

Pseudo-Hermitian Systems with \mathcal{PT} -Symmetry: Degeneracy and Krein Space

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Abstract We show in the present paper that pseudo-Hermitian Hamiltonian systems with even \mathcal{PT} -symmetry ($\mathcal{P}^2 = 1, \mathcal{T}^2 = 1$) admit a degeneracy structure. This kind of degeneracy is expected traditionally in the odd \mathcal{PT} -symmetric systems ($\mathcal{P}^2 = 1, \mathcal{T}^2 = -1$) which is appropriate to the fermions (Scolarici and Solombrino, Phys. Lett. A **303**, 239 2002; Jones-Smith and Mathur, Phys. Rev. A **82**, 042101 2010). We establish that the pseudo-Hermitian Hamiltonians with even \mathcal{PT} -symmetry admit a degeneracy structure if the operator \mathcal{PT} anticommutes with the metric operator η_σ which is necessarily indefinite. We also show that the Krein space formulation of the Hilbert space is the convenient framework for the implementation of unbroken \mathcal{PT} -symmetry. These general results are illustrated with great details for four-level pseudo-Hermitian Hamiltonian with even \mathcal{PT} -symmetry.

Keywords Pseudo-Hermiticity · \mathcal{PT} -symmetry · Degeneracy · Krein space

1 Introduction

The research works which deal with pseudo-Hermitian and \mathcal{PT} -symmetric Hamiltonians have received a great deal of interest over the last two decades [3–10, 12, 13]. In this

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context, Sato et al. [14] established a generalization of the Kramers degeneracy to pseudo-Hermitian Hamiltonians admitting even time-reversal symmetry ($\mathcal{T}^2 = 1$). This extension is achieved using the mathematical structure of split-quaternions¹ instead of quaternions, usually adopted in the case of Hermitian Hamiltonians with odd time-reversal symmetry ($\mathcal{T}^2 = -1$) [15]. In a recent paper [16], we have found that the metric operator for the pseudo-Hermitian Hamiltonian H that allows the realization of the generalized Kramers degeneracy is necessarily indefinite. We have further shown that such H with real spectrum also possesses odd antilinear symmetry induced from the existing odd time-reversal symmetry of its Hermitian counterpart h , so that the generalized Kramers degeneracy of H is in fact crypto-Hermitian Kramers degeneracy [16].

On the other hand, as it is well known in \mathcal{PT} quantum theory [5, 12], due to the anti-linearity of \mathcal{PT} , the eigenstates of a \mathcal{PT} -symmetric Hamiltonian H , may or may not be the eigenstates of \mathcal{PT} . If every eigenstate of H is also an eigenstate of \mathcal{PT} , then we have an unbroken \mathcal{PT} -symmetry, which corresponds to real eigenvalues. Conversely, if some of the eigenstates of a H are not simultaneously eigenstates of \mathcal{PT} , then we have a broken \mathcal{PT} -symmetry, which corresponds to complex conjugate eigenvalues. The \mathcal{PT} -broken and \mathcal{PT} -unbroken phases are separated by exceptional points (EPs) [17]. For \mathcal{PT} -symmetric systems, the exceptional points are an obligatory passage in the \mathcal{PT} -broken and \mathcal{PT} -symmetric phase transitions. In this context, Berry has shown [18] that non-Hermitian physics differs radically from Hermitian physics in the presence of degeneracies. This non-Hermitian behavior has been further illustrated in several physical examples [18]. Recently, the role of degeneracy in \mathcal{PT} -symmetry breaking has been also subject of investigations [19]. The degeneracy for odd \mathcal{PT} -symmetric systems ($\mathcal{P}^2 = 1, \mathcal{T}^2 = -1$) appropriate to the fermions was first proven by Sclariaci and Solombrino in [1], and studied afterwards by Jones-Smith and Mathur [2] who extended \mathcal{PT} -symmetric quantum mechanics to the case of odd time-reversal symmetry. It has been established that an analog of Kramers degeneracy exists also for odd \mathcal{PT} -symmetric systems, and an unbroken \mathcal{PT} -symmetry can exist if we assemble the two column vectors of the \mathcal{PT} doublets in a single quaternionic column [2].

The purpose of the present paper is to extend the non-Hermitian degeneracy behavior developed for odd \mathcal{PT} -symmetric systems [2] to even \mathcal{PT} -symmetric ones ($\mathcal{P}^2 = 1, \mathcal{T}^2 = 1$).

The paper is organized as follows. In Section 2, we first analyze the existence of the degeneracy structure for pseudo-Hermitian Hamiltonian systems with even \mathcal{PT} -symmetry. We will establish that the degeneracy structure exists if the \mathcal{PT} operator anticommutes with the metric operator η_σ , which is necessarily indefinite metric. Moreover, we emphasize in Section 3 the role of the Krein space formulation of the Hilbert space for restoring an unbroken \mathcal{PT} -symmetry. This degeneracy structure is thus well implemented in the Krein space formulation of the Hilbert space. In the purpose of illustration of the above general results, we study in great details in Section 4, the pseudo-Hermitian four-level Hamiltonian invariant under the even \mathcal{PT} -symmetry. The paper ends with conclusion and outlook.

¹The quaternion algebra is generated by the 2×2 unit matrix σ_0 and the pure imaginary complex number i multiplied by the SU(2) Pauli matrices ($i\sigma_x, i\sigma_y, i\sigma_z$), while the split-quaternion algebra is generated by the 2×2 unit matrix σ_0 and the pure imaginary complex number i multiplied by the SU(1,1) Pauli matrices ($-\sigma_x, -\sigma_y, i\sigma_z$).

2 Pseudo-Hermiticity, \mathcal{PT} -Symmetry and Degeneracy

We start our analysis by considering a diagonalizable non-Hermitian Hamiltonian H with a real discrete spectrum. The associated complete biorthonormal eigenbasis $\{|\psi_{n,a}\rangle, |\phi_{n,a}\rangle\}$ satisfies by definition [6, 9, 10, 20, 21],

$$H|\psi_{n,a}\rangle = E_n|\psi_{n,a}\rangle, H^\dagger|\phi_{n,a}\rangle = E_n^*|\phi_{n,a}\rangle, \tag{1}$$

$$\langle\phi_{m,b}|\psi_{n,a}\rangle = \delta_{mn}\delta_{ab}, \tag{2}$$

$$\sum_n \sum_{a=1}^{d_n} |\phi_{n,a}\rangle \langle\psi_{n,a}| = \sum_n \sum_{a=1}^{d_n} |\psi_{n,a}\rangle \langle\phi_{n,a}| = \mathbf{1}, \tag{3}$$

$$H = \sum_n \sum_{a=1}^{d_n} E_n |\psi_{n,a}\rangle \langle\phi_{n,a}|, \tag{4}$$

where n and a, b are, respectively, the spectral and degeneracy labels, d_n is the multiplicity (degree of degeneracy) of E_n . The Hamiltonian H is taken to be pseudo-Hermitian, this means that H satisfies the relation [6, 9]

$$H^\dagger = \eta H \eta^{-1}, \tag{5}$$

where η is Hermitian, linear invertible operator. As we deal with real eigenvalues of H , for such H there exists a positive definite metric operator η_+ given by [10],

$$\eta_+ = \sum_n \sum_{a=1}^{d_n} |\phi_{n,a}\rangle \langle\phi_{n,a}|, \tag{6}$$

and its inverse

$$\eta_+^{-1} = \sum_n \sum_{a=1}^{d_n} |\psi_{n,a}\rangle \langle\psi_{n,a}|, \tag{7}$$

with the properties [6, 10]

$$|\phi_{n,a}\rangle = \eta_+ |\psi_{n,a}\rangle, |\psi_{n,a}\rangle = \eta_+^{-1} |\phi_{n,a}\rangle. \tag{8}$$

As established in [10], besides η_+ , there are Hermitian invertible operators η_σ that are also associated with the same biorthonormal system (1)–(4) and satisfy the pseudo-Hermitian relation (5). η_σ and its inverse are given by [10]:

$$\eta_\sigma = \sum_n \sum_{a=1}^{d_n} \sigma_n^a |\phi_{n,a}\rangle \langle\phi_{n,a}|, \tag{9}$$

and

$$\eta_\sigma^{-1} = \sum_n \sum_{a=1}^{d_n} \sigma_n^a |\psi_{n,a}\rangle \langle\psi_{n,a}|, \tag{10}$$

where $\sigma = (\sigma_n^a)$ is a sequence of signs $\sigma_n^a = \pm 1$. Thus, η_σ and η_+ are both metric operators associated to the Hamiltonian H . Moreover, η_σ and η_+ are linked via the following relation [11],

$$\mathcal{C}_\sigma = \eta_+^{-1} \eta_\sigma, \tag{11}$$

or

$$\eta_+ = \eta_\sigma \mathcal{C}_\sigma, \tag{12}$$

where C_σ is the grading operator, which is a linear invertible involution operator² associated with H , given by [9, 10],

$$C_\sigma = \sum_n \sum_{a=1}^{d_n} \sigma_n^a |\psi_{n,a}\rangle \langle \phi_{n,a}|, \text{ with } C_\sigma^2 = \mathbf{1}, \tag{13}$$

with the properties [9],

$$C_\sigma |\psi_{n,a}\rangle = \sigma_n^a |\psi_{n,a}\rangle. \tag{14}$$

Since C_σ is an involution operator ($C_\sigma^2 = \mathbf{1}$), in view of (14) one deduce that $\sigma_n^{a^2} = 1$. From the equations (11) and (14), one can express the relations between $|\psi_{n,a}\rangle$ and $|\phi_{n,a}\rangle$ given in (8) as follow,

$$|\phi_{n,a}\rangle = \eta_\sigma C_\sigma |\psi_{n,a}\rangle = \sigma_n^a \eta_\sigma |\psi_{n,a}\rangle, \quad |\psi_{n,a}\rangle = \sigma_n^a \eta_\sigma^{-1} |\phi_{n,a}\rangle. \tag{15}$$

Furthermore, we assume that H is invariant under the even \mathcal{PT} -symmetry, i.e $[H, \mathcal{PT}] = 0$, where \mathcal{P} and \mathcal{T} are parity and even time-reversal operators respectively, ($\mathcal{P}^2 = 1, \mathcal{T}^2 = 1$). Here we define the parity operator \mathcal{P} by following the definition given in [2]. The action of \mathcal{P} to any state ψ is to multiply ψ by the matrix S which is a real N -dimensional matrix given by [2],

$$S = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{16}$$

where I denotes the $(N/2)$ -dimensional identity matrix. For the time-reversal operator, one follows the definition given in [14] as $\mathcal{T} = ZK$, with K being a complex conjugation operator, Z is an unitary matrix which can be chosen as a real matrix with all the diagonal terms equal to a 2×2 Pauli matrix σ_x and all the off-diagonal terms equal to zero [14], namely

$$Z = \begin{pmatrix} \sigma_x & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \sigma_x \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{17}$$

We have $\mathcal{T}^2 = 1$. It is useful to note that as established [2], the non-Hermitian Hamiltonians which are invariant under the odd \mathcal{PT} -symmetry, admit Kramers degeneracy implemented by the mathematical structure of quaternions [15]. This degeneracy is expected because the time-reversal operator is odd ($\mathcal{T}^2 = -1$).

We propose to show in the following that a degeneracy exists also for pseudo-Hermitian Hamiltonians with even \mathcal{PT} -symmetry. In order to show the degeneracy in the eigenvalues of H , we show that the eigenstates $|\psi_{n,a}\rangle$ and $\mathcal{PT}|\psi_{n,a}\rangle$ which correspond to the same eigenvalue E_n are linearly independent. As shown in [22], any antiunitary operator θ can be written as $\theta = UK$, where U is an unitary operator and K is the conjugation operator. Here we have

$$\theta = \mathcal{PT} = SZK = UK, \tag{18}$$

where $U = SZ$ is the unitary operator. Moreover, according to [14], one deduce from the antiunitary property³ of \mathcal{PT} , that pseudo-Hermiticity is consistent with the \mathcal{PT} -symmetry if the metric operator commutes or anticommutes with the \mathcal{PT} operator. Thus, as in [14],

²The operator C_σ is denoted by \mathcal{S}_σ in reference [10]. See equation (43) of [10] and the proposition 4 just before this equation.

³The antiunitary property means that for any operator antiunitary denoted θ , we have $\langle \theta x | \theta y \rangle = \langle y | x \rangle$.

the metric operators are classified into two categories: The first which commutes with \mathcal{PT} and the second which anticommutes with \mathcal{PT} .

Now we compute $\langle \phi_{n,a} | \mathcal{PT} \psi_{n,a} \rangle$, where $|\psi_{n,a}\rangle$ and $|\phi_{n,a}\rangle$ form a complete biorthonormal eigenbasis (1)–(3). By using the antiunitary property of \mathcal{PT} , with the fact that $(\mathcal{PT})^2 = 1$ and the relations between $|\psi_{n,a}\rangle$ and $|\phi_{n,a}\rangle$ given in (15), we have

$$\begin{aligned} \langle \phi_{n,a} | \mathcal{PT} \psi_{n,a} \rangle &= \langle (\mathcal{PT})^2 \psi_{n,a} | \mathcal{PT} \phi_{n,a} \rangle = \langle \psi_{n,a} | \mathcal{PT} \phi_{n,a} \rangle = \langle \phi_{n,a} \sigma_n^a \eta_\sigma^{-1} | \mathcal{PT} \sigma_n^a \eta_\sigma \psi_{n,a} \rangle \\ &= \sigma_n^{a2} \langle \phi_{n,a} \eta_\sigma^{-1} | \mathcal{PT} \eta_\sigma \psi_{n,a} \rangle = \langle \phi_{n,a} \eta_\sigma^{-1} | \mathcal{PT} \eta_\sigma \psi_{n,a} \rangle. \end{aligned} \tag{19}$$

Here we have the term σ_n^{a2} which appears in fact in the last step of (19), this term has no effect since $\sigma_n^{a2} = 1$. If \mathcal{PT} and η_σ anticommutes, one finds from the last equation that

$$\langle \phi_{n,a} | \mathcal{PT} \psi_{n,a} \rangle = -\langle \phi_{n,a} \eta_\sigma^{-1} | \eta_\sigma \mathcal{PT} \psi_{n,a} \rangle. \tag{20}$$

The Hermiticity of η_σ leads to

$$\langle \phi_{n,a} | \mathcal{PT} \psi_{n,a} \rangle = -\langle \phi_{n,a} | \mathcal{PT} \psi_{n,a} \rangle. \tag{21}$$

We have thereby $\langle \phi_{n,a} | \mathcal{PT} \psi_{n,a} \rangle = 0$. On the other hand, from (2) we have $\langle \phi_{n,a} | \psi_{n,a} \rangle = 1$, so, one deduces that $|\psi_{n,a}\rangle$ and $\mathcal{PT}|\psi_{n,a}\rangle$ are linearly independent. Consequently, as in the odd \mathcal{PT} -symmetric case [1, 2], we have also two fold degeneracy in the eigenstates of H although the \mathcal{PT} -symmetry is even. Here we point out that the \mathcal{PT} doublet $|\psi_{n,a}\rangle$ and $\mathcal{PT}|\psi_{n,a}\rangle$ are linearly independent but they are not orthogonal as in the odd \mathcal{PT} -symmetric case. Now we show that in the above described scheme the indefiniteness feature of the metric operator η_σ . In this aim, we shall show that the η_σ -norm related to the η_σ -inner product $\langle \cdot | \cdot \rangle_{\eta_\sigma} = \langle \cdot | \eta_\sigma \cdot \rangle$ of the eigenstates of H is indefinite [26–29]. Thus we compute the norms $\langle \psi_{n,a} | \psi_{n,a} \rangle_{\eta_\sigma}$ and $\langle \mathcal{PT} \psi_{n,a} | \mathcal{PT} \psi_{n,a} \rangle_{\eta_\sigma}$. We have

$$\langle \mathcal{PT} \psi_{n,a} | \mathcal{PT} \psi_{n,a} \rangle_{\eta_\sigma} = \langle \mathcal{PT} \psi_{n,a} | \eta_\sigma \mathcal{PT} \psi_{n,a} \rangle. \tag{22}$$

By using the anticommutation relation between η_σ and \mathcal{PT} , the antiunitary property of \mathcal{PT} and $(\mathcal{PT})^2 = 1$, the Hermiticity of η_σ , one obtains from the last equation that

$$\begin{aligned} \langle \mathcal{PT} \psi_{n,a} | \mathcal{PT} \psi_{n,a} \rangle_{\eta_\sigma} &= -\langle \mathcal{PT} \psi_{n,a} | \mathcal{PT} \eta_\sigma \psi_{n,a} \rangle = -\langle (\mathcal{PT})^2 \eta_\sigma \psi_{n,a} | (\mathcal{PT})^2 \psi_{n,a} \rangle \\ &= -\langle \eta_\sigma \psi_{n,a} | \psi_{n,a} \rangle = -\langle \psi_{n,a} | \eta_\sigma \psi_{n,a} \rangle = -\langle \psi_{n,a} | \psi_{n,a} \rangle_{\eta_\sigma}. \end{aligned} \tag{23}$$

Thereby, the norms $\langle \psi_{n,a} | \psi_{n,a} \rangle_{\eta_\sigma}$ and $\langle \mathcal{PT} \psi_{n,a} | \mathcal{PT} \psi_{n,a} \rangle_{\eta_\sigma}$ are of opposite signs, which confirm the indefiniteness feature η_σ . Due to this degeneracy behavior, a question arises: Is the \mathcal{PT} -symmetry in the broken or unbroken phase? Traditionally, the \mathcal{PT} -broken and \mathcal{PT} -unbroken phases are separated by exceptional points which are non-Hermitian degenerate points with coalesced eigenvalues and eigenvectors. In our case, we deal with an other kind of degeneracy different from the exceptional points degeneracy.

In conclusion of this section, we have established that the pseudo-Hermitian Hamiltonians with even \mathcal{PT} -symmetry admit a degeneracy structure if the operator \mathcal{PT} anticommutes with the metric operator η_σ which is necessarily indefinite.

3 Krein Space Formulation

Let us show how we achieve an unbroken \mathcal{PT} -symmetry. The idea is the passage to the Krein space formulation of the Hilbert space. We recall that in the case of even \mathcal{PT} -symmetry, \mathcal{PT} is said to be unbroken if the states $|\psi_{n,a}\rangle$ are invariant under \mathcal{PT} , i.e

$\mathcal{PT}|\psi_{n,a}\rangle = |\psi_{n,a}\rangle$. However, the situation is different here, although the \mathcal{PT} -symmetry is even, the eigenstates $|\psi_{n,a}\rangle$ are not invariant under \mathcal{PT} , i.e. $\mathcal{PT}|\psi_{n,a}\rangle \neq |\psi_{n,a}\rangle$ because we have established previously that the \mathcal{PT} doublets $|\psi_{n,a}\rangle$ and $\mathcal{PT}|\psi_{n,a}\rangle$ are linearly independent. Then the \mathcal{PT} -symmetry is broken in the Hilbert space spanned by the eigenstates of H . The way out from this dead end is in fact to introduce the Krein space formulation of the Hilbert space [24, 25]. Indeed, let us define new eigenstate $|\chi_{n,a}\rangle$ spanned by the \mathcal{PT} doublets $|\psi_{n,a}\rangle$ and $\mathcal{PT}|\psi_{n,a}\rangle$ as follows,

$$|\chi_{n,a}\rangle = \frac{i}{\sqrt{2}} (|\psi_{n,a}\rangle - \mathcal{PT}|\psi_{n,a}\rangle), \quad (24)$$

with the following properties, (i) the $|\chi_{n,a}\rangle$ are eigenstates of H with the same eigenvalue E_n , (ii) the states $|\chi_{n,a}\rangle$ are invariant under \mathcal{PT} , i.e

$$\mathcal{PT}|\chi_{n,a}\rangle = -\frac{i}{\sqrt{2}} \mathcal{PT} (|\psi_{n,a}\rangle - \mathcal{PT}|\psi_{n,a}\rangle) = |\chi_{n,a}\rangle, \quad (25)$$

we have used the anti-linearity and the involution properties of \mathcal{PT} . Now we show that the Hilbert space \mathcal{K} spanned by the states $|\chi_{n,a}\rangle$ is a Krein space [24, 25]. \mathcal{K} possess the following properties: (i) \mathcal{K} is endowed with the indefinite inner product $\langle \cdot | \cdot \rangle_{\eta_\sigma}$ defined in Section 2. (ii) \mathcal{K} can be decomposed in a pair of vector subspaces \mathcal{H}_\pm , where \mathcal{H}_+ and \mathcal{H}_- are spanned by the \mathcal{PT} doublets $|\psi_{n,a}\rangle$ and $\mathcal{PT}|\psi_{n,a}\rangle$ respectively; $\mathcal{K} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where \oplus means direct sum which means that for all element $f \in \mathcal{K}$ there are unique $f_\pm \in \mathcal{H}_\pm$ such that $f = f_+ + f_-$. (iii) In the purpose of showing that \mathcal{H}_+ and \mathcal{H}_- are orthogonal, we calculate the inner product $\langle \psi_{n,a} | \mathcal{PT} \psi_{n,a} \rangle_{\eta_\sigma}$ for any elements $|\psi_{n,a}\rangle \in \mathcal{H}_+$ and $\mathcal{PT}|\psi_{n,a}\rangle \in \mathcal{H}_-$, we have

$$\begin{aligned} \langle \psi_{n,a} | \mathcal{PT} \psi_{n,a} \rangle_{\eta_\sigma} &= \langle \psi_{n,a} | \eta_\sigma \mathcal{PT} \psi_{n,a} \rangle = -\langle \psi_{n,a} | \mathcal{PT} \eta_\sigma \psi_{n,a} \rangle = -\langle (\mathcal{PT})^2 \eta_\sigma \psi_{n,a} | \mathcal{PT} \psi_{n,a} \rangle \\ &= -\langle \eta_\sigma \psi_{n,a} | \mathcal{PT} \psi_{n,a} \rangle = -\langle \psi_{n,a} | \eta_\sigma \mathcal{PT} \psi_{n,a} \rangle = -\langle \psi_{n,a} | \mathcal{PT} \psi_{n,a} \rangle_{\eta_\sigma}, \end{aligned} \quad (26)$$

thereby $\langle \psi_{n,a} | \mathcal{PT} \psi_{n,a} \rangle_{\eta_\sigma} = 0$. This means that \mathcal{H}_+ and \mathcal{H}_- are orthogonal. Here, we have again used the anticommutation relation between η_σ and \mathcal{PT} and the antiunitary property of \mathcal{PT} and $(\mathcal{PT})^2 = 1$. The Hilbert space \mathcal{K} is therefore a Krein space. It should be noticed that the condition of unbroken \mathcal{PT} -symmetry in the odd case is different to our even \mathcal{PT} -symmetry case. Indeed, in the odd case the \mathcal{PT} doublets are assembled in two column vectors ($|\psi_{n,a}\rangle, \mathcal{PT}|\psi_{n,a}\rangle$) which form a single component column of quaternions [2]. It is not possible to assemble our \mathcal{PT} doublets in two column vectors, because the single component column obtained is a split-quaternion which possesses a structure different from the quaternion [14].

4 Illustration

In this section we illustrate the above general results with an example. We consider the four-level model described by the following non-Hermitian Hamiltonian:

$$H = \begin{pmatrix} a & ib \\ ic & -a \end{pmatrix}, \quad (27)$$

where $b = b_0\sigma_0 + b_1\sigma_x + b_2\sigma_y - ib_3\sigma_z$ and $c = b_0\sigma_0 - b_1\sigma_x - b_2\sigma_y + ib_3\sigma_z$ are real split-quaternions⁴, and $a = a_0\sigma_0$ is the real split-quaternion proportional to the identity, the σ_k ($k = x, y, z$) are the Pauli matrices. By setting $A = b_1 + ib_2$ and $B = b_0 + ib_3$, this Hamiltonian H can also be written as a four-level Hamiltonian as follows:

$$H = \begin{pmatrix} a_0 & 0 & iB^* & iA^* \\ 0 & a_0 & iA & iB \\ iB & -iA^* & -a_0 & 0 \\ -iA & iB^* & 0 & -a_0 \end{pmatrix}, \tag{28}$$

where A^* and B^* are complex conjugates of A and B respectively. The Hamiltonian H , in (28), satisfies the following properties: (i) H is pseudo-Hermitian, this means that H satisfies the relation $H^\dagger = \eta H \eta^{-1}$, (ii) H is invariant under the even \mathcal{PT} -symmetry, i.e $[H, \mathcal{PT}] = 0$, with $\mathcal{P}^2 = 1, \mathcal{T}^2 = 1$, where \mathcal{P} and \mathcal{T} are given in the case of the four-level system by [2, 16],

$$\mathcal{P} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \mathcal{T} = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} K, \tag{29}$$

where I_2 is the 2×2 identity matrix, σ_x is Pauli matrix, K being the complex conjugation operator. Moreover, H admits an indefinite metric operator η_σ which anticommutes with \mathcal{PT} , η_σ is given explicitly by,

$$\eta_\sigma = \begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix}. \tag{30}$$

The eigenvalues of H are

$$E_\pm = \pm\Omega, \quad \text{with} \quad \Omega = \sqrt{a_0^2 + |A|^2 - |B|^2}.$$

This eigenvalues are indeed twofold degenerate. The Hamiltonian H represents a new class of *even \mathcal{PT} -symmetric Hamiltonians with degeneracy*. We remark that H is asymmetric. We deal with real eigenvalues, i.e.

$$a_0^2 + |A|^2 > |B|^2. \tag{31}$$

The \mathcal{PT} doublets $(|\psi_{-+}\rangle, |\psi_{--}\rangle)$ and $(|\psi_{++}\rangle, |\psi_{+-}\rangle)$ associated to the negative and positive eigenvalues respectively are given as follows:

For the negative eigenvalue $E_- = -\Omega$:

$$|\psi_{-+}\rangle = k \begin{pmatrix} iA^* \\ iB \\ 0 \\ -(\Omega + a_0) \end{pmatrix}, |\psi_{--}\rangle = \mathcal{PT} |\psi_{-+}\rangle = k \begin{pmatrix} -iB^* \\ -iA \\ (\Omega + a_0) \\ 0 \end{pmatrix}, \tag{32}$$

For the positive eigenvalue $E_+ = \Omega$:

$$|\psi_{++}\rangle = k \begin{pmatrix} (\Omega + a_0) \\ 0 \\ iB \\ -iA \end{pmatrix}, |\psi_{+-}\rangle = \mathcal{PT} |\psi_{++}\rangle = k \begin{pmatrix} 0 \\ \Omega + a_0 \\ -iA^* \\ iB^* \end{pmatrix}. \tag{33}$$

⁴The split-quaternion algebra is generated by the 2×2 unit matrix σ_0 and the pure imaginary complex number i multiplied by the $SU(1,1)$ Pauli matrices $(-\sigma_x, -\sigma_y, i\sigma_z)$.

The eigenstates $(|\phi_{-+}\rangle, |\phi_{--}\rangle)$ and $(|\phi_{++}\rangle, |\phi_{+-}\rangle)$ associated to H^\dagger are obtained by the action of η_σ given in (30) to the eigenstates of H , they are given for the negative eigenvalue $E_- = -\Omega$ by:

$$|\phi_{-+}\rangle = k \begin{pmatrix} iA^* \\ -iB \\ 0 \\ -(\Omega + a_0) \end{pmatrix}, |\phi_{--}\rangle = k \begin{pmatrix} -iB^* \\ iA \\ -(\Omega + a_0) \\ 0 \end{pmatrix}, \tag{34}$$

and for the positive eigenvalue $E_+ = \Omega$ by:

$$|\phi_{++}\rangle = k \begin{pmatrix} \Omega + a_0 \\ 0 \\ -iB \\ -iA \end{pmatrix}, |\phi_{+-}\rangle = k \begin{pmatrix} 0 \\ -(\Omega + a_0) \\ iA^* \\ iB^* \end{pmatrix}, \tag{35}$$

where k is the normalization constant fixed by the requirement that:

$$2\Omega (\Omega + a_0) |k|^2 = 1. \tag{36}$$

These states satisfy the abnormal relations which is a consequence of the indefinite metric η_σ given in (30),

$$\langle \psi_{++} | \phi_{++} \rangle = 1, \langle \psi_{+-} | \phi_{+-} \rangle = -1, \langle \psi_{-+} | \phi_{-+} \rangle = 1, \langle \psi_{--} | \phi_{--} \rangle = -1, \\ \langle \psi_{m\alpha} | \phi_{\bar{m}\bar{\alpha}} \rangle = 0, \langle \psi_{m\alpha} | \phi_{\bar{m}\alpha} \rangle = 0, \langle \psi_{m\alpha} | \phi_{m\bar{\alpha}} \rangle = 0. \tag{37}$$

where $\alpha = \pm, m = \pm, \bar{m}$ and $\bar{\alpha}$ are the opposite signs of m and α respectively. These states satisfy also the relations

$$|\psi_{++}\rangle\langle\phi_{++}| - |\psi_{+-}\rangle\langle\phi_{+-}| + |\psi_{-+}\rangle\langle\phi_{-+}| - |\psi_{--}\rangle\langle\phi_{--}| = \mathbf{1}. \tag{38}$$

The (indefinite) η_σ -norms of the \mathcal{PT} doublets are given by

$$\langle \psi_{++} | \psi_{++} \rangle_{\eta_\sigma} = \langle \psi_{++} | \phi_{++} \rangle = 1, \quad \langle \psi_{+-} | \psi_{+-} \rangle_{\eta_\sigma} = \langle \psi_{+-} | \phi_{+-} \rangle = -1, \tag{39}$$

$$\langle \psi_{-+} | \psi_{-+} \rangle_{\eta_\sigma} = \langle \psi_{-+} | \phi_{-+} \rangle = 1, \quad \langle \psi_{--} | \psi_{--} \rangle_{\eta_\sigma} = \langle \psi_{--} | \phi_{--} \rangle = -1. \tag{40}$$

We see that the eigenstates $|\psi_{n,a}\rangle$ are not invariant under \mathcal{PT} , i.e $\mathcal{PT}|\psi_{n,a}\rangle \neq |\psi_{n,a}\rangle$, the \mathcal{PT} -symmetry is therefore broken. In order to restore an unbroken \mathcal{PT} -symmetry, one introduces the Krein space formulation of the Hilbert space. The Krein space \mathcal{K} is spanned by the states $|\chi_{-+}\rangle$ and $|\chi_{++}\rangle$ which are linear combination of the \mathcal{PT} doublets $(|\psi_{-+}\rangle, \mathcal{PT}|\psi_{-+}\rangle)$ and $(|\psi_{++}\rangle, \mathcal{PT}|\psi_{++}\rangle)$ associated to the negative and positive eigenvalue respectively. Thus, in the Krein space \mathcal{K} , the eigenstates are given by,

For the negative eigenvalue $E = -\Omega$:

$$|\chi_{-+}\rangle = \frac{i}{\sqrt{2}} (|\psi_{-+}\rangle - \mathcal{PT}|\psi_{-+}\rangle), \tag{41}$$

for the positive eigenvalue $E_+ = \Omega$:

$$|\chi_{++}\rangle = \frac{i}{\sqrt{2}} (|\psi_{++}\rangle - \mathcal{PT}|\psi_{++}\rangle). \tag{42}$$

We see that states $|\chi_{-+}\rangle$ and $|\chi_{++}\rangle$ are invariant under \mathcal{PT} , i.e

$$\mathcal{PT}|\chi_{-+}\rangle = -\frac{i}{\sqrt{2}} \mathcal{PT} (|\psi_{-+}\rangle - \mathcal{PT}|\psi_{-+}\rangle) = |\chi_{-+}\rangle, \tag{43}$$

and

$$\mathcal{PT}|\chi_{++}\rangle = -\frac{i}{\sqrt{2}} \mathcal{PT} (|\psi_{++}\rangle - \mathcal{PT}|\psi_{++}\rangle) = |\chi_{++}\rangle. \tag{44}$$

One has therefore achieved an unbroken \mathcal{PT} -symmetry in the Krein space \mathcal{K} .

5 Conclusion and Outlook

We have established in the present paper a new kind of degeneracy structure which is due to the non-Hermitian behavior of the system. We have shown that the pseudo-Hermitian Hamiltonians with real eigenvalues and even \mathcal{PT} -symmetry admit a degeneracy structure if the operator \mathcal{PT} anticommutes with the metric operator η_σ which is necessarily indefinite. We have also shown that the Krein space formulation of the Hilbert space is the convenient framework for the implementation of unbroken \mathcal{PT} -symmetry for our system. Let us discuss some implications and outlook of our analysis.

- (1): It is useful to point out that the \mathcal{PT} -symmetric four-level Hamiltonian H given in (27) represents a new class of \mathcal{PT} -symmetric Hamiltonians written in split-quaternionic form which represent the most general traceless Hamiltonian matrix in the case of even \mathcal{PT} -symmetry by generalizing the time reversal operator to include a matrix multiplying the complex conjugation operator. The study of further features of this Hamiltonian will be interesting. For instance the construction of the \mathcal{CPT} inner product in the form of Mostafazadeh η_+ inner product [30], with $\eta_+ \equiv \mathcal{PC}$.
- (2): The generalization to even \mathcal{PT} -symmetry of the \mathcal{PT} - and \mathcal{CPT} -symmetric representations of fermionic algebras developed by Bender [31] and one of the actual authors [23] in the odd \mathcal{PT} -symmetric case, will be also a part of our future investigations.
- (3): It is useful to point out that the *graded time-reversal operator in fermion Fock space* is also an interesting outlook. In this case, the Hilbert space is neither even nor odd with respect to time-reversal symmetry but rather has a graded structure with regard to time reversal. The action of time reversal operator \mathcal{T} may as usual be represented by $\mathcal{T}\varphi = L\varphi^*$, where the unitary operator has however a block diagonal structure

$$L = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}, \quad (45)$$

with $L_+^2 = 1$, $L_-^2 = -1$. Thus time-reversal is even in the bosonic subspace; it is odd in the fermionic subspace. It is a general feature of the *graded Fock space of fermions* that decomposes into two subspaces which are even and odd with respect to time-reversal. Thus it will be interesting to extend our analysis and all the works in the literature which deal with time-reversal symmetry to the case of time-reversal symmetry in Fock space of fermions with a graded form which can be a subject of interest for \mathcal{PT} community.

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