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The Dynamical Evolution of Quantum Correlations in the Two Isolate Spin Particles Coupled to a Common Bath

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Abstract We study the dynamical evolution of quantum correlations between two central spins independently coupled to a common bath, which are represented by quantum entanglement and quantum discord. According to the results of the exact solution, we show that quantum discord is more robust and includes richer correlation than quantum entanglement due to the nonvanishing quantum correlation in the region of entanglement death, i.e., the separable states maybe contain nonclassical correlations. We discuss the effects of the intrinsic properties of the bath on quantum correlation between the two central spins in the XY and XXZ model baths. At the low temperature, the central system can keep the good quantum correlation. With the more spin number in the bath, the dynamical evolution of quantum correlation can be bounded with the small oscillation and finally approaches a stable value. In addition, we find that the interaction between the central spins and the bath in the z direction has the significant effects on quantum correlation of the central spin system.

Keywords Quantum entanglement · Quantum discord · Heisenberg model

Xiang-Dong Zhai and Li-Guo Qin contributed equally to this work and are considered co-first authors.

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1 Introduction

Quantum correlation (QC) as the characterization of nonclassical correlation between different particles in a composite system has been playing a crucial role in quantum information and communication theory [1, 2]. A central and fundamental issue in quantum information theory is to quantify QC [3]. At present, quantum entanglement (QE) [4] and quantum discord (QD) [5–11] are two main ways to measure quantum correlations. QE that represents correlations related to non-separability of the state of a composite quantum system is a key resource in quantum information [12, 13]. Zurek et al. [8], found that entanglement does not include all nonclassical correlations and that even separable states usually contain nonclassical correlations that are not entirely classical. These correlations are named QD. Therefore quantum discord is beyond quantum entanglement, especially the separable mixed states. Recently, the comparisons between QE and QD have been studied and shown that QD plays an important role in investigating some quantum tasks without QE, such as quantum computation [14, 15].

As is well known, the realistic quantum systems unavoidably interact with surrounding environments. The interaction with environment causes quantum systems decoherence and leads to the loss of correlations [16]. These unavoidable interactions represent the main obstacle to quantum computing, quantum communication and quantum information processing. The environment not only destroys QC of the central system, but also induces the revivals of the QC in a non-Markovian quantum environment, e.g., the death and birth of QE [17, 18]. Hence, the environment has the important effects on the dynamical evolution of the central quantum system. Great scientific works [16, 19, 20] have been devoted to study the influence of the environment on the correlation dynamics of the central system. Researchers have investigated the non-Markovian dynamics of a central spin system coupled to a spin environment [21]. The spin environment plays a very important role in our study of the quantum system. Such spin environment of the interaction with the system is studied extensively, i.e., quantum dot [22], nitrogen-vacancy centers in diamond [23]. Here, we study the dynamics of QC between two isolated spin systems coupled to a common spin environment. In general, the effects of the bath on QC of the central system depend on the intrinsic properties of the bath such as temperature and the number of spin in bath.

We organize the work as follows. In Section 2 we introduce the model of the two spin-$\frac{1}{2}$ particles and the common spin bath. In Section 3, it is obtained that the reduced density matrix of two spin-$\frac{1}{2}$ particles in the initial conditions of X-structured. And the Section 4, we analyse quantum correlations of entanglement and quantum discord in two regimes. Finally, we conclude the paper in the last section.

2 The Model

We consider that two central spin-$\frac{1}{2}$ particles (spin particles A and B) are initially prepared in a quantum state, then separated and couple to a common Heisenberg-type environment composed of N interacting spin-$\frac{1}{2}$ particles in thermal equilibrium at temperature T, as shown in Fig. 1. Two same external magnetic field with the strength μ is locally applied to the two spins along the z direction, respectively. We assume that two spins, respectively, couples to all other spins in the environment through long-range anisotropic Heisenberg interactions. In this model, the total Hamiltonian of the system can be written as

$$H = H^A \otimes I^B + I^A \otimes H^B. \tag{1}$$

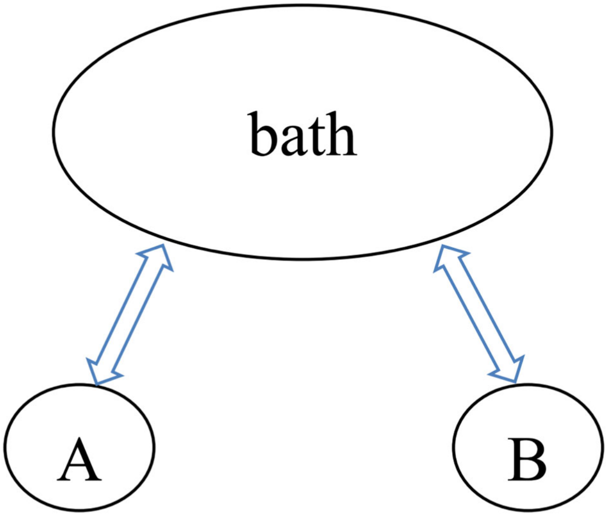

Fig. 1 The model: two spin-$\frac{1}{2}$ particles interact with the common spin bath. There is no direct interaction between spin particle A and spin particle B, and the two spin particles are initially prepared in a quantum state, then separated

where H^A and H^B, respectively, is the Hamiltonian of the subsystem A and B, and H^A and H^B have the same structure. I^A and I^B are the unit matrix. The Hamiltonian H^A is described by [24]

$$H^A = H_S^A + H_{SE}^A + H_E^A, \tag{2}$$

$$H_S^A = 2\mu S_z^A, \tag{3}$$

$$H_{SE}^A = \frac{2\gamma}{\sqrt{N}} S_z^A \sum_{k=1}^{N} S_z^k + \frac{2\alpha}{\sqrt{N}} \left(S_x^A \sum_{k=1}^{N} S_x^k + S_y^A \sum_{k=1}^{N} S_y^k \right), \tag{4}$$

$$H_E^A = \frac{g}{N} \left[\sum_{k\neq l}^{N} \left(S_x^k S_x^l + S_y^k S_y^l \right) + \Delta \sum_{k\neq l}^{N} S_z^k S_z^l \right], \tag{5}$$

where H_S^A and H_E^A are the Hamiltonian of the central spin A and the local surrounding environment, respectively. The interaction Hamiltonian between the central spin particle A and the common bath is described by the Hamiltonian H_{SE}^A. The bath constituents are denoted by S_i^k (where $k = 1, 2, \ldots, N$ and $i = x,\ y,\ z$). And, γ and α are the coupling constants of the central spin to the environment along the different axes, g stands for the strength of interactions between spins in the bath, and Δ is the anisotropy constant. By introducing the total angular momentum of the spin environment $J = \sum_{i=1}^{N} S^i$ and neglecting the constant $(2 + \Delta)\, g/4$, Eqs. (4) and (5) can be rewritten respectively as

$$H_{SE}^A = \frac{2\gamma}{\sqrt{N}} S_z^A J_z + \frac{\alpha}{\sqrt{N}} \left(S_+^A J_- + S_-^A J_+ \right), \tag{6}$$

$$H_E^A = \frac{g}{2N} \left(K + 2\Delta J_z^2 \right), \tag{7}$$

where the raising(J_+) and lowering(J_-) operators $J_\pm = J_x \pm i J_y$, and $K = J_+ J_- + J_- J_+$.

3 The Reduced Dynamics of System

We derive the exact time evolution of two spin-$\frac{1}{2}$ particles for open quantum system. According to *Schrödinger* equation, the time evolution operator $U(t)$ satisfies

$$i\frac{d}{dt}U(t) = HU(t), \tag{8}$$

where H is the total Hamiltonian of system and Planck's constant $\hbar$ has been set equal to 1. For the initial condition of the system the time evolution operator is

$$U(0) = I. \tag{9}$$

Total density matrix operator of open quantum system $\rho_{tot}(t)$ is given by

$$\rho_{tot}(t) = U(t)\rho_{tot}(0)U^{\dagger}(t), \tag{10}$$

where $\rho_{tot}(0)$ denotes the initial value of the total density matrix operator. The reduced density matrix of two controlled qubits in the open quantum system can be calculated by tracing $\rho_{tot}(t)$ with respect to the environmental degrees of freedom, namely,

$$\rho(t) = tr_E\{\rho_{tot}(t)\}. \tag{11}$$

This can be explicitly written in terms of environment states as

$$\rho(t) = \sum_{j,m} v(N, j)\langle j, m|\rho_{tot}(t)|j, m\rangle, \tag{12}$$

where $v(N, j)$ is degeneracy degree, the same to Ref. [24].

3.1 Time Evolution Operator

In our model, the time evolution operator of $U(t)$ is given by

$$U(t) = U^A(t) \otimes U^B(t), \tag{13}$$

$$U^A(t) = U^B(t) = \begin{pmatrix} U_{00} & U_{10} \\ U_{01} & U_{11} \end{pmatrix}, \tag{14}$$

By calculating [24], we can obtain

$$U_{00}(t) = e^{-iG_1 t}\left(\cos\sqrt{M_1}t + \frac{iF_1}{\sqrt{M_1}}\sin\sqrt{M_1}t\right), \tag{15}$$

$$U_{10}(t) = -i\frac{\alpha J_+}{\sqrt{NM_1}}e^{-iG_1 t}\sin\sqrt{M_1}t, \tag{16}$$

$$U_{11}(t) = e^{-iG_2 t}\left(\cos\sqrt{M_2}t + \frac{iF_2}{\sqrt{M_2}}\sin\sqrt{M_2}t\right), \tag{17}$$

$$U_{01}(t) = -i\frac{\alpha J_-}{\sqrt{NM_2}}e^{-iG_2 t}\sin\sqrt{M_2}t, \tag{18}$$

where $\mu + \frac{\gamma J_z}{\sqrt{N}} = H_a$,

$$F_1 = -\left\{\mu + \left[\frac{\gamma}{\sqrt{N}} + \frac{g}{N}(1-\Delta)\right]\left(J_z + \frac{1}{2}\right)\right\}, \tag{19}$$

$$M_1 = F_1^2 + \frac{\alpha^2}{N} J_- J_+, \tag{20}$$

$$F_2 = \mu + \left[\frac{\gamma}{\sqrt{N}} + \frac{g}{N}(1-\Delta)\right]\left(J_z - \frac{1}{2}\right), \tag{21}$$

$$M_2 = F_2^2 + \frac{\alpha^2}{N} J_+ J_-, \tag{22}$$

$$G_1 = F_1 + H_a + H_E, \tag{23}$$

$$G_2 = F_2 - H_a + H_E, \tag{24}$$

Furthermore, using the commutation relations of operators, we can easily prove that $J_- F_2 = -F_1 J_-$, $J_- M_2 = M_1 J_-$, $J_- G_2 = G_1 J_-$ and $J_+ F_1 = -F_2 J_+$, $J_+ M_1 = M_2 J_+$, $J_+ G_1 = G_2 J_+$.

3.2 Reduced Density Matrix

Initially, the two spins are assumed to be uncorrelated with the environment. The total density matrix of the system is given by $\rho_{tot}(0) = \rho(0) \otimes \rho_E$ where $\rho(0)$ and ρ_E are, respectively, the initial density matrices of the central spin and the bath. At time $t = 0$, the spin bath in thermal equilibrium at the inverse temperature β has canonical equilibrium distribution

$$\rho_E = \frac{e^{-\beta H_E^A}}{tr_E\left\{e^{-\beta H_E^A}\right\}}, \tag{25}$$

where $k_B = 1$. We consider initially entangled two-qubit states within the class of X states, in the standard computational basis $\mathcal{B} = \{|1\rangle \equiv |11\rangle, |2\rangle \equiv |10\rangle, |3\rangle \equiv |01\rangle, |4\rangle \equiv |00\rangle\}$, whose general structure is

$$\rho(0) = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{23}^* & \rho_{33} & 0 \\ \rho_{14}^* & 0 & 0 & \rho_{44} \end{pmatrix}. \tag{26}$$

By Eq. (11), the reduced density matrix can be yield and holds the X structured.

4 Dynamics of Quantum Correlations

To measure the dynamical evolution of QC between two central spins, we use the concurrence and QD to analyse our results and compare the differences between two kinds of measure methods of QC. The definition of the concurrence [25–27] is given by

$$C(t) = max\left\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right\}, \tag{27}$$

where λ_S are the eigenvalues of the non-Hermitian matrix $\rho(t)\left(\sigma_y^{(1)} \otimes \sigma_y^{(2)}\right)\rho^*(t)\left(\sigma_y^{(1)} \otimes \sigma_y^{(2)}\right)$ arranged in non-increasing order of magnitude, where σ_y is Pauli matrix. Note that the concurrence varies from $C = 0$ for a separable state to $C = 1$ for a maximally entangled state.

QD, as introduced by Zurek et.al [8], is used to describe the correlation characteristics of two quantum systems. It is to take advantage of the observation that in pursuing quantum analogs of classical notions [9]. QD is defined by the difference of the total correlations and the classical correlations as

$$\mathcal{Q}\left(\rho^{AB}\right) = S\left(\rho^{B}\right) - S\left(\rho^{AB}\right) + \min_{\{B_k\}} S\left(\rho^{AB} | \{B_k\}\right), \tag{28}$$

where $S\left(\rho^{AB}\right)$ and $S\left(\rho^{B}\right)$ are the von Neumann entropy of density matrix ρ^{AB} and ρ^{B}, respectively. ρ^{B} is the partial trace of ρ^{AB}. $S\left(\rho^{AB} | \{B_k\}\right)$ is the conditional entropy and $\{B_k\}$ is the project operator. Here, we choose

$$\{B_k\} = \{\cos\theta\,|0\rangle + e^{i\phi}\sin\theta\,|1\rangle\,, e^{-i\phi}\sin\theta\,|0\rangle - \cos\theta\,|1\rangle\}, \tag{29}$$

where $\phi = 1rad$ [8].

As an important family of two-qubit states, X-state can be prepared easily through a unitary transformation to a two qubit state [28] and has attracted the extensive attentions. QD of the X-state has been evaluated analytically and becomes significant in QC of the

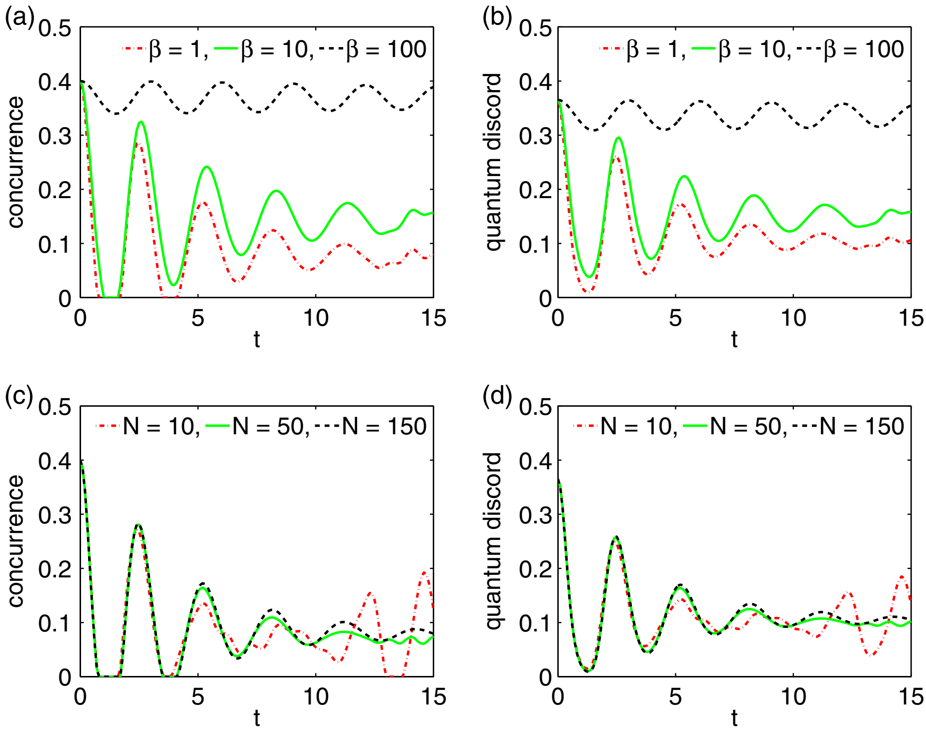

Fig. 2 (*Color online*) The concurrence and quantum discord as a function of time. The initial state of two spin particles is Werner state, and the parameter about Werner state is $p = 0.6$. **a** concurrence and **b** quantum discord in different β, $\beta = 1$ (*red dash-dot line*), $\beta = 10$ (*green solid line*) and $\beta = 100$ (*black dashed line*). **c** concurrence and **d** quantum discord in different N, $N = 10$ (*red dash-dot line*), $N = 50$ (*green solid line*) and $N = 150$ (*black dashed line*). In **a** and **b**, $N = 50$, and in **c** and **d**, $\beta = 1$. The other parameters are $\mu = 1, \alpha = 1, g = 1, \gamma = 0$, and $\Delta = 0$

two-qubit system [5, 11]. In our model, we choose an X-state as the initial state prepared in a Werner state, which is given by

$$\rho_W = \frac{1-p}{4} I + p \left|\Psi_-\right\rangle \left\langle\Psi_-\right|, \tag{30}$$

where $p \in [0, 1]$ and $\left|\Psi_-\right\rangle = \left(|10\rangle - |01\rangle\right)/\sqrt{2}$, one of the four Bell states, is a maximal entanglement state. The initial concurrence and QD of the two spins are, respectively,

$$C(t) = \begin{cases} 0, & p < \frac{1}{3} \\ \frac{1}{2}(3p-1), & p \geq \frac{1}{3} \end{cases}, \tag{31}$$

and,

$$\begin{aligned} Q\left(\rho^{AB}\right) = & \; 1 + \tfrac{3}{4}(1-p)\log_2 \tfrac{1-p}{4} + \tfrac{1}{4}(1+3p)\log_2 \tfrac{1+3p}{4} \\ & -\tfrac{1}{2}(1-p)\log_2 \tfrac{1-p}{2} - \tfrac{1}{2}(1+p)\log_2 \tfrac{1+p}{2}. \end{aligned} \tag{32}$$

The effects of the bath on the dynamical evolution of two central spins are discussed in the case of XY model with $\Delta = 0$ and XXZ model with $\Delta \neq 0$.

4.1 XY Model

When $\Delta = 0$, the environment is reduced to a XY model. When $\gamma = 0$, the concurrence and QD between the two spins are plotted as a function of time for different the inverse

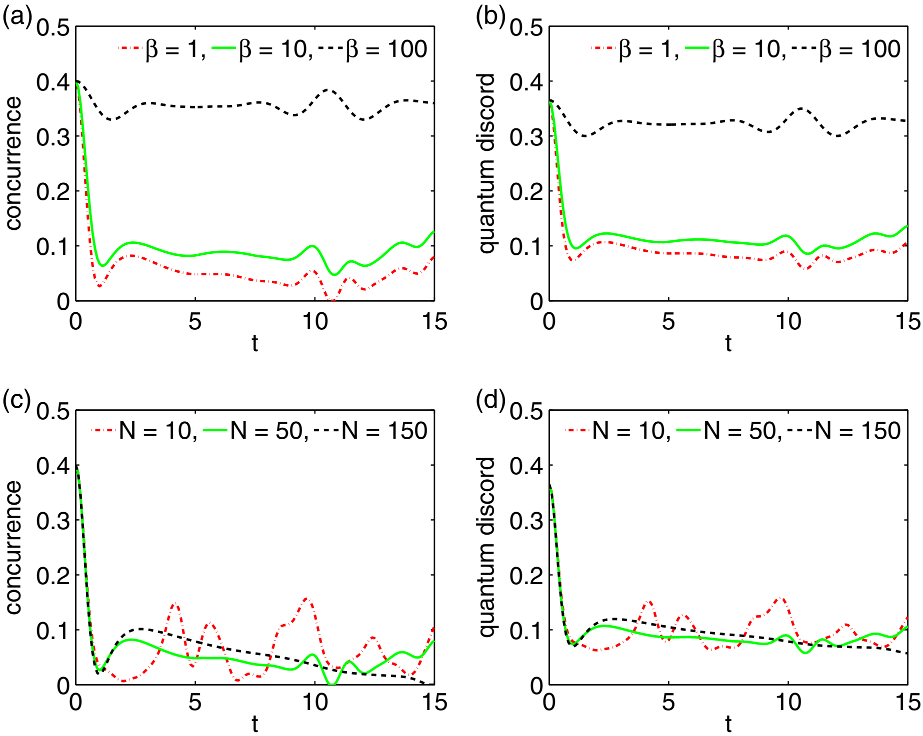

Fig. 3 (*Color online*) The concurrence and quantum discord as a function of time when $\gamma = 2$, and $\Delta = 0$. Other parameters are given in Fig. 2

temperature β in Fig. 2a and b. At the low temperature, we can find that the evolution of concurrence is the periodic oscillation with little decay as the time increases. When the temperature increases, concurrence has a obvious decay with the periodic oscillation as the time increases, then tends to a stable value. And entanglement appears the sudden death, then a revival as shown in Fig. 2a. Contrary to the behavior of concurrence, QD has the similar variation trend in the different temperature value, but does not appear the behavior of the sudden death. Hence, QD contains the more QC than entanglement[3]. In Fig. 2c and d, we investigate concurrence and QD in the different particle numbers N of spin bath. When $N = 50$ or 150, concurrence has a decay in the periodic oscillation as the time increases, and finally trends a stable value. Concurrence has a decay in the periodic oscillation in the initial time, then a strengthening process in the periodic oscillation for $N = 10$. And we can see the entanglement sudden death with $N = 10$, as shown in Fig. 2c. Contrary to concurrence, QD also has a similar dynamics process. Environment can make concurrence and quantum discord more stable, when N increases. This indicates that the bigger of N, the system can be bound in a more stable state.

In Fig. 3, when the $\gamma \neq 0$, Concurrence and QD have a distinct difference compared with Fig. 2 with the periodic oscillation. Concurrence and QD with the time evolution have a rapidly decay without the periodic oscillation as shown in Fig. 3 . We can know that the interaction of environment and system in the z direction has great influence on the QE and

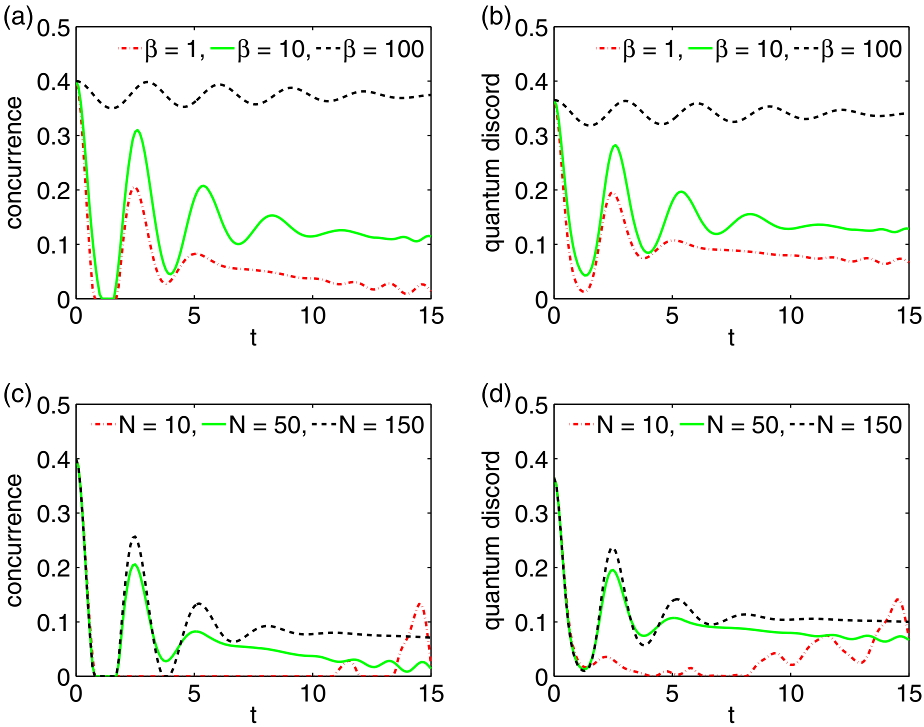

Fig. 4 (*Color online*) The concurrence and quantum discord as a function of time when $\gamma = 0$, and $\Delta = 6$. Other parameters are given in Fig. 2

QD. Otherwise, we can get the same conclusion that the lower temperature, concurrence; QD is bigger and the bigger of N, the system can be bound in a more stable state.

4.2 XXZ Model

When the interaction between the central spins and the bath in z direction is absent, i.e., $\gamma = 0$, the time evolution of concurrence and QD has the obvious differences in Fig. 4 compared with the case of XY model in Fig. 2. At the low temperature, the entanglement and QD show the obvious decay, e.g. $\beta = 100$ in Fig. 4a and b. When the temperature increases, the decay of concurrence and QD becomes more obvious, e.g. $\beta = 1$ or 10 in Fig. 4a and b. For the small N in Fig. 4c and d, the bath can make the two spins lose entanglement or QD, then induce two spins to generate entanglement or QD. For the big N, the bath can make the central system bound in the relatively stable state. In addition, QD still keeps the certain correlation in the region of entanglement death.

Compared with Figs. 4 and 5 with $\gamma \neq 0$ shows that the concurrence and QD has the greater decay as the temperature changes. When the number of the spin particles N is small, the oscillation of QE and QD all becomes large. However, when N increases, concurrence and QD of the two spins can be bounded and the amplitude of concurrence and QD becomes small. Hence, the interaction between the central spins and the bath in the z direction can reduce concurrence and QD.

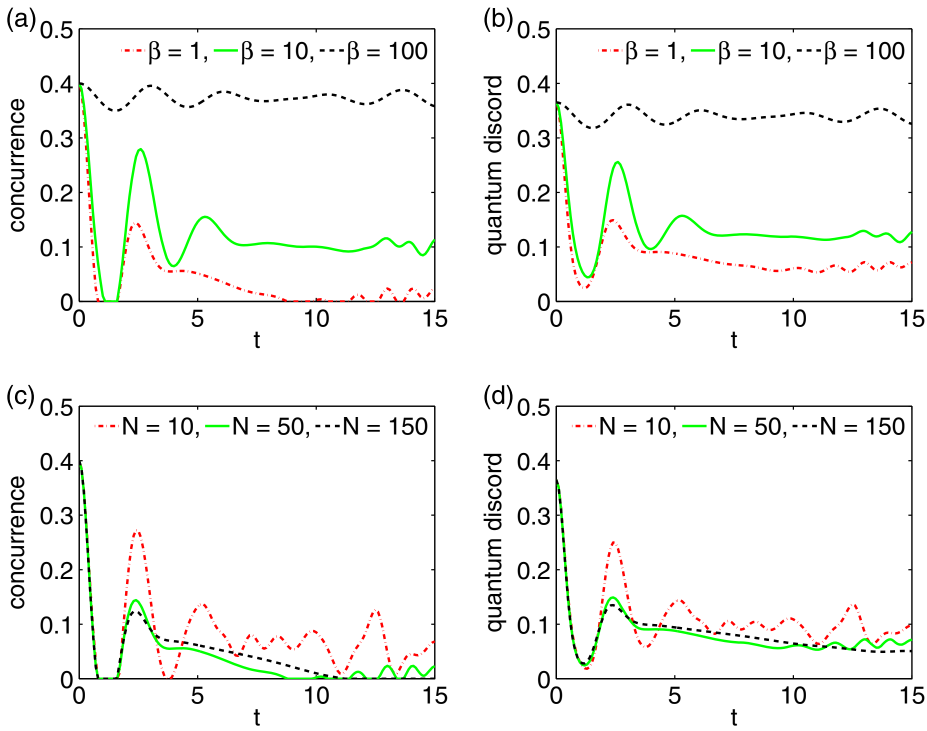

Fig. 5 (*Color online*) The concurrence and quantum discord as a function of time when $\gamma = 2$, and $\Delta = 6$. Other parameters are given in Fig. 2

5 Conclusion

In summary, we obtained the exact solution of the reduced density matrix of the two central spins. Then we studied the dynamical evolution of the quantum correlations represented by QE and QD in two central spins coupled to a common spin bath in thermal equilibrium. By discussing the effects of the temperature and the spin number of the bath on the concurrence and QD, we can find that QC maintains a long time with the small decay at the low temperature, and the time evolution of QC can be bounded and has a small oscillation for the more spin number in the bath. Compared with the XXZ model-bath, the central spins can keep the strong QC in the XY model-bath. In addition, we find that the interaction between the central spins and the bath in the z direction has the significant effects on the time evolution of QC. Further, by comparing QE and QD, QD contains richer correlation than QE due to keep the some QD in the region of entanglement death.

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