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A Note on Orthomodular Lattices

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Abstract We introduce a new identity equivalent to the orthomodular law in every ortholattice.

Keywords Lattice with complementation · Ortholattice · Orthomodular law

1 Introduction

It is well known that every modular ortholattice is orthomodular [1, 2]. In the present work, we introduce a simple identity being in two variables, which easily follows from the orthomodular law in an ortholattice and, on the other hand, can be in fact equivalent to the orthomodular law in every ortholattice.

2 Preliminaries

A *bounded lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, 0, 1\rangle$ of type $\langle 2, 2, 0, 0\rangle$ such that:

i) $\langle L, \wedge, \vee\rangle$ is a lattice,
ii) $x \wedge 0 \approx 0$ and $x \vee 1 \approx 1$.

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A *complementation* on **L** is a map $'$: $L \to L$ such that:

a) $x \wedge x' \approx 0$,
b) $x \vee x' \approx 1$.

An *orthocomplementation* on **L** is a complementation satisfying the further following requirements:

1) $(x')' \approx x$,
2) if $x \leq y$ then $y' \leq x'$, for each $x, y \in L$.

Definition 1 An *ortholattice* is a bounded lattice equipped with orthocomplementation.

It can be easily checked that every ortholattice $(L, \vee, \wedge, ', 0, 1)$ satisfies the De Morgan laws:
$(x \wedge y)' \approx x' \vee y'$ and $(x \vee y)' \approx x' \wedge y'$.

Definition 2 An *orthomodular lattice* is an ortholattice $(L, \vee, \wedge, ', 0, 1)$ satisfying the following quasi-identity:

(OML) if $x \leq y$ then $y = x \vee (y \wedge x')$,

for each x, y.

Condition (OML) is usually referred to as *orthomodular law* or *orthomodularity*. It has always be a challenge to find equivalent expressions for the orthomodular law, in particular in the form of an identity. It is useful to recall some of them.

Remark 1 If $(L, \vee, \wedge, ', 0, 1)$ is an ortholattice, then the following are equivalent, for each $x, y \in L$:

(OML) if $x \leq y$ then $y = x \vee (y \wedge x')$,
(†) $x \vee y = x \vee ((x \vee y) \wedge x')$,
(∗) $y = (x \wedge y) \vee (x \wedge (x \wedge y)')$.

The above equivalences are evident (proofs can be found for example in [1]). It follows from Remark 1 that orthomodularity is definable by means of an identity, hence the class of orthomodular lattices forms a variety, which we denote by $\mathcal{OML}$.
We introduce the notion of commutativity as it is useful to understand Foulis-Holland Theorem which will be used later on. An element x of an ortholattice **L** *commutes* with another element $y \in$ **L** if $x = (x \wedge y) \vee (x \wedge y')$. For commuting elements, the following holds

Theorem 1 (Foulis-Holland [3, 4]) *Let a, b, c be elements of an orthomodular lattice. Suppose that at least one of them commutes with other two. Then*

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

and

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

In the following section, we establish a new equivalent characterization of the orthomodular law, in terms of an identity.

3 A New Identity for Orthomodularity

Theorem 2 *Let* $\mathbf{L} = \langle L, \wedge, \vee, ', 0, 1\rangle$ *be an ortholattice. Then* $\mathbf{L}$ *is orthomodular if and only if it satisfies the following identity:*
$x \vee (y' \wedge (x \vee y)) \approx (x \vee y') \wedge (x \vee y)$. *(I)*

Proof We start by showing that if $\mathbf{L}$ is an ortholattice satisfying (I), then it satisfies also (OML). In an ortholattice every identity is equivalent to its dual version. Hence (I) is equivalent to:

$$x \wedge (y' \vee (x \wedge y)) \approx (x \wedge y') \vee (x \wedge y). \tag{3.1}$$

Suppose $x \leq y$. Then

$$\begin{aligned} x \vee (x' \wedge y) &= (x \wedge y) \vee (x' \wedge y) \\ &= y \wedge (x' \vee (y \wedge x)) \\ &= y \wedge (x' \vee x) \\ &= y \wedge 1 \\ &= y, \end{aligned}$$

i.e. (OML) holds.

For the converse implication, suppose $\mathbf{L}$ is an orthomodular lattice. Since $x \vee y$ commutes with both x and y, then it also does with x and y'. Therefore (I) can be derived using Theorem 1, i.e. $x \vee (y' \wedge (x \vee y)) = (x \vee y') \wedge (x \vee (x \vee y)) = (x \vee y') \wedge (x \vee y)$. □

The reader may wonder whether the assumption in Theorem 2 on the lattice $\mathbf{L}$ to be an ortholattice could be weakened. Unfortunately this is not the case.

Remark 2 A lattice with complementation satisfying (I) may fail to satisfy the orthomodular law, as witnessed by the lattice in Fig. 1, where $0' = 1$, $d' = a$, $(a')' = a$, $(b')' = b$, $(c')' = c$ and $1' = 0$.

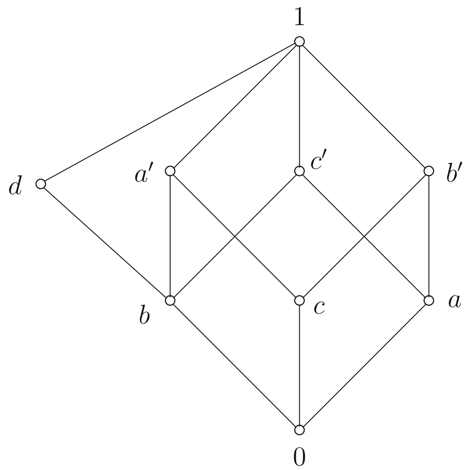

Fig. 1 Hasse diagram of a lattice with complementation satisfying (I) but not (OML)

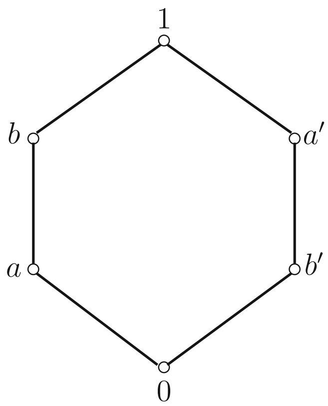

Fig. 2 The lattice $\mathbf{O}_6$ (Benzene ring), a typical example of non-orthomodular lattice

It can be easily checked that it is a lattice with complementation satisfying (I). However, (OML) fails to hold, as $b \leq d$, but $d \neq b \vee (d \wedge b') = b \vee 0 = b$.

Theorem 3 *Let* $\mathbf{L} = \langle L, \wedge, \vee, ', 0, 1\rangle$ *be an ortholattice. Then* $\mathbf{L}$ *is orthomodular if and only if it satisfies the identity*

$$x \vee (y' \wedge (x \vee y)) \approx x \vee (y \wedge (x \vee y')). \tag{3.2}$$

Proof Suppose $\mathbf{L}$ is orthomodular, then, by Theorem 2, it satisfies (I), i.e. $x \vee (y' \wedge (x \vee y)) = (x \vee y') \wedge (x \vee y) = (x \vee y) \wedge (x \vee y') = x \vee (y \wedge (x \vee y'))$.
For the converse, suppose $\mathbf{L}$ is an ortholattice satisfying (3.2) but $\mathbf{L}$ is not orthomodular. Since every ortholattice is orthomodular if and only if it does not contain a subalgebra isomorphic to $\mathbf{O}_6$ in Fig. 2 ([1, Theorem II.5.4]), then $\mathbf{L}$ contains $\mathbf{O}_6$ as subalgebra.

In such a case, identity (3.2) yields: $a = a \vee 0 = a \vee (b \wedge b') = a \vee (b' \wedge (a \vee b)) = a \vee (b \wedge (a \vee b')) = a \vee (b \wedge 1) = a \vee b = b$, a contradiction. Therefore, $\mathbf{L}$ is orthomodular. □

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