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Hawking Radiation of the Charged Particle via Tunneling from the Reissner-Nordström Black Hole

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Abstract Since Parikh and Wilczek proposed a semiclassical tunneling method to investigate the Hawking radiation of static and spherically symmetric black holes, the method has been extensively developed to study various black holes. However, in almost all of the subsequent papers, there exists a important shortcoming that the geodesic equation of the massive particle is defined inconsistently with that of the massless particle. In this paper, we propose a new idea to reinvestigate the tunneling radiation from the event horizon of the Reissner-Nordström black hole. In our treatment, by starting from the Lagrangian analysis on the action, we redefine the geodesic equation of the massive and massless particle via tunneling from the event horizon of the Reissner-Nordström black hole, which overcomes the shortcoming mentioned above. The highlight of our work is a new and important development for the Parikh-Wilczek's semiclassical tunneling method.

Keywords Hawking radiation · Lagrangian analysis · Geodesic equation

1 Introduction

About forty years ago, Hawking made an astounding discovery that, from the viewpoint of quantum mechanics, black hole can radiate thermally like a black body [1, 2]. According to this scenario, due to vacuum fluctuations near the event horizon, a pair of particles is spontaneously created near the horizon. If the particle/antiparticle pair is created just inside

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the horizon, the positive energy particle then tunnels out to the infinity, and the negative energy "partner" falls into the singularity and effectively lowers the mass of the black hole. Moreover, if the the particle/antiparticle pair is created just outside the horizon, the negative particle tunnels into the horizon because the negative energy orbit exists only inside the horizon, the positive energy partner is radiated to the infinity. Nevertheless, in above two sight, there is a tunneling process, but there is no tunneling barrier. With the emission of thermal radiation, black holes could lose energy, shrink and eventually evaporate away completely. The result implies the loss of information of black hole after it has evaporated away and disappeared completely.

In 2000, Parikh and Wilczek have proposed a semiclassical tunneling picture near the event horizon of the black hole to study the Hawking radiation [3–6]. In this framework, a particle takes across the classically forbidden trajectories, and the tunneling barrier is just created by the outgoing particle itself. Moreover, they took into account the energy conservation and allowed the background geometry of the black holes to be dynamical and fluctuant. It means that the black hole lost mass while radiating, but maintained a constant energy for the total system. By applying a WKB (s-wave) approximation, the emission spectrum have been calculated for the Schwarzschild and Reissner-Nordström black holes. This result reveals that the derived spectrum of black hole radiation deviates from the pure thermal under the consideration of energy conservation and unfixed spacetime background, which may be a correct amendment to Hawking radiation spectrum and open the way to a possible resolution of the information loss paradox. Subsequently, this tunneling method has been extensively developed to study the Hawking radiation of various black hole [7–115]. The results are are very successful to support the Parikh-Wilczeks picture. Nevertheless, Parikh-Wilczek's tunneling are limited to the spherically symmetric black holes and most of them are confined only to discuss the tunneling process of the uncharged massless particles.

As an interesting development, Zhang and Zhao have generalized the Parikh-Wilczek's method to study the charged massive particle's tunneling from the Reissner-Nordström black hole horizon [16]. In this approach, the outgoing particle is treated as a massive shell (de Broglie s-wave). According to de Broglie hypothesis, this massive shell is a sort of de Broglie s-wave. The form of the approximative wave equation is given by $\psi(r,t) = Ce^{i(\int_{r_i-\epsilon}^{r} p_r dr - \omega t)}$, where $r_i - \epsilon$ represents the initial location of the particle. If letting $\int_{r_i-\epsilon}^{r} p_r dr - \omega t = \phi_0$, $dr/dt = \dot{r} = \omega/k$ is obtained, where k is the de Broglie wave number, and $\dot{r}$ is the phase velocity of the de Broglie wave. The definitions of the phase velocity v_p and the group velocity v_g are given by

$$v_p = \frac{dr}{dt} = \dot{r} = \frac{\omega}{k}, v_g = \frac{dr_c}{dt} = \frac{d\omega}{dk}, \quad v_p = \frac{1}{2}v_g, \tag{1}$$

where r_c denotes the location of the tunneling particle. According to Landau's theory of the coordinate clock synchronization, the group velocity is

$$v_g = \frac{dr_c}{dt} = -\frac{g_{00}}{g_{01}}, \tag{2}$$

and the phase velocity is therefore

$$\dot{r} = v_p = \frac{1}{2}v_g = -\frac{1}{2}\frac{g_{00}}{g_{01}}. \tag{3}$$

Thus, the geodesic equation of the charged massive particle has been given by investigating the relation between the group and phase velocity of the tunneling particle. The radial null geodesic, i.e. the geodesic equation of the massless particle, is given by $ds^2 = 0$. Since then, this definition for the geodesic equation of the charged massive particle has been received a wide attention, and lots of efforts have followed this definition to study the Hawking radiation of the charged massive particle via tunneling from various types of black holes [47–57].

However, in the work [16], the definition for the geodesic equation of the charged massive particle by investigating the relation between the group and phase velocity of the tunneling particle, appears to be unnatural and flawed. First, it is inconsistent with the first principle - the variation principle. In general relativity, the geodesics equation is always defined by applying the variation principle on the Lagrangian action. Secondly, the geodesic equations of the massless particle and the massive particle are not uniformly derived. In previous work, the geodesic equation of the massless particle is instead obtained by $ds^2 = 0$. The geodesic equation of the massive particle is obtained from (3). Obviously, the geodesic equation of the massless particle is defined inconsistently with that of the massive particle. Most importantly, it is flawed because it has been derived by using inconsistent foundations - mixing together relativistic and non-relativistic descriptions. The equation $v_p = 1/2v_g$ is obtained by combining the relativistic and non-relativistic descriptions. In quantum mechanics, we have $E = \hbar\omega$ and $P = \hbar k$. In the non-relativistic context, we also have $E = P^2/2m$. We can easily obtain $\omega = \hbar k^2/2m$. Thus, we can easily derive $v_g = d\omega/dk = \hbar k/m = 2\hbar kE/p^2 = 2\omega/k = 2v_p$. The equation $E = P^2/2m$ is valid only in the non-relativistic context. But for the black hole (i.e. the strong gravitational field), the physical law should be described in the relativistic context. So, the definition for the geodesics of the massive particle mixes together the relativistic and non-relativistic descriptions.

In view of above-mentioned reasons, we attempt to propose a new approach to naturally and uniformly redefine the geodesic equation of the massless particle and the charged massive particle, and then apply it to restudy the Hawking radiation of the charged massive particle via tunneling from the event horizon of the Reissner-Nordström black hole. This new definition for the geodesic equation of the particle comes from the Lagrangian analysis on the action, which overcomes the shortcomings of its previous definition, and is more suitable for the tunneling mechanism. On the one hand, the geodesic equation of the massive particle and massless particle can be defined by a uniform and self-consistent way. On the other hand, the definition for the geodesic equation of the charged massive particle is consistent with the first principle. It is noted that, our work in this paper is a new and important development of the Parikh-Wilczek's tunneling method.

The remainders of this paper are outlined as follows. In Section 2, we redefine the geodesic equation of the charged massive particle from the Lagrangian analysis on the action. By applying the new definition for the geodesics, Section 3 is devoted to study the Hawking radiation of the charged massive particle via tunneling from the event horizon of the Reissner-Nordström black hole. Section 4 ends up with some conclusions and discussions.

2 The Geodesic Equation of the Charged Massive Particle

In this section, we attempt to redefine the geodesic equation of the charged massive particle by applying the Lagrangian analysis on the action. For the Reissner-Nordström black hole, the motion equation of the particles can be absolutely determined with the three conserved integral constant. There are two Killing vector ∂_t and ∂_φ, respectively corresponding to energy E and angular momentum L. The last one is Hamiltonian $\mathcal{H}$, which can be constrained a constant by the normalizing condition of the 4-velocity of the timelike geodesic. We should first find the Lagrangian function governing the geodesic equation.

Apart from the energy conservation and the particle's self-gravitation are considered, a key point in the Parikh-Wilczeks approach is to introduce a coordinate system that is well-behaved at the event horizon in order to conveniently calculate the emission probability. The coordinate is not only time independent and regular at the horizon, but for which time reversal is manifestly asymmetric. It means that the coordinate is stationary but not static, and called "Painlevé-Gullstrand coordinate" [116]. For the Reissner-Nordström black hole, the Painlevé line element is obtained from the standard Reissner-Nordström line element by the coordinate transformation [3–6], and can be expressed as

$$ds^2 = -\Delta dt^2 \pm 2\sqrt{1-\Delta}dtdr + dr^2 + r^2 d\Omega^2, \tag{4}$$

where $\Delta = 1 - 2M/r + Q^2/r^2$, and the sign $+(-)$ corresponds to the line element of the outgoing (ingoing) particle at the event horizon of the black hole. Moreover, from the coordinate transformation, the 4-dimensional electromagnetic potential is given by

$$A_\mu = (A_t, 0, 0, 0), \tag{5}$$

where $A_t = -Q/r$. This Painlevé line element (4) has many interesting properties: i) The metric is regular at the event horizon; ii) The time direction remains to be a Killing vector; iii) The new form of the line element is stationary, but not static; iv) It satisfies the Landau's condition of the coordinate clock synchronization. All these features provide some comfortable surroundings for the tunneling particle across the event horizon of the black hole.

From (4), the equation of motion of the tunneling particle can be completely determined by three conservative integral constants. The three constants are obtained by using Lagrangian analysis on the action. In the Painlevé-Reissner-Nordström spacetime (4), when a charged massive particle with mass m and charge q tunnels across the event horizon, the effect of the electromagnetic field should be taken into account. Obviously, the matter-gravity system consists of the black hole and the electromagnetic field outside the hole. The Lagrangian function of the matter-gravity system is given by

$$\mathcal{L} = \mathcal{L}_m + \mathcal{L}_e, \tag{6}$$

where $\mathcal{L}_e = -1/4F_{\mu\nu}F^{\mu\nu}$ is the Lagrangian function of the electromagnetic field corresponding to the generalized coordinates $A_\mu = (A_t, 0, 0, 0)$. It is note that $A_\mu = (A_t, 0, 0, 0)$ is an ignorable coordinate from the expression of $\mathcal{L}_e$. According to (4) and (6), the Lagrangian quantity is given by [117]

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}mg_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - qA_\mu\frac{dx^\mu}{d\tau},\\ &= \frac{m}{2}[-\Delta\dot{t}^2 + 2\sqrt{1-\Delta}\dot{t}\dot{r} + \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2] + \frac{qQ}{r}\dot{t}.\end{aligned} \tag{7}$$

Thus, the corresponding canonical momenta are given by

$$\begin{aligned} P_t &= \frac{\partial \mathcal{L}}{\partial \dot{t}} = m[-\Delta \dot{t} + \sqrt{1-\Delta}\dot{r}] + \frac{qQ}{r}, \\ P_r &= \frac{\partial \mathcal{L}}{\partial \dot{r}} = m[\sqrt{1-\Delta}\dot{t} + \dot{r}], \\ P_\theta &= \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}, \\ P_\varphi &= \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = mr^2 \sin^2\theta\dot{\varphi}. \end{aligned} \tag{8}$$

According to the Legendre transformation, the Hamiltonian of the charged massive particle can be obtained by

$$\begin{aligned} \mathcal{H} &= P_t\dot{t} + P_r\dot{r} + P_\theta\dot{\theta} + P_\varphi\dot{\varphi} - \mathcal{L}, \\ &= \frac{m}{2}[-\Delta\dot{t}^2 + 2\sqrt{1-\Delta}\dot{t}\dot{r} + \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2]. \end{aligned} \tag{9}$$

Here, we can easily obtain $\mathcal{H} = \frac{1}{2}mg_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$ from (7) and (9). Moreover, the normalizing condition of the 4-velocity is given by

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -k, \tag{10}$$

where k is defined as a constant, when $k = 0$ denotes a photon, when $k = 1$ denotes a massive particle. Thus, by rescaling the affine parameter τ, we can always set

$$\mathcal{H} = -\frac{mk}{2}, k = \begin{cases} 0, & \text{(for the massless particle).} \\ 1, & \text{(for the charged massive particle).} \end{cases} \tag{11}$$

Obviously, the Hamiltonian quantity of the massless particle can be written in a unified way with that of the charged massive particle. In view of this, it is encouraging to obtain a unified form for the geodesic equation of the massless particle and charged massive particle by analyzing the Hamiltonian quantity. In the Lagrangian quantity (7), t and φ are the cyclic coordinates, so we have

$$\frac{dP_t}{d\tau} = \frac{\partial \mathcal{L}}{\partial t} = 0, \frac{dP_\varphi}{d\tau} = \frac{\partial \mathcal{L}}{\partial \varphi} = 0, \tag{12}$$

which means

$$\begin{aligned} P_t &= m[-\Delta\dot{t} + \sqrt{1-\Delta}\dot{r}] + \frac{qQ}{r} = const = E, \\ P_\varphi &= mr^2\sin^2\theta\dot{\varphi} = const = L, \end{aligned} \tag{13}$$

where E and L are integral constants. Combining (11) and (13), if the geodesics is described in an invariant plane $\theta = \pi/2$ without loss of generality, we have

$$\dot{r} \equiv \frac{dr}{d\tau} = \pm\sqrt{\left(\frac{E - qQ/r}{m}\right)^2 - \Delta\left(k + \frac{L^2}{m^2r^2}\right)}, \tag{14}$$

and

$$\dot{t} \equiv \frac{dt}{d\tau} = \frac{1}{\Delta}\left[-\frac{E - qQ/r}{m} \pm \sqrt{1-\Delta}\sqrt{\left(\frac{E - qQ/r}{m}\right)^2 - \Delta\left(k + \frac{L^2}{m^2r^2}\right)}\right]. \tag{15}$$

Hence, the geodesic equation of the tunneling particle across the event horizon of the Reissner-Nordström black hole is given by

$$\bar{r} \equiv \frac{\dot{r}}{\dot{t}} = \Delta\left[\sqrt{1-\Delta} \pm \frac{E-qQ/r}{\sqrt{(E-qQ/r)^2-\Delta(m^2k+L^2/r^2)}}\right]^{-1}, \tag{16}$$

where the $+(-)$ sign can be identified with the outgoing(ingoing) radial motion under the implicit assumption that time t increase towards the future. Obviously, in (16), the geodesic equation of the charged massive particle (i.e. $k=1$) has been defined consistently with that of the massless particle (i.e. $k=0$). Next, we will remark on it.

i) When $k=0$ for the geodesics moving by the massless particle, considering the s-wave approximation ($P_\varphi = L = 0$) yields

$$\bar{r} = \pm 1 - \sqrt{1-\Delta}, \tag{17}$$

where the $+(-)$ sign is identified with the outgoing(ingoing) radial geodesics. It is noted that, our definition for the geodesic equation of the massless particle is precisely consistent with that defined in Ref. [3–6] by using the relation $ds^2 = 0$.

ii) When $k=1$ for the geodesics moving by the charged massive particle, considering the s-wave approximation ($P_\theta = L = 0$), we have

$$\bar{r} = \Delta\Big[\sqrt{1-\Delta} \pm \frac{E-qQ/r}{\sqrt{(E-qQ/r)^2-\Delta m^2}}\Big]^{-1}, \tag{18}$$

where the $+(-)$ sign denotes the outgoing(ingoing) radial geodesics. Near the event horizon, it is interestingly found that our new definition for the geodesic equation of the charged massive particle is precisely consistent with that defined in Ref. [16] by investigating the relation between the group and phase velocity of the tunneling particle.

In a word, we obtain the three conserved integral constants by applying the Lagrangian analysis on the action, and then use the above integral constants to redefine the geodesic equation of the charged massive particle. Comparing with previous definition in Ref. [16] by investigating the relation between the group and phase velocity of the tunneling particle, our new definition for the geodesic equation of the charged massive particle appears to be more suitable for the tunneling mechanism. On the one hand, the geodesic equation of the massless and charged massive particle can be uniformly defined by letting $k=0$ and $k=1$ respectively corresponding to the massless particle and the massive particle. On the other hand, the geodesic equation of the charged massive particle is defined consistently with the first principle. In the next section, we use this new definition for the geodesic equation to restudy the Hawking radiation of the charged massive particle via tunneling from the event horizon of the Reissner-Nordström black hole.

3 The Hawking Radiation at the Event Horizon

In the section, we turn to discuss Hawking radiation of charged particle as a semiclassical tunneling process. We adopt the picture of a pair of virtual particles spontaneously created just inside the horizon. The positive energy virtual particle can tunnel out and materialize as a real particle escaping classically to infinity, its negative energy partner is absorbed by the black hole, resulting in a decrease in the mass of the black hole and the surface of the event horizon. We should take into account the self-gravitation of the tunneling particle.

According to the energy and charge conservation, we assume that the total ADM mass and charge of the hole-particle system are held fixed, whereas the mass and charge of the hole are allowed to vary, the black hole mass and charge will become $M \to M - \omega$ and $Q \to Q - q$ when a particle with energy ω and charge q has evaporated from the event horizon. Now, the geodesic equation of the charged massive particle is modified as

$$\bar{r} = \tilde{\Delta}\left[\sqrt{1-\tilde{\Delta}} + \frac{\tilde{E} - q\tilde{Q}/r}{\sqrt{\left(\tilde{E} - q\tilde{Q}/r\right)^2 - \tilde{\Delta}m^2}}\right]^{-1} \tag{19}$$

where $\tilde{\Delta} = 1 - 2(M-\omega)/r + (Q-q)^2/r^2$, and $\tilde{E}$ and $q\tilde{Q}/r$ correspond to the integral constant of the canonical momentum P_t and the electromagnetic energy after taking into account the tunneling particle's self-gravitation effect. When the charged massive particle tunnels out from the event horizon of the Reissner-Nordström black hole, the effect of the electromagnetic field should be taken into account. When a charged particle tunnels out, the system transit from one state to another. But from the expression of $\mathcal{L}_e$ we find that the generalized coordinate $A_\mu = (A_t, 0, 0, 0)$ is an ignorable coordinate. In order to eliminate the freedom corresponding to A_μ, the imaginary part of the action for the charged massive particle should be written as

$$\begin{aligned} \mathrm{Im}S &= \mathrm{Im}\int_{r_i}^{r_f} P_r - \frac{P_{A_t}\bar{A}_t}{\bar{r}}dr \\ &= \mathrm{Im}\int_{r_i}^{r_f}\left(\int_{(0,0)}^{(P_r,P_{A_t})} dP'_r - \frac{\bar{A}_t}{\bar{r}}dP'_{A_t}\right)dr, \end{aligned} \tag{20}$$

where r_i and r_f represent the locations of the event horizon before and after the charged massive particle with energy ω and charge q tunnels out, and P_{A_t} is the electromagnetic field's canonical momentum conjugated to A_t. To proceed with an explicit calculation, we apply the Hamilton's equation

$$\bar{r} = \frac{dH}{dP_r}\Big|_{(r;A_t,P_{A_t})} = \frac{d(M-\omega)}{dP_r}, \tag{21}$$

$$\bar{A}_t = \frac{dH}{dP_{A_t}}\Big|_{(A_t;r,P_r)} = \frac{(Q-q)}{r}\frac{d(Q-q)}{dP_{A_t}}. \tag{22}$$

Substituting (21) and (22) into (20) yields

$$\begin{aligned} \mathrm{Im}S &= \mathrm{Im}\int_{r_i}^{r_f}\int_{(0,0)}^{(\omega,q)}\left[d(M-\omega') - \frac{Q-q'}{r}d(Q-q')\right]\frac{dr}{\bar{r}}, \\ &= \mathrm{Im}\int_{r_i}^{r_f}\int_{(0,0)}^{(\omega,q)}\frac{1}{\tilde{\Delta}}\left[\sqrt{1-\tilde{\Delta}'} + \frac{\tilde{E}' - q\tilde{Q}'/r}{\sqrt{(\tilde{E}' - q\tilde{Q}'/r)^2 - \tilde{\Delta}'m^2}}\right] \\ &\times\left[d(M-\omega') - \frac{Q-q'}{r}d(Q-q')\right]dr, \end{aligned} \tag{23}$$

where

$$\begin{aligned}
&\tilde{\Delta}' = 1 - 2(M-\omega')/r + (Q-q')^2/r^2 = (r-r'_+)(r-r'_-)/r^2,\\
&r'_\pm = (M-\omega') \pm \sqrt{(M-\omega')^2 - (Q-q')^2},\\
&r_i = M + \sqrt{M^2 - Q^2},\\
&r_f = (M-\omega') + \sqrt{(M-\omega')^2 - (Q-q')^2}.
\end{aligned} \tag{24}$$

Obviously, at the event horizon (i.e. $r = r'_+$), there is a single pole in (23). Let us switch the order of integration and do the r integral first. Then, deforming the contour around the pole yields

$$\mathrm{Im}S = -\pi \int_{r_i}^{r_f} r'_+ dr'_+ = -\frac{\pi}{2}(r_f^2 - r_i^2) = -\frac{1}{2}\Delta S_{EH}, \tag{25}$$

where $\Delta S_{EH} = S_{EH}(M-\omega, Q-q) - S_{EH}(M, Q)$ is the change of the Bekenstein-Hawking entropy before and after the charged massive particle tunnels out. Therefore, the tunneling rate of the charged massive particle via tunneling from the event horizon of the Reissner-Nordström black hole is given by

$$\Gamma \propto e^{-2\mathrm{Im}S} = e^{\Delta S_{EH}}. \tag{26}$$

This tunneling rate is accurately consistent with that in Ref. [16]. In Ref. [16], the geodesic equation of the charged massive particle is unnaturally and even defectively given by investigating the relation between the group and phase velocity of the tunneling particle. In our treatment, the geodesic equation of the charged massive particle is well defined by applying the Lagrangian analysis on the action. At the event horizon, both of them are asymptotically taking the same form. That's why we get the same tunneling rate in spite of different definitions for the geodesic equation of the charged massive particle.

4 Conclusion and Discussion

In this paper, we first redefine the geodesic equation of the charged massive particle by applying the Lagrangian analysis on the action of the variation principle. Then, basing on the new definition for the geodesics, we revisit the Hawking radiation of the charged massive particle via tunneling from the event horizon of the Reissner-Nordström black hole. It is worth noting that, our new definition for the geodesic equation of the charged massive particle overcomes some shortcomings of its previous definition in Ref. [16], and is more suitable for the tunneling mechanism. On the one hand, the geodesic equation of the massless and charged massive particle can be uniformly defined by letting the parameter k in the normalizing condition of the 4-velocity. On the other hand, the geodesic equation of the charged massive particle is defined consistently with the first principle - the variation principle. In addition to, the method of redefining the geodesic equation can be extended to other types of black holes since our derivation of the geodesic equation basic on the Lagrangian analysis is universal. Our work in this paper is a new and important development of the Parikh-Wilczek's tunneling method.

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