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# Concrete Quantum Logics and Δ-Logics, States and Δ-States

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**Abstract** By a concrete quantum logic (in short, by a logic) we mean the orthomodular poset that is set-representable. If $L = (\Omega, \mathcal{L})$ is a logic and $\mathcal{L}$ is closed under the formation of symmetric difference, $\Delta$, we call $L$ a $\Delta$-logic. In the first part we situate the known results on logics and states to the context of $\Delta$-logics and $\Delta$-states (the $\Delta$-states are the states that are subadditive with respect to the symmetric difference). Moreover, we observe that the rather prominent logic $\mathcal{E}_\Omega^{\mathrm{even}}$ of all even-coeven subsets of the countable set $\Omega$ possesses only $\Delta$-states. Then we show when a state on the logics given by the divisibility relation allows for an extension as a state. In the next paragraph we consider the so called density logic and its $\Delta$-closure. We find that the $\Delta$-closure coincides with the power set. Then we investigate other properties of the density logic and its factor.

**Keywords** Concrete quantum logic · Symmetric difference · Δ-logic · State · Density logic · Banach limit

## 1 Notions and Results

The (concrete) logics and $\Delta$-logics have been investigated by several authors [2–6, 8, 15, 16, 19, 21, 24, 25]. In this note, we extend on this investigation.

Let us review the basic notions as we shall use them in the sequel (by $\exp \Omega$ we mean a collection of all subsets of $\Omega$).

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**Definition 1** A concrete quantum logic (abbr., a logic) is a pair $(\Omega, \mathcal{L})$ where $\Omega$ is a set and $\mathcal{L}$, $\mathcal{L} \subset \exp\Omega$ is such a collection of sets that is subject to the following conditions:

1. $\Omega \in \mathcal{L}$,
2. if $A, B \in \mathcal{L}$ and $A \cap B = \emptyset$, then $A \cup B \in \mathcal{L}$.

A logic is said to be a $\Delta$-logic if it is closed under the formation of the symmetric difference: If $A, B \in \mathcal{L}$, then $A\Delta B = (A\backslash B) \cup (B\backslash A) \in \mathcal{L}$.

Let us observe that if $L = (\Omega, \mathcal{L})$ is a logic then the $\Delta$-logic generated in $\Omega$ by $\mathcal{L}$ consists of all elements, $D$, of the type $D = A_1\Delta A_2\Delta \ldots \Delta A_n$, where $A_i \in \mathcal{L}$. Let us denote by $(\Omega, \Delta\mathcal{L})$ the $\Delta$-logic generated by $\mathcal{L}$ in $\Omega$. Obviously, if a collection $\mathcal{K}$ is closed under the formation of the symmetric difference and $\Omega \in \mathcal{K}$, then $\mathcal{K}$ is a logic.

The previous research revealed a large variety of concrete logics, including Boolean algebras, of course. It is easily seen that $(\Omega, \mathcal{L})$ is a Boolean algebra exactly when $A \cap B \in \mathcal{L}$ for any pair $A, B \in \mathcal{L}$. (It may be noted that some authors—including the inventor of $\Delta$-logics P. G. Ovchinnikov [19]—preferred the expression “symmetric logic” to $\Delta$-logic, we feel that $\Delta$-logic is more suggestive and short.)

**Definition 2** Let $L = (\Omega, \mathcal{L})$ be a logic. A mapping $s : \mathcal{L} \to [0, 1]$ is said to be a state on $L$ (or, alternatively, $s$ is said to be a state on $\mathcal{L}$ if we do not need to refer to $\Omega$) if

1. $s(\Omega) = 1$,
2. if $A, B \in \mathcal{L}$ and $A \cap B = \emptyset$, then $s(A \cup B) = s(A) + s(B)$.

If $(\Omega, \mathcal{L})$ is a $\Delta$-logic and $s$ is a state on $\mathcal{L}$, then $s$ is called a $\Delta$-state if $s(A\Delta B) \le s(A) + s(B)$ for any $A, B \in \mathcal{L}$.

In the first part of the paper we ask when a state on $(\Omega, \mathcal{L})$ can be extended over $(\Omega, \Delta\mathcal{L})$ as a $\Delta$-state. This question is related to “discrete integration” as pursued e.g. in [12, 14, 17]—if a state on $(\Omega, \mathcal{L})$ allows for an extension over $(\Omega, \Delta\mathcal{L})$ as a $\Delta$-state, the corresponding integral is $\Delta$-subadditive. In the ideal case when the state $s$ extends over $\exp\Omega$, the corresponding integral is additive. The degree of additivity of the integral can be a significant matter in e.g. coarse-grained measurement or in economic theories (see [5, 13]).

The first instance to be taken up is the situation when the logic $(\Omega, \mathcal{L})$ is already a $\Delta$-logic. A most natural question then reads as follows: When a state on $(\Omega, \mathcal{L})$ is automatically a $\Delta$-state? In [2] the authors asked this question. They showed that if $(\Omega, \mathcal{L})$ is a non-Boolean logic and $\mathcal{L}$ is finite then there is always a state on $(\Omega, \mathcal{L})$ that is not a $\Delta$-state. In a certain contrast, they proved that if $\mathcal{L} = \mathcal{E}_\Omega^{\mathrm{even}}$ is a logic of all even-coeven subsets of $\Omega$ and $\Omega$ is *uncountable*, then each state on $(\Omega, \mathcal{E}_\Omega^{\mathrm{even}})$ is automatically a $\Delta$-state. The authors of [2] omit the case of $\Omega$ countable. In the following theorem we take care of this case.

**Theorem 1** *Let $\Omega$ be an (infinite) countable set and let $\mathcal{E}_\Omega^{even}$ be the quantum logic of all even-coeven subsets of $\Omega$. Let $s$ be a state on $\mathcal{E}_\Omega^{even}$. Then $s$ is a $\Delta$-state.*

*Proof* The proof makes use of the insight taken from [2] plus a few new observations. Let $A, B \in \mathcal{E}_\Omega^{\mathrm{even}}$. We have to show that $s(A\Delta B) \le s(A) + s(B)$. First, if $A \cap B \in \mathcal{E}_\Omega^{\mathrm{even}}$, then both $A\backslash B$ and $B\backslash A$ belong to $\mathcal{E}_\Omega^{\mathrm{even}}$ and the inequality is obvious: $s(A\Delta B) = s((A\backslash B) \cup (B\backslash A)) = s(A\backslash B) + s(B\backslash A) \le s(A) + s(B)$. Suppose therefore that $A \cap B \notin \mathcal{E}_\Omega^{\mathrm{even}}$. Then neither of the sets $A\backslash B$ and $B\backslash A$ belong to $\mathcal{E}_\Omega^{\mathrm{even}}$. Let us discuss the situation by cases. Suppose first that both $A$ and $B$ are infinite. Thus, $A = \Omega\backslash\{a_1, a_2, \ldots, a_{2k}\}$ and $B = \Omega\backslash\{b_1, b_2, \ldots, b_{2l}\}$, where $k, l$ are positive integers. Then $A\backslash B = \{b_1, b_2, \ldots, b_{2l}\}\backslash\{a_1, a_2, \ldots, a_{2k}\}$ and $B\backslash A = \{a_1, a_2, \ldots, a_{2k}\}\backslash\{b_1, b_2, \ldots, b_{2l}\}$. By our assumption, both $A\backslash B$ and $B\backslash A$ are of odd cardinalities. Then there is an $x \in A$ and a $y \in B$ such that both $(A\backslash B) \cup \{x\}$ and $(B\backslash A) \cup \{y\}$ belong to $\mathcal{E}_\Omega^{\text{even}}$. If $x \neq y$, which is easy to satisfy, then $s(A\Delta B) = s(((A\backslash B) \cup \{x\}) \cup ((B\backslash A) \cup \{y\})) \leq s((A\backslash B) \cup \{x\}) + s((B\backslash A) \cup \{y\}) \leq s(A) + s(B)$. Secondly, suppose that $A$ is infinite and $B$ is finite. Then $A = \Omega\backslash\{a_1, a_2, \ldots, a_{2k}\}$ and $B = \{b_1, b_2, \ldots, b_{2l}\}$. It means that $A\backslash B = \Omega\backslash(\{a_1, a_2, \ldots, a_{2k}\} \cup \{b_1, b_2, \ldots, b_{2l}\})$ and $B\backslash A = \{a_1, a_2, \ldots, a_{2k}\} \cap \{b_1, b_2, \ldots, b_{2l}\}$. Suppose that the cardinality of $A \cap B$ is greater than or equal to 3. Then we can easily find two distinct points $x, y \in A \cap B$ such that $((A\backslash B) \cup \{x\}) \in \mathcal{E}_\Omega^{\text{even}}$ and $((B\backslash A) \cup \{y\}) \in \mathcal{E}_\Omega^{\text{even}}$. Then $s(A\Delta B) = s((A\backslash B) \cup (B\backslash A)) \leq s(((A\backslash B) \cup \{x\}) \cup ((B\backslash A) \cup \{y\})) = s((A\backslash B) \cup \{x\}) + s((B\backslash A) \cup \{y\}) \leq s(A) + s(B)$. Suppose therefore that $A \cap B$ is a singleton. Write $A \cap B = \{c\}$. We are going to show that for any $\varepsilon$, $\varepsilon > 0$, we have the inequality $s(A\Delta B) \leq s(A) + s(B) + \varepsilon$. This implies that $s(A\Delta B) \leq s(A) + s(B)$. Since $A\backslash B$ is infinite, there is an infinite number of disjoint two-point sets in $A\backslash B$. Among these two-point sets there must be one, say $\{u, v\}$, such that $s(\{u, v\}) \leq \varepsilon$ (otherwise we have a contradiction with the additivity of $s$). Then we have the following inequalities: $s(A\Delta B) = s((A\backslash B)\cup(B\backslash A)) = s((A\backslash\{c\})\cup(B\backslash\{c\})) = s((A\backslash\{c, u\})\cup((B\backslash\{c\})\cup\{u\})) = s((A\backslash\{c, u\})) + s((B\backslash\{c\}) \cup \{u\})) \leq s(A) + s(B \cup \{u, v\}) = s(A) + s(B) + s(\{u, v\}) = s(A) + s(B) + \varepsilon$. Finally, suppose that both $A$ and $B$ are finite. So $A = \{a_1, a_2, \ldots, a_{2k}\}$ and $B = \{b_1, b_2, \ldots, b_{2l}\}$. If the cardinality of $A \cap B$ is greater than or equal to 3, we can again find two distinct points $x, y \in A \cap B$ such that $((A\backslash B) \cup \{x\}) \in \mathcal{E}_\Omega^{\text{even}}$ and $((B\backslash A) \cup \{y\}) \in \mathcal{E}_\Omega^{\text{even}}$. As before, $s(A\Delta B) = s((A\backslash B) \cup (B\backslash A)) \leq s(((A\backslash B) \cup \{x\}) \cup ((B\backslash A) \cup \{y\})) = s((A\backslash B) \cup \{x\}) + s((B\backslash A) \cup \{y\}) \leq s(A) + s(B)$. The only case that remains is when $A \cap B$ is a singleton. Then we will show that for any $\varepsilon$, $\varepsilon > 0$, we have the inequality $s(A\Delta B) \leq s(A) + s(B) + 2\varepsilon$. Consider again infinitely many two-point sets in $\Omega\backslash(A \cup B)$. If an $\varepsilon$, $\varepsilon > 0$ is given, there must be a two-point set $\{u, v\}$, $\{u, v\} \subset \Omega\backslash(A \cup B)$ with $s(\{u, v\}) \leq \varepsilon$. Consider the sets $A \cup \{u, v\}$ and $B \cup \{u, v\}$. Then $(A\cup\{u, v\})\cap(B\cup\{u, v\})$ has 3 elements and, moreover, $(A\cup\{u, v\})\Delta(B\cup\{u, v\}) = A\Delta B$. So we obtain $s(A\Delta B) = s((A \cup \{u, v\})\Delta(B \cup \{u, v\})) \leq s(A \cup \{u, v\}) + s(B \cup \{u, v\}) \leq s(A) + \varepsilon + s(B) + \varepsilon \leq s(A) + s(B) + 2\varepsilon$. The proof is complete. □

The result above supports the conjecture that a state on $\mathcal{E}_\Omega^{\text{even}}$ extends over the Boolean algebra of finite-cofinite sets. This question seems to be open so far.

It should be noted that in [24] the author shows that there is a $\Delta$-logic $(\Omega, \mathcal{L})$ on which each state is subadditive (a state on $(\Omega, \mathcal{L})$ is said to be subadditive if for any $A, B \in \mathcal{L}$ there is a $C \in \mathcal{L}$ such that $A \cup B \subset C$ and $s(C) \leq s(A) + s(B)$). Since each subadditive state is a $\Delta$-state, this example somewhat strengthens the uncountable example of [2] and, in addition, it enjoys several other algebraic properties (for instance, it is pseudocomplemented). The example of [24] does require the set $\Omega$ uncountable.

Another conceptually important example is the case of the divisibility logics. Suppose that $n = mk$ with numbers $m, n, k \in N$ and $k \geq 2$. Let $\Omega = \{1, 2, \ldots, n\}$ and let us denote by $Div_k$ the logic of all subsets of $\Omega$ whose number of elements is divisible by $k$. Thus, the cardinality of $Div_k$ is $\sum_{i=0}^{m} \binom{n}{ik}$. Consider the logic $(\Omega, Div_k)$. If $m = 2$, then problem trivializes—there is always a state on $(\Omega, Div_k)$ that cannot be extended over $(\Omega, \Delta Div_k)$ (see [4, 22]). A rather interesting situation occurs when $k \geq 3$ and $m \geq 3$. Since the case of $k$ even reduces to the situation covered by $k = 2$ (in this case $\Delta Div_k = Div_2$), let us assume that $k$ is odd. In this case $\Delta Div_k = \exp \Omega$ and we therefore ask whether a state on $(\Omega, Div_k)$ extends over $\exp\Omega$. This question was investigated in a nice paper [22] with the answer that there are always states that do not extend over $\exp\Omega$ as states but that each state on $(\Omega, Div_k)$ always allows for an extension over $\exp\Omega$ as a *signed* state. In other words, there is always a set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of real (not necessarily non-negative) numbers with $\sum_{i=1}^{n} \alpha_1 = 1$ and with the property that the combination of the Dirac states on $\exp\Omega$ with the coefficients $\alpha_i$, $(i \le n)$, gives us the original state $s$ when restricted to $(\Omega, Div_k)$. We would like to contribute to this result by formulating—in the line of Farkas lemma (see [9, 10])—a necessary condition for a state to be extended as a state (we could strengthen it to obtain a necessary and sufficient condition but the formulation is then perhaps less satisfactory and less elegant). To do that, we have to express the result of [22] in more detail and fix some terminology.

Let $n = mk$, when $k \ge 3$, $m \ge 3$ and $k$ is odd. Let us assign to the logic $(\Omega, Div_k)$ a matrix, $P(Div_k)$, in the following manner. The matrix $P(Div_k)$ is an $(n-1)\times(n-1)$ matrix such that the rows of $P(Div_k)$ are the following vectors (each vector contains $k$ many of 1's and $(n-1-k)$ many of 0's):

$$\begin{aligned}
r_1 &= (1, 1, \ldots, 1, 0, 0, \ldots, 0, 0, 0),\\
r_2 &= (0, 1, 1, \ldots, 1, 1, 0, \ldots, 0, 0),\\
&\vdots\\
r_{n-k-1} &= (0, \ldots, 0, 1, 1, \ldots, 1, 1, 1, 0),\\
r_{n-k} &= (0, 0, \ldots, 0, 1, 1, \ldots, 1, 1, 1),\\
r_{n-k+1} &= (1, 0, 0, \ldots, 0, 1, 1, \ldots, 1, 1),\\
r_{n-k+2} &= (1, 1, 0, 0, \ldots, 0, 1, 1, \ldots, 1),\\
&\vdots\\
r_{n-2} &= (1, 1, \ldots, 1, 0, 0, \ldots, 0, 1, 1),\\
r_{n-1} &= (1, 1, 1, \ldots, 1, 0, 0, \ldots, 0, 1).
\end{aligned}$$

Let us assign to each vector of the row the set $A_i$ $(i \le n-1)$ of $Div_k$ which "copies the coordinates" (for instance, to the vector $r_1 = (1, 1, \ldots, 1, 0, 0, \ldots, 0, 0, 0)$ we assign $A_1 = \{1, 2, \ldots, k\}$, to the vector $r_2 = (0, 1, 1, \ldots, 1, 1, 0, \ldots, 0, 0)$ we assign $A_2 = \{2, 3, \ldots, k+1\}$, etc.). Write $s_i = s(A_i)$. The author of [22] shows that the system $P(Div_k)\cdot x^T = s_i$ has a precisely one solution $(\det(P(Div_k)) = k)$. In fact, by the Cramer rule we obtain $x_i = \frac{\det(P_i)}{k}$, $i \le n-1$, where $P_i$ is the $i$-th Cramer matrix associated to $P(Div_k)$. The coordinates of the vector determine the values of the extension of $s$ over $\exp\Omega$. This follows from the fact proved in [22] that the sets $A_i$ are generators of $Div_k$. It further shows that there is always an extension of $s$ over $\exp\Omega$ as a signed state. A necessary condition for a non-negative extension is given by the following version of Farkas lemma (by an additional condition, we can even arrive to a characterization).

**Theorem 2** *Let $n = mk$, where $k \ge 3$, $m \ge 3$ and $k$ is odd. Write $\Omega = \{1, 2, \ldots, n\}$ and consider the logic $(\Omega, Div_k)$. Then*

1) $(\Omega, \Delta Div_k) = (\Omega, \exp\Omega)$,
2) *if $s$ is a state on $(\Omega, Div_k)$ and $P(Div_k)$ is the matrix associated to $(\Omega, Div_k)$, then the validity of the following implication is a necessary condition for $s$ to be extended over $\exp\Omega$ as a state: If $(P(Div_k))^T \cdot p^T \ge 0$ for a vector $p$ with all coordinates integer, then $(s_1, s_2, \ldots, s_{n-1})\cdot p^T \ge 0$,*

3) *suppose that the implication in the condition 2) above is valid and suppose that* $s(\{1, 2, \ldots, k-1, n\}) - \sum_{i=1}^{k-1} \frac{\det(P_i)}{k} \geq 0$. *Then s extends over* $\exp \Omega$ *as a state.*

*Proof* Since $P(Div_k)$ is a matrix with the entries 0 and 1 only, we can apply the variant of Farkas lemma proved in [7]. The condition 3) guarantees that we can find the non-negative extension for the singleton $\{n\}$, too. □

Let us note that the system of linear equations considered above may indeed have a "properly signed" solution (thus, there is a state on $(\Omega, Div_k)$ that cannot be extended over $\exp \Omega$ as a state. Take, for instance, $n = 9$, $k = 3$ and $m = 3$. Thus $\Omega = \{1, 2, \ldots, 9\}$. Consider the evaluation $e : \Omega \to R$ such that $e(1) = -\frac{1}{7}$, $e(2) = e(3) = \ldots = e(9) = \frac{1}{7}$. This evaluation uniquely determines a (non-negative) state on $(\Omega, Div_k)$ by setting $s(A) = \sum_{a \in A} e(a)$. The state $s$ cannot be extended over $\exp \Omega$ as a state. This can be verified directly or it suffices to take, in our condition 2), the vector $p = (3, -2, 0, 2, 0, -2, 3, -1)$.

It should be noted, in connection with the theme of our paper, that an analogous question about extensions of states has been asked and investigated in [13] for so called coarse-grained logics and fully answered in [20] (for a further extension on this type of research, see [5] and [6]). Recalling briefly the definition, if we again write $n = mk$ and $\Omega = \{1, 2, \ldots, n\}$, then the coarse-grained logic is the one generated by consecutive $k$-tuples in $\Omega$ understood $\mod k$. Hence the generating sets are $\{1, 2, \ldots, k\}$, $\{2, 3, \ldots, k+1\}$, ..., $\{n-k+1, n-k+2, \ldots, n\}$, $\{n-k+2, \ldots, n, 1\}$, ..., $\{n, 1, 2, \ldots, k-1\}$. So the number of generators is $n$ (this number could be lowered but this is not a matter of our interest in this paper). In a rather interesting manner, the nature of the extension problem differs considerably from the previous situation. If $m \geq 3$ and $(\Omega, \mathcal{L})$ is a coarse-grained logic on $\Omega$, then a state on $(\Omega, \mathcal{L})$ *always* allows for an extension over $\exp \Omega$ as a state, and therefore the state always allows for an extension over $(\Omega, \Delta\mathcal{L})$ as a state. (In order to expose the structural difference of the two situations, let us again consider the example of the previous paragraph given by the evaluation $e : \Omega \to R$ such that $e(1) = -\frac{1}{7}$, $e(2) = e(3) = \ldots = e(9) = \frac{1}{7}$. If understood as a state of $(\Omega, Div_3)$, it cannot be extended over $\exp \Omega$ as a state. However, if understood as a state on the coarse-grained logic $(\Omega, \mathcal{L})$, $k = 3$, it does allow for an extension as a state (indeed, it suffices to take $s(\{1\}) = \frac{1}{7}$, $s(\{4\}) = s(\{7\}) = \frac{3}{7}$, $s(\{2\}) = s(\{3\}) = s(\{5\}) = s(\{6\}) = s(\{8\}) = s(\{9\}) = 0$).

Let us introduce the final area of questions which we want to take up (and contribute to) in this paper. Let $N = \{1, 2, \ldots, n, \ldots\}$ be the set of all natural numbers and let $\mathcal{L}$ be the collection of all subsets $A$, $A \subset N$ such that $\lim_{n\to\infty} \frac{\text{card}(A \cap \{1,2,\ldots,n\})}{n}$ exists. Put $\Omega = N$ and let us consider $(\Omega, \mathcal{L})$. Let us call $(\Omega, \mathcal{L})$ a *d-logic* (the letter d indicates "density" as sometimes referred to in the literature). This classical structure of number theory and analysis has apparently not been considered from the point of view of quantum logics (in the paper [27] this example was mentioned without any further discussion). Let us formulate and prove certain properties of $(\Omega, \mathcal{L})$ for a potential further investigation within quantum logics.

**Theorem 3** *Let* $(\Omega, \mathcal{L})$ *be a d-logic. Thus,* $\Omega = N$ *and* $\mathcal{L}$ *consists of all subsets of* $\Omega$ *that are determined by the limit condition introduced in the paragraph above. Then*

1) $(\Omega, \mathcal{L})$ *is a (concrete) quantum logic,*
2) *if we write, for any* $A \in \mathcal{L}$, $s(A) = \lim_{n\to\infty} \frac{\text{card}(A \cap \{1,2,\ldots,n\})}{n}$, *then s is a state on* $\mathcal{L}$,

3) *$\mathcal{L}$ is not a lattice (and therefore $\mathcal{L}$ is not Boolean). In fact, any couple $A,\ B \in \mathcal{L}$ such that $A \cap B \notin \mathcal{L}$ and $A \cap B$ is infinite does not have an infimum,*
4) *$\Delta\mathcal{L} = \exp\Omega$. More explicitly, for each $A$, $A \subset \exp\Omega$ there are sets $B, C \in \mathcal{L}$ such that $s(B) = s(C) = \frac{1}{2}$ and $A = B\Delta C$,*
5) *the state $s$ can be extended over $\exp\Omega$ as a state,*
6) *there is a family of $2^{\aleph_0}$ almost disjoint subsets of $\Omega$, $\{A_\alpha, \alpha < 2^{\aleph_0}\}$, such that $s(A_\alpha) = 0$ for each $\alpha, \alpha < 2^{\aleph_0}$. A consequence: Let us consider the quantum logic $\mathcal{K} = \mathcal{L}/\mathcal{F}$ obtained as the factor of $(\Omega, \mathcal{L})$ with respect to the ideal $\mathcal{F}$ of all finite sets. Then $\mathcal{K}$ has $2^{\aleph_0}$ elements and $\mathcal{K}$ is atomless. Moreover, this factor logic $\mathcal{K}$ is pseudocomplemented (i.e., the elements $A,\ B \in \mathcal{K}$ are compatible exactly when $A \wedge B$ exists).*

*Proof* The statements 1) and 2) can be proved by routine verifications. Let us consider the statement 3). It is easy to check that for any couple referred to in statement 3) the infimum does not exists (the logic $\mathcal{L}$ contains all finite sets). What remains to show is that such a couple exists at all. Indeed, it suffices to take for $A$ the set of all odd numbers and to construct the set $B$ as follows. First we put into the set $B$ the elements 2 and 3, then precisely all even numbers from the segment $(2^k + 1)$ up to $(\frac{3}{2}2^k)$ and precisely all odd numbers from the segment $(\frac{3}{2}2^k + 1)$ up to $(2^{k+1} + 1)$ for all natural $k,\ k \geq 2$. It is easy to see that $s(A) = s(B) = \frac{1}{2}$ and that the sequence $d_n = \frac{\text{card}((B\cap C)\cap\{1,2,\ldots,n\})}{n}$ has the values $\frac{1}{4}$ and $\frac{1}{6}$ for its cluster points (thus, $\lim_{n\to\infty} d_n$ does not exist and hence $B \cap C \notin \mathcal{L}$).

Let us take up the proof of statement 4). The formal expression of $B$ and $C$ would be rather difficult and cumbersome, we will indicate the construction idea which is sufficiently intuitive. Let us consider $A$ expressed as a union of subsets, $A = \bigcup_{i=1}^{\infty} I_i$, where each $I_i$ $(i \in N)$ is a segment of consecutive points. Also, let us express the set $\Omega\backslash A$ as a union of subsets, $\Omega\backslash A = \bigcup_{i=1}^{\infty} H_i$ $(i \in N)$, where each $H_i$ is a segment of consecutive points. In our argument, let us refer to an $I_i$ as an "island" in $A$ and to an $H_i$ as a "hole" in $A$. If either of $I_i$ of $H_i$ is equal to $\Omega$ up to a finite set, then the proof is easy. Suppose therefore that both the families $I_i$ and $H_i$ $(i \in N)$ are infinite and each $I_i$ and $H_i$ is a finite set. We can consider $I_i$ and $H_i$ with its order inherited from $N(= \Omega)$. Call this order the natural order of $I_i$ and $H_i$. Let us construct the sets $B$ and $C$. Firstly, consider those islands $I_i$ which consist of an even number of elements. In this case the set $B$ to be constructed contains precisely the odd elements in the $I_i$ considered in the natural order and, analogously, the set $C$ to be constructed contains precisely the even elements in the $I_i$. Secondly, consider the holes $H_i$ which consist of an even number of elements. Then we put the same points into both sets $B$ and $C$, and these sets will consist precisely of the odd elements in the $H_i$. It remains to take up the odd-elements sets $I_i$ and $H_i$. Then the situation is slightly more complicated. Let us first consider the set of all the odd-elements islands $I_i$. If the latter set is empty, we do not have anything to do. Otherwise, there is the first $i$ $(i \in N)$, some $i_1$, such that $I_{i_1}$ is the first odd-elements island. Further, we construct the set $B$ from all the odd-ordered elements of $I_{i_1}$ and the set $C$ from all the even-ordered elements of $I_{i_1}$. Then all odd-elements islands $I_{i_{2k+1}}$ will be treated equally. Next, we are to take the points from the islands of the type $I_{i_{2k}}$. In this case we distribute the odd-ordered points of $I_{i_{2k}}$ into the set $C$ and the even-ordered elements to the set $B$. Finally, let us consider the set of all the odd-elements holes $H_i$. If the latter set is empty, we do not have anything to do. Otherwise there is the first $i$ $(i \in N)$, some $i_1$, such that $H_{i_1}$ is the first odd-elements hole. Then we construct both the sets $B$ and $C$ from the odd-ordered elements of $H_{i_1}$. Then all holes $H_{i_{2k+1}}$ will be treated equally. Further, we are to take points from the holes of the type $H_{i_{2k}}$. In this case we construct the both sets $B$ and $C$ from the even-ordered points of $H_{i_{2k}}$. By the construction of the sets $B$ and $C$, it is not difficult to check that $s(B) = s(C) = \frac{1}{2}$.

The statement 5) can be proved by the classical result on the Banach limits (see e.g. [1], p. 41).

In order to show the statement 6), let us first see that there is a collection of $2^{\aleph_0}$ almost disjoint subsets of $\Omega$. An easy proof of this known result can be obtain as follows (see also [11]). Identify $\Omega$ with the set of all rational numbers $Q$. For each irrational number $r \in R$, let us choose a sequence $(q_n^r)_{n \in N}$ of rational numbers that converges to $r$. Consider the family of the previously constructed sequences $S_r = \{(q_n^r), n \in N\}$. Let us take the collection $\mathcal{S} = \{S_r, r \text{ is an irrational number}\}$. Then this collection is an almost disjoint family of subsets of $Q$ with the cardinality $2^{\aleph_0}$. Going back to $\Omega$, we have the required almost disjoint collection. Continuing our argument let us first observe that each infinite subset of $\Omega$ contains a subset $M$ with $s(M) = 0$. For each $S_r$ choose such a set $M_r$. Write $\mathcal{M} = \{M_r, r \text{ is an irrational number}\}$. Since the sets of $\mathcal{S}$ are pairwisely eventually almost disjoint, so are the sets of $\mathcal{M}$. This proves the first part of statement 6). To complete the proof, we only need to observe that $\mathcal{F}$ consists of central elements of $(\Omega, \mathcal{L})$ and hence the factor $\mathcal{L}/\mathcal{F}$ gives us a quantum logic [26]. One only takes into account that the pseudocomplementedness ($a \leq b' \iff a \wedge b = 0$) can be equivalently expressed by the equivalence ($a \wedge b$ exists $\iff$ $a$ is compatible with $b$, see e.g. [18]). The rest is easy. □

The results above indicate certain potential for the interpretation of the d-logic in the realm of quantum logics. In concluding our paper, let us for instance note a link of the d-logic with the projection logic $L(H)$. Let us take an orthonormal basis, $\mathcal{E} = \{v_i, i \in N\}$, in $H$. Then $\mathcal{E}$ understood as elements of $L(H)$ generates a Boolean subalgebra $B_{\mathcal{E}}$ of $L(H)$. Obviously, $B_{\mathcal{E}}$ is Boolean isomorphic to $\exp N$. Consider a state $t$ on $L(H)$. Let us call it an *$\mathcal{E}$-d-state* if the restriction of $t$ on $B_{\mathcal{E}}$ is a Banach extension of the state $s$ on the d-logic understood as being underlied by the set $\mathcal{E}$. Observe that the $\mathcal{E}$-d-states exist. Indeed, a Banach extension of $s$ considered as a state on $\exp \mathcal{E}$ can be extended over the entire $L(H)$ [23]. It may be interesting to see what the size of the closure of the convex hull, $\overline{\text{conv}(T)}$, comes to ($T$ is the set of all $\mathcal{E}$-d-states for all choices of orthonormal bases $\mathcal{E}$). How smaller this $\overline{\text{conv}(T)}$ is than the entire state space of $L(H)$?

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