_{Q'}(L)$ is closed in view of Proposition 3.1. □

In connection with the further considerations, let us introduce the following notation.

Notation 3.3 Let $a, b \in L$, $H \subseteq [0, 1]$ and Q be a condition. Let us write $\mathcal{S}_Q(L; H, a, b) = \{s \in \mathcal{S}_Q(L);\ s(a_1) = 1$ for any $a_1 \in [a, 1]$ and $s(b_1) \in H$ for any $b_1 \in [0, b]\}$.

Proposition 3.4 *Suppose that $a_1, a_2, b_1, b_2 \in L$ with $a_1 \leq a_2, b_2 \leq b_1$. Then $\mathcal{S}_Q(L; H, a_1, b_1) \subseteq \mathcal{S}_Q(L; H, a_2, b_2)$.*

Proof Obvious. □

Proposition 3.5 *Suppose that $a, b \in L$. Suppose further that $H \subseteq [0, 1]$ is a closed set and Q is a condition. Then $\mathcal{S}_Q(L; H, a, b)$ is a closed subset of the space* $[0, 1]^L$.

Proof It is immediately seen that $\mathcal{S}_Q(L; H, a, b) = \mathcal{S}_Q(L) \cap \bigcap_{x\in[a,1]} \pi_x^{-1}(\{1\}) \cap \bigcap_{x\in[0,b]} \pi_x^{-1}(H)$. Since π_x is continuous and $\{1\}$ and H are closed, we infer that $\pi_x^{-1}(\{1\})$

and $\pi_x^{-1}(H)$ are closed in $[0,1]^L$. By Proposition 3.2, the set $\mathcal{S}_Q(L)$ is also closed in $[0,1]^L$ and therefore so is the set $\mathcal{S}_Q(L; H, a, b)$. □

Prior to formulating a main result of this section, let us recall that the next theorem can be viewed as a generalization of the classical Stone lemma from the theory of Boolean algebras on the distinguishing disjoint filters and ideals by means of two-valued states.

Theorem 3.6 *(a generalized Stone's lemma) Suppose that $L \in \mathcal{W}$, Q is a condition and $H \subseteq [0,1]$ is a closed set. Suppose that F is a filter and I is an ideal in L. Suppose that the following condition holds true: If $a \in F$ and $b \in I$ then $\mathcal{S}_Q(L; H, a, b) \neq \emptyset$. Then there exists an $s \in \mathcal{S}_Q(L)$ such that $s(a) = 1$ for any $a \in F$ and $s(b) \in H$ for any $b \in I$.*

Proof Let us first show that the system $\{\mathcal{S}_Q(L; H, a, b)\}_{a\in F, b\in I}$ is a centered system of sets (meaning that its each finite subsystem has a non-empty intersection). Indeed, consider the sets $\mathcal{S}_Q(L; H, a_1, b_1), \dots, \mathcal{S}_Q(L; H, a_m, b_m)$, where $m \geq 1$, $a_1, \dots, a_m \in F$ and $b_1, \dots, b_m \in I$. Write $a = a_1 \wedge \dots \wedge a_m$, $b = b_1 \vee \dots \vee b_m$. Since F is a filter, we have $a \in F$. Analogously, $b \in I$. Thus, $\mathcal{S}_Q(L; H, a, b) \neq \emptyset$. Let us choose $s \in \mathcal{S}_Q(L; H, a, b)$. Because $a \leq a_k$ and $b_k \leq b$ for each $k \in \{1, \dots, m\}$, Proposition 3.4 implies that $s \in \bigcap_{k=1}^{m} \mathcal{S}_Q(L; H, a_k, b_k)$. Further, Proposition 3.5 establishes that $\{\mathcal{S}_Q(L; H, a, b)\}_{a\in F, b\in I}$ is a centered system of closed subsets of $[0,1]^L$. The compactness of $[0,1]^L$ yields that $\bigcap_{a\in F, b\in I} \mathcal{S}_Q(L; H, a, b) \neq \emptyset$. Finally, each statoid s, $s \in \bigcap_{a\in F, b\in I} \mathcal{S}_Q(L; H, a, b)$, is a statoid we looked for. □

4 Full Algebras

The following definition is a kind of 'fullness' dealt with in the quantum logic theory ([8]).

Definition 4.1 Let Q be a condition and let $H \subseteq [0,1]$. Let us set $\mathcal{X} = (Q, H)$. Then an algebra $L \in \mathcal{W}$ is called *$\mathcal{X}$-full* if for any $a, b \in L$ the following condition holds true: If $a \not\leq b$, then $\mathcal{S}_Q(L; H, a, b) \neq \emptyset$.

Let us denote by $\mathcal{W}_{\mathcal{X}}$ the class of all $\mathcal{X}$-full algebras from the variety $\mathcal{W}$.

Remark 4.2 (a) Let us suppose that we set $H = [0,1]$ in Definition 4.1. Then an algebra $L \in \mathcal{W}$ is $\mathcal{X}$-full exactly when for any $a \in L$, $a \neq 0$ there exists a Q-statoid s on L such that $s(a_1) = 1$ for any $a_1 \in [a, 1]$. An algebra L with this property will be called *Q-unital*.
(b) Let us suppose that we set $H = \{0\}$ in Definition 4.1. Then an algebra $L \in \mathcal{W}$ is $\mathcal{X}$-full exactly when for any $a, b \in L$ the following condition holds true: If $a \not\leq b$, then there exists a Q-statoid s on L such that $s(a_1) = 1$ for any $a_1 \in [a, 1]$ and $s(b_1) = 0$ for any $b_1 \in [0, b]$. An algebra L with this property will be called *Q-rich*.

In this section we shall prove - under assumptions on closedness of H - that the class $\mathcal{W}_{\mathcal{X}}$ is a subvariety of $\mathcal{W}$. We shall need the following auxiliary result.

Proposition 4.3 *Suppose that $K, L \in \mathcal{W}$ and $f : K \to L$ is a surjective homomorphism. Suppose that $a, b \in K$. Suppose further that $H \subseteq [0,1]$ and Q is a condition. Finally, suppose that $s_1 \in \mathcal{F}(K)$, $s_2 \in \mathcal{F}(L)$ and $s_1 = f \circ s_2$. Then $s_1 \in \mathcal{S}_Q(K; H, a, b)$ exactly when $s_2 \in \mathcal{S}_Q(L; H, f(a), f(b))$.*

Proof Assume first $s_1 \in \mathcal{S}_Q(K; H, a, b)$. Choose elements $c, d \in L$ with $c \in [f(a), 1]_L$ and $d \in [0, f(b)]_L$. Since f is surjective, there are elements $c_1, d_1 \in K$ such that $f(c_1) = c$ and $f(d_1) = d$. Let us set $a_1 = a \vee c_1, b_1 = b \wedge d_1$. Since $c \geq f(a)$, we obtain $f(a_1) = f(a \vee c_1) = f(a) \vee f(c_1) = f(a) \vee c = c$. Analogously, $f(b_1) = f(b \wedge d_1) = f(b) \wedge f(d_1) = f(b) \wedge d = d$. So we have $s_2(c) = s_2(f(a_1)) = s_1(a_1) = 1$ and $s_2(d) = s_2(f(b_1)) = s_1(b_1) \in H$, in view of $s_1 \in \mathcal{S}_Q(K; H, a, b)$ and $a \leq a_1, b_1 \leq b$. Conversely, suppose that $s_2 \in \mathcal{S}_Q(L; H, f(a), f(b))$. Choose elements $a_1 \in [a, 1]_K$ and $b_1 \in [0, b]_K$. Since $s_2 \in \mathcal{S}_Q(L; H, f(a), f(b))$ and $f(a) \leq f(a_1), f(b_1) \leq f(b)$, we see that $s_1(a_1) = s_2(f(a_1)) = 1$ and $s_1(b_1) = s_2(f(b_1)) \in H$. □

Theorem 4.4 *Suppose that $\mathcal{X} = (Q, H)$, where Q is a condition and $H \subseteq [0, 1]$ is a closed set. Then the class $\mathcal{W}_\mathcal{X}$ forms a variety.*

Proof We shall show that the class $\mathcal{W}_\mathcal{X}$ is closed under subalgebras, products and homomorphic images (Birkhoff's theorem, [3]).

(a) Suppose that $L \in \mathcal{W}_\mathcal{X}$ and K is a subalgebra of L. Then $K \in \mathcal{W}$. Suppose $a, b \in K$ and $a \not\leq_K b$. Since K is a subalgebra of L, we have the inequality $a \not\leq_L b$. Since L is $\mathcal{X}$-full, there exists an $s \in \mathcal{S}_Q(L)$ such that $s(a_1) = 1$ and $s(b_1) \in H$ for any $a_1 \in [a, 1]_L, b_1 \in [0, b]_L$. It suffices to observe that $[a, 1]_K \subseteq [a, 1]_L$, $[0, b]_K \subseteq [0, b]_L$ and the restriction of s to K is a Q-statoid on K.

(b) Suppose that $L_i \in \mathcal{W}_\mathcal{X}, i \in J$. Let us denote by L the product $\prod_{i \in J} L_i$. For any $i \in J$, let us denote by σ_i the i-th projection $L \to L_i$ (i.e., $\sigma_i(\mathbf{x}) = \mathbf{x}(i)$ for any $\mathbf{x} \in L$). Suppose that $\mathbf{a}, \mathbf{b} \in L$ and $\mathbf{a} \not\leq_L \mathbf{b}$. Then there exists an index $j \in J$ such that $\mathbf{a}(j) \not\leq_{L_j} \mathbf{b}(j)$ Write $a = \mathbf{a}(j), b = \mathbf{b}(j)$. Then $a, b \in L_j$ and $a \not\leq_{L_j} b$. Since L_j is $\mathcal{X}$-full, there exists $s \in \mathcal{S}_Q(L_j; H, a, b)$. Consider the Q-statoid $\sigma_j \circ s$ on L. By Proposition 4.3, we see that $\sigma_j \circ s \in \mathcal{S}_Q(L; H, \mathbf{a}, \mathbf{b})$.

(c) Finally, suppose that $K \in \mathcal{W}_\mathcal{X}$ and $f : K \to L$ is a surjective homomorphism. Suppose that $a, b \in L$ and $a \not\leq_L b$. Choose arbitrary elements $a_1, b_1 \in K$ such that $f(a_1) = a, f(b_1) = b$. Write $F = f^{-1}([a, 1_L]) = \{x \in K;\ a \leq f(x)\}$ and $I = [0_K, b_1]$. It is easily seen that F is a filter and I is an ideal in K. We are going to show the assumption of Theorem 3.6. Choose elements $c \in F, d \in I$. We have $a \leq_L f(c)$ and $f(d) \leq_L f(b_1) = b$. Since $a \not\leq_L b$, we see that $f(c) \not\leq_L f(d)$ and therefore $c \not\leq_K d$. Further, since K is $\mathcal{X}$-full, we have $\mathcal{S}_Q(K; H, c, d) \neq \emptyset$. This means that the assumption of Theorem 3.6 is fulfiled. As a result, there exists an $s \in \mathcal{S}_Q(L)$ such that $s(x) = 1$ and $s(y) \in H$ for any $x \in F, y \in I$. Going on with the proof, if $x \in f^{-1}(1_L)$ then $x \in F$ and therefore $s(x) = 1$. Making use of Proposition 2.10, there exists $\widetilde{s} \in \mathcal{S}_Q(L)$ such that $s = f \circ \widetilde{s}$. Finally, since $s \in \mathcal{S}_Q(K; H, a_1, b_1)$, we can apply Proposition 4.3 to obtain $\widetilde{s} \in \mathcal{S}_Q(L; H, f(a_1), f(b_1)) = \mathcal{S}_Q(L; H, a, b)$. We conclude that $\mathcal{S}_Q(L; H, a, b) \neq \emptyset$. This completes the proof.

□

Our result of Theorem 4.4 can be further generalized. For an algebraically oriented reader, let us include here this generalized formulation.

Theorem 4.5 *Let Q be a condition, let $p(x_1, \ldots, x_m), q(x_1, \ldots, x_m)$ be $\mathcal{L}$-terms ($m \geq 2$) and let $H \subseteq [0, 1]^{m-1}$ be a closed set. Let us set $\mathcal{X} = (Q, p, q, H)$. Let us denote by $\mathcal{W}_\mathcal{X}$ the class of all algebras $L \in \mathcal{W}$ such that for any $a_1, \ldots, a_m \in L$ the following condition* *holds true: If* $p(a_1, \ldots, a_m) \neq q(a_1, \ldots, a_m)$*, then there exists an* $s \in \mathcal{S}_Q(L)$ *such that* $(s(b_1), \ldots, s(b_m)) \in \{1\} \times H$ *for any* $b_1 \in [a_1, 1], \ldots, b_m \in [a_m, 1]$*. Then the class* $\mathcal{W}_\mathcal{X}$ *forms a variety.*

5 Applications

In this section we show (mostly without proofs) how some known varieties can be alternatively obtained from Theorem 4.4. Several new varieties appear in this way as well, some natural and some slightly artifactual. However, these varieties may seem less relevant to quantum theories. We intend to pursue them elsewhere. Our first application concerns Boolean algebras (BAs), the second and third application some Mayet's varieties. We shall make use of the notation of Example 2.5.

1. BAs: In this case one takes for $\mathcal{W}$ the class of all OMLs, $Q = \{(\phi_{(1)}, M_{(1)}), (\phi_\perp, M_+), (\phi_\vee, M_+^\leq)\}$ and $H = [0, 1]$ (the result of [22] is crucial therein).
2. Unital OMLs: In this case one takes for $\mathcal{W}$ the class of all OMLs, $Q = \{(\phi_{(1)}, M_{(1)}), (\phi_\perp, M_+)\}$ (see Remark 2.7) and $H = [0, 1]$ (see Remark 4.2, (a)).
3. Set-representable OMLs: In this case one takes for $\mathcal{W}$ the class of all OMLs, $Q = \{(\phi_{(1)}, M_{(1)}), (\phi_\perp, M_+), (\phi_=, M_{0,1})\}$ and $H = \{0\}$.
4. Set-representable ODLs (ODLs - orthocomplemented lattices with a symmetric difference): The class ODLs has been investigated in [4, 10, 13] and [14]. We would like to show, as an application of Theorem 4.4 again, that the set-representable ODLs form a variety. Since here we find ourselves in a less exploited terrain (and here the language properly extends $\mathcal{L}_0$), let us indicate the proof. Let us start off with the definition.

Definition 5.1 Let $L = (X, \wedge, \vee, ^\perp, 0, 1, \triangle)$, where $(X, \wedge, \vee, ^\perp, 0, 1)$ is an orthomodular lattice and $\triangle : X^2 \to X$ is a binary operation. Then L is said to be an *orthocomplemented difference lattice* (abbr., an ODL) if the following formulas hold in L:
(D$_1$)$x \triangle (y \triangle z) = (x \triangle y) \triangle z$,
(D$_2$)$x \triangle 1 = x^\perp, 1 \triangle x = x^\perp$,
(D$_3$)$x \triangle y \leq x \vee y$.

We shall employ the following lemma.

Lemma 5.2 *Let* L *be an ODL. Let* $x, y \in L$ *and* $x \perp y$*. Then* $x \triangle y = x \vee y$*.*

Proof The inequality $x^\perp \triangle y \leq x^\perp \vee y$ gives us that $x \wedge y^\perp \leq (x^\perp \triangle y)^\perp$. Since $x \perp y$, we have $x \leq y^\perp$ and therefore $x \wedge y^\perp = x$. Conversaly, $(x^\perp \triangle y)^\perp = ((1 \triangle x) \triangle y)^\perp = (1 \triangle (x \triangle y))^\perp = (x \triangle y)^{\perp\perp} = x \triangle y$. So, $x \leq x \triangle y$. Analogously, $y \leq x \triangle y$. It implies that $x \vee y \leq x \triangle y$. According to (D$_3$) we obtain $x \vee y = x \triangle y$. □

In a certain analogy with BAs we introduce the notion of an evaluation in an ODL.

Definition 5.3 Let L be an ODL and let $e : L \to \{0, 1\}$. Then e is said to be an *ODL-evaluation* on L if the following three properties are fulfilled ($x, y \in L$):
(E$_1$)$e(1_L) = 1$, (E$_2$) if $x \leq y$, then $e(x) \leq e(y)$, (E$_3$)$e(x \triangle y) = e(x) \oplus e(y)$.

Let $\mathcal{E}(L)$ be the set of all ODL-evaluations on L. In contrast to Boolean algebras, an ODL generaly fails to be set-representable ([13]). However, the following result is in force.

Lemma 5.4 *There exists a condition Q such that $\mathcal{E}(L) = \mathcal{S}_Q(L)$ for any ODL L.*

Proof Let L be an ODL. Then $\mathcal{E}(L) \subseteq \mathcal{S}(L)$. Indeed, suppose that $e \in \mathcal{E}(L)$. We have to show that e enjoys the property (s_2) of Definition 1.1. Suppose that $a, b \in L$ and $a \perp b$. By Lemma 5.2 we have $a \triangle b = a \vee b$. Hence, $e(a \vee b) = s(a \triangle b) = e(x) \oplus e(y)$. Thus, $e \in \mathcal{S}(L)$.

Let us denote by $\phi_\triangle(x, y, z)$ the ca-formula $z = x \triangle y$. Let us consider the condition $Q = \{(\phi_=, M_{0,1}), (\phi_{(1)}, M_{(1)}), (\phi_\leq, M_\leq), (\phi_\triangle, M_\oplus)\}$. Let us show that $\mathcal{E}(L) = \mathcal{F}_Q(L)$. We have checked that $\mathcal{E}(L) \subseteq \mathcal{S}(L)$. Therefore $\mathcal{E}(L) = \mathcal{S}(L) \cap \mathcal{E}(L) = \mathcal{S}(L) \cap \mathcal{F}_Q(L) = \mathcal{S}_Q(L)$. □

The following 'Boolean-like' result has been proved in [13].

Proposition 5.5 *Let L be an ODL. Then L is set-representable if and only if for any couple $a, b \in L$ with $a \not\leq b$ there exists an $e \in \mathcal{E}(L)$ such that $e(a) = 1$ and $e(b) = 0$.*

Theorem 5.6 *The class of all set-representable ODLs forms a variety.*

Proof Let $\mathcal{W}$ be the class of all ODLs. It is easily seen that the class of all set-representable ODLs equals to $\mathcal{W}_\mathcal{X}$, where $\mathcal{X} = (Q, \{0\})$, with Q taken from Lemma 5.4. □

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