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Evolution Equation for a Joint Tomographic Probability Distribution of Spin-1 Particles

Ya. A. Korennoy[1] **· V. I. Man'ko**[1]

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Abstract The nine-component positive vector optical tomographic probability portrait of quantum state of spin-1 particles containing full spatial and spin information about the state without redundancy is constructed. Also the suggested approach is expanded to symplectic tomography representation and to representations with quasidistributions like Wigner function, Husimi $Q-$function, and Glauber-Sudarshan $P-$function. The evolution equations for constructed vector optical and symplectic tomograms and vector quasidistributions for arbitrary Hamiltonian are found. The evolution equations are also obtained in special case of the quantum system of charged spin-1 particle in arbitrary electro-magnetic field, which are analogs of non-relativistic Proca equation in appropriate representations. The generalization of proposed approach to the cases of arbitrary spin is discussed. The possibility of formulation of quantum mechanics of the systems with spins in terms of joint probability distributions without the use of wave functions or density matrices is explicitly demonstrated.

Keywords Quantum tomography · Spin tomography · Evolution equation · Proca equation · Non-negative vector portrait of state

1 Introduction

The proposition of optical tomographic description of states of spinless quantum systems was formulated in [1, 2]. Generalising the optical tomography technique the symplectic tomography was suggested, and evolution equation for symplectic tomograms of spinless

✉ Ya. A. Korennoy
abc772211@mail.ru

V. I. Man'ko
manko@sci.lebedev.ru

[1] P.N. Lebedev Physical Institute, Leninskii prospect 53, 119991, Moscow, Russia

quantum systems was found in [3, 4] providing a bridge between classical and quantum worlds (for review see [5]). Evolution equations for optical tomograms of spinless quantum systems were obtained in Refs. [6, 7].

In Ref. [8] the spin density matrix for particles of arbitrary intrinsic angular momentum is explicitly expressed in terms of directly measurable expectation values of components of multipole moments, or the relative weights of partial beams split-up by a Stern-Gerlach apparatus.

Extending the tomographic approach in Ref. [9] the spin tomography was formulated based on the description of spin states with the help of positive distribution functions depending on continuous variables like Euler's angles, a spin state reconstruction procedure similar to the symplectic tomography was considered, and quantum evolution equation of spin dynamics was found for continuous spin tomogram.

In Ref. [10] spin dynamics was expressed for expectation values of spin projections along a discrete set of fixed directions. The spin tomography was also studied in [11–19] and in other papers.

The first attempt of tomographic formulation of the Pauli equation simultaneously describing both spatial and spin dynamics, apparently, was done in [20]. The evolution equation obtained is extremely complicated, because it uses redundant tomogram depending on continuous Euler's angles and on symplectic variables.

In our resent paper [21] we introduced the positive vector optical tomogram fully describing both spatial and spin characteristics of the quantum state of spin-1/2 particle without any redundancy. We obtained the evolution equation for this vector optical tomogram and considered examples of evolution of quantum systems in proposed representation. Also we discussed the expansion of our approach to representations of Wigner and Husimi quasidistributions (and pointed out the possibility of dissemination of the discussed scheme to the Glauber-Sudarshan representation).

The aim of our work is the construction of spin-1 particle quantum state vector tomography without redundancy of information representing the joint vector distribution for space coordinates and spin projections; and derivation of the evolution equation for such distribution, which would be an analogue of non-relativistic Proca equation. The latter will allow to explicitly demonstrate the possibility of formulation of quantum mechanics of the systems with spins in terms of joint probability distributions without application of wave functions or density matrices.

The paper is organized as follows. In Section 2 we give basic formulas of tomographic and quasiprobability representations of quantum mechanics for spinless particles. In Section 3 we introduce a positive nine-component vector probability and quasiprobability description of spin-1 particles and give the evolution equations for such vector-portraits of quantum state with arbitrary Hamiltonian. In Section 4 charged spin-1 in arbitrary electromagnetic field is considered in proposed representations, and evolution equations, which are analogs of nonrelativistic limit of Proca equation, are obtained. The conclusion is presented in Section 5.

2 Probability Representation and Evolution of spinless Quantum Systems

Let us review the constructions of tomographyc or quasidistribution representations in general case for spinless systems.

If the state of the quantum system is described by the density matrix $\hat{\rho}$ normalized by the condition $\mathrm{Tr}\hat{\rho} = 1$, then in accordance with general scheme the tomographic distribution function or quasidistribution $w(x, \eta)$ is related with the density matrix as follows (see [22]):

$$w(x, \eta, t) = \mathrm{Tr}\{\hat{\rho}(t)\hat{U}(x, \eta)\}, \quad \hat{\rho}(t) = \int w(x, \eta, t)\hat{D}(x, \eta)\mathrm{d}x\,\mathrm{d}\eta, \tag{1}$$

where x is a set of distribution (quasidistribution) variables, η is a set of parameters of corresponding tomography, and $\hat{U}(x, \eta)$, $\hat{D}(x, \eta)$ are dequantizer and quantizer operators for appropriate tomographic scheme or corresponding quasidistribution representation.

For Wigner [23], Husimi [24], and Glauber-Sudarshan [25, 26] quasidistributions the corresponding dequantizers $\hat{U}(x)$ and quantizers $\hat{D}(x)$ depend only on the sets of quasidistributions variables x and do not depend on the sets of parameters η. Therefore, for these representations the letter η vanishes in all of the formulas and the integration over $d\eta$ is omitted in the second formula in (1) and in subsequent (3), (16), (17).

Notion of quantizer and dequantizer is related to star product quantization schemes (see recent review [27]).

Quantizer and dequantizer are constrained by the duality relation

$$\mathrm{Tr}\{\hat{U}(x, \eta)\hat{D}(x', \eta')\} = \delta(x - x')\delta(\eta - \eta').$$

The von-Neumann equation without interaction with the environment

$$i\hbar\frac{\partial}{\partial t}\hat{\rho} = [\hat{H}, \hat{\rho}] \tag{2}$$

with the help of maps of type (1) transforms to evolution equations for tomograms [7], or to Moyal equation [28] for the Wigner function [23], or to evolution equation for other quasidistribution

$$\partial_t w(x, \eta, t) = \frac{2}{\hbar}\int \mathrm{Im}\left[\mathrm{Tr}\left\{\hat{H}(t)\hat{D}(x', \eta')\hat{U}(x, \eta)\right\}\right] w(x', \eta', t)\mathrm{d}x'\mathrm{d}\eta'\,. \tag{3}$$

If we have spinless quantum system in the $N-$dimensional space, then dequantizer and quantizer for optical tomography equal

$$\hat{U}_w(\mathbf{X}, \boldsymbol{\theta}) = |\mathbf{X}, \boldsymbol{\theta}\rangle\langle\mathbf{X}, \boldsymbol{\theta}| = \prod_{\sigma=1}^{N}\delta\left(X_\sigma - \hat{q}_\sigma\cos\theta_\sigma - \hat{p}_\sigma\frac{\sin\theta_\sigma}{m_\sigma\omega_\sigma}\right), \tag{4}$$

$$\hat{D}_w(\mathbf{X}, \boldsymbol{\theta}) = \int\prod_{\sigma=1}^{N}\frac{\hbar|\eta_\sigma|}{2\pi m_\sigma\omega_\sigma}\exp\left\{i\eta_\sigma\left(X_\sigma - \hat{q}_\sigma\cos\theta_\sigma - \hat{p}_\sigma\frac{\sin\theta_\sigma}{m_\sigma\omega_\sigma}\right)\right\}\mathrm{d}^N\eta, \tag{5}$$

where m_σ and ω_σ are constants that have the dimensions of mass and frequency and are chosen for reasons of convenience for the Hamiltonian of a quantum system under study, $|\mathbf{X}, \boldsymbol{\theta}\rangle$ [22] is an eigenfunction of the operator $\hat{\mathbf{X}}(\boldsymbol{\theta})$ with components $\hat{X}_\sigma = \hat{q}_\sigma\cos\theta_\sigma + (\hat{p}_\sigma\sin\theta_\sigma)/(m_\sigma\omega_\sigma)$ corresponding to the eigenvalue $\mathbf{X}$, where $\hat{q}_\sigma$ and $\hat{p}_\sigma$ are the canonical position and momentum operators.

For symplectic tomography quantizer and dequantizer can be written as

$$\hat{U}_M(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = |\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}\rangle\langle\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}| = \prod_{\sigma=1}^{N}\delta(X_\sigma - \hat{q}_\sigma\mu_\sigma - \hat{p}_\sigma\nu_\sigma), \tag{6}$$

$$\hat{D}_M(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \prod_{\sigma=1}^{N}\frac{m_\sigma\omega_\sigma}{2\pi}\exp\left\{i\sqrt{\frac{m_\sigma\omega_\sigma}{\hbar}}\left(X_\sigma - \hat{q}_\sigma\mu_\sigma - \hat{p}_\sigma\nu_\sigma\right)\right\}, \tag{7}$$

where $|\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}\rangle$ is an eigenfunction of the operator $\hat{\mathbf{X}}(\boldsymbol{\mu}, \boldsymbol{\nu})$ with components $\hat{X}_\sigma = \mu_\sigma \hat{q}_\sigma + \nu_\sigma \hat{p}_\sigma$ corresponding to the eigenvalue $\mathbf{X}$.

For Wigner representation we have

$$\hat{U}_W(\mathbf{q}, \mathbf{p}) = \frac{1}{(2\pi\hbar)^N} \int |\mathbf{q} - \mathbf{u}/2\rangle \exp(-i\mathbf{pu}/\hbar)\langle \mathbf{q} + \mathbf{u}/2|\, \mathrm{d}^N u, \tag{8}$$

$$\hat{D}_W(\mathbf{q}, \mathbf{p}) = 2^N \int \mathrm{d}^N u\, \exp(2i\mathbf{pu}/\hbar)|\mathbf{q} + \mathbf{u}\rangle\langle \mathbf{q} - \mathbf{u}|\, ; \tag{9}$$

for Husimi representation (see [29, 30])

$$\hat{U}_Q(\mathbf{q}, \mathbf{p}) = (2\pi\hbar)^{-N}|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|, \quad \boldsymbol{\alpha} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\mathbf{q} + \frac{i}{\sqrt{\hbar m\omega}}\mathbf{p}\right), \tag{10}$$

where $|\mathbf{q}\rangle$ is an eigenvalue of the position operator, $|\boldsymbol{\alpha}\rangle$ is a standard boson coherent state,

$$\begin{aligned} \hat{D}_Q(\mathbf{q}, \mathbf{p}) \;=\; & \left(\frac{m\omega}{\pi\hbar}\right)^{N/2} \int \mathrm{d}^N x \mathrm{d}^N y \Bigg\{ |\mathbf{x}\rangle\langle\mathbf{y}| \exp\left(\frac{m\omega}{2\hbar}(\mathbf{x}-\mathbf{y})^2\right) \\ & \times \exp\left[-\frac{m\omega}{\hbar}\left(\mathbf{q} - \frac{\mathbf{x}+\mathbf{y}}{2}\right)^2 - \frac{m\omega}{\hbar}(\mathbf{x}-\mathbf{y})^2 + \frac{i}{\hbar}\mathbf{p}(\mathbf{x}-\mathbf{y})\right] \\ & \times \prod_{\sigma=1}^{N}\left[\sum_{n_\sigma=0}^{\infty} \frac{(-1)^{n_\sigma}}{n_\sigma! 2^{n_\sigma}} H_{2n_\sigma}\left(\sqrt{\frac{m\omega}{\hbar}} q_\sigma - \frac{m\omega}{2\hbar}(x_\sigma + y_\sigma)^2\right)\right]\Bigg\}. \end{aligned} \tag{11}$$

Likewise, the Glauber-Sudarshan P-function [25, 26] (see, also [31]) can be introduced with the help of corresponding dequantizer and quantizer

$$\hat{U}_P(\boldsymbol{\alpha}) = \left(\frac{e^{|\boldsymbol{\alpha}|^2}}{\pi^{2N}} \int |\boldsymbol{\beta}\rangle\langle -\boldsymbol{\beta}| e^{|\boldsymbol{\beta}|^2} - \boldsymbol{\beta}\boldsymbol{\alpha}^* + \boldsymbol{\beta}^*\boldsymbol{\alpha} \mathrm{d}^{2N}\beta\right), \quad \hat{D}_P(\boldsymbol{\alpha}) = |\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|\, , \tag{12}$$

and so on for the other tomographic schemes.

3 Probability Description of Spin-1 Particles

In general case the evolution of charged spin-1 particle in the external electro-magnetic field is determined by Proca equation [32, 33]. This is a relativistic wave equation of four-component wave function $(\varphi_0, \varphi_1, \varphi_2, \varphi_3)$. But in the case of weak relativism it can be reduced to the Schrödinger type equation with the Hermitian Hamiltonian [34, 35] for the three-component spinor wave function (ψ_1, ψ_2, ψ_3).

For quantum system of charged spin-1 particles without electrical quadrupole moment, with charge e and mass m in the electro-magnetic field with vector and scalar potentials $\mathbf{A}(\mathbf{q}, t)$, $\varphi(\mathbf{q}, t)$ this Hamiltonian has the form:

$$\hat{H} = \frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2 + e\varphi - \frac{\varkappa}{s}\hat{\mathbf{s}}\,\mathbf{H} = \hat{H}_0 - \frac{\varkappa}{s}\hat{\mathbf{s}}\,\mathbf{H}, \tag{13}$$

where $\hat{H}_0$ is an independent on spin part of Hamiltonian, $\mathbf{H} = \mathrm{rot}\mathbf{A}$ is a magnetic field, and $\varkappa$ is a magnetic moment of the particle.

The wave function satisfy by the normalization condition

$$\int (|\psi_1(\mathbf{q})|^2 + |\psi_2(\mathbf{q})|^2 + |\psi_3(\mathbf{q})|^2)\mathrm{d}^3 q = 1.$$

So, mixed states are described by the density matrix $\hat{\rho}_{ij}$ with dimension 3×3, which is actually defined by nine real scalar components.

Analogously with the case of spin-1/2 particles [21], to construct the vector portrait of such density matrix we must solve the state reconstruction problem, i.e., we have to find the inverse map, which transforms the set of expectation values of observables constituting a quorum to the density matrix.

For this purpose we should choose nine spin-1 states $|\sigma_\beta, \mathbf{n}_\beta\rangle$ with definite spin projections σ_β along the directions $\mathbf{n}_\beta$, which define nine-component dequantizer vector $\hat{\boldsymbol{\mathcal{U}}}$ of 3×3 spin matrices with components $\hat{\mathcal{U}}_\beta = |\sigma_\beta, \mathbf{n}_\beta\rangle\langle\sigma_\beta, \mathbf{n}_\beta|$ and quantizer 3×3 matrix $\hat{\boldsymbol{\mathcal{D}}}$ of nine-component vectors so, that

$$\mathrm{Tr}_{kl}\left\{\mathcal{U}_{\beta(kl)}\mathcal{D}_{(kl)\beta'}\right\} = \sum_{k,l=1}^{2s+1} \mathcal{U}_{\beta(kl)}\mathcal{D}_{(kl)\beta'} = \delta_{\beta\beta'}, \qquad \sum_{\beta=1}^{(2s+1)^2} \mathcal{U}_{\beta(kl)}\mathcal{D}_{(k'l')\beta} = \delta_{kk'}\delta_{ll'}. \quad (14)$$

Here greek letters $\beta, \beta' = 1, 2, ..., 9$ are the indexes of numbers of the components of the nine-component vectors, and roman letters with parentheses (kl) are the indexes of 3×3 matrices. It is obvious that the set of matrices $\{\hat{\mathcal{U}}_\beta\}$ must be linearly independent. With the help of $\hat{\boldsymbol{\mathcal{U}}}$ the nine-component vector tomogram or quasidistribution is defined as

$$\mathbf{w}(x, \eta, t) = \mathrm{Tr}\left\{\hat{\rho}(t)\left[\hat{U}(x, \eta) \otimes \hat{\boldsymbol{\mathcal{U}}}\right]\right\}, \quad (15)$$

where the trace is calculated also over spin indexes, and $\hat{U}(x, \eta)$ is defined by formula (4), (6), (8), (10), or (12). Here $\mathbf{w}(x, \eta, t)$ is the aggregate designation of the vector optical $\mathbf{w}(\mathbf{X}, \boldsymbol{\theta}, t)$ or symplectic $\mathbf{M}(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}, t)$ tomogram, vector Wigner function $\mathbf{W}(\mathbf{q}, \mathbf{p}, t)$, vector Husimi function $\mathbf{Q}(\mathbf{q}, \mathbf{p}, t)$, or vector Glauber-Sudarshan function $\mathbf{P}(\boldsymbol{\alpha}, t)$. For optical and symplectic vector tomograms and for the Husimi vector quasidistribution each of the function $w_\beta(x, \eta, t)$ is the probability distribution of the operator $\hat{x}(\eta)$ at time t under the condition that the particle has the corresponding value of spin projection along the appropriate direction. Consequently, the components of the vector $\vec{w}(x, \eta, t)$ must be integrable over $\mathrm{d}x$ and must satisfy the inequalities

$$0 \le w_\beta(x, \eta, t) \le 1, \quad 0 \le \int w_\beta(x, \eta, t)\mathrm{d}x \le 1, \quad \beta = 1, ..., 9.$$

The components of the constructed vector Wigner function and vector Glauber-Sudarshan function corresponding definite spin projection along the appropriate direction are not obligatory non-negative, but definition (15) guarantee that they are definitely real.

Such a definition (15) for our vector Wigner and Husimi functions differs from those usually given in literature by many authors, when the Wigner function $W_{jk}(\mathbf{q}, \mathbf{p}, t)$ and Husimi function $Q_{jk}(\mathbf{q}, \mathbf{p}, t)$ become $(2s + 1) \times (2s + 1)$ matrices dependent on position and momentum, but their non-diagonal elements over the spin indexes are not surely real. So, the main advantage of such quasidistributions with respect to density matrix disappears.

The inverse map of (15) is written by means of spin quantizer $\hat{\boldsymbol{\mathcal{D}}}$ as follows

$$\hat{\rho}_{jk}(t) = \int \hat{D}(x, \eta) \otimes \boldsymbol{\mathcal{D}}_{(jk)}\mathbf{w}(x, \eta, t)\mathrm{d}x\mathrm{d}\eta. \quad (16)$$

Generalizing (17) to the case of spin particles we can write the evolution equation for the components of the tomogram or vector quasidistribution

$$\partial_t w_\beta(x,\eta,t) = \frac{2}{\hbar}\sum_{\beta'=1}^{(2s+1)^2}\int \mathrm{Im}\left[\mathrm{Tr}\left\{\hat{U}(x,\eta)\otimes\hat{\mathcal{U}}_\beta \hat{H}\,\hat{D}(x',\eta')\otimes\hat{\mathcal{D}}_{\beta'}\right\}\right]$$
$$\times w_{\beta'}(x',\eta',t)\mathrm{d}x'\mathrm{d}\eta', \quad \beta = 1, 2, ..., (2s+1)^2, \tag{17}$$

where the designation “Im” signifies the imaginary part of the subsequent expression. Thus, (17) is real-valued equation for the nine-component real vector-function $\mathbf{w}(x,\eta,t)$.

Let us point out that the scheme proposed admits a generalization to the case of arbitrary spin s. If we have the evolution equation of the quantum system with spin s for the nonnegative, hermitian, and normalized density matrix, then we can introduce $(2s+1)^2$-component vector of $(2s+1)\times(2s+1)$ matrices dequantizer $\hat{\mathcal{U}}$ and dual $(2s+1)\times(2s+1)$ matrix of $(2s+1)^2$-component vectors $\hat{\mathcal{D}}$, which are related by conditions (14). After that we can define $(2s+1)^2$-component vector tomogram (or quasidistribution) in accordance with analog of (15), and with the help of the formula analogous to (16) we can write the evolution equation of type (17) for the $(2s+1)^2$-component vector tomogram or quasidistribution.

4 Example of Vector Tomography Representation for Spin-1 Particle

To determine the dequantizer $\hat{\mathcal{U}}$ we should choose nine positive projections of the quantum state, which completely define the density matrix. Let us choose such spin projectors as follows:

$$\hat{\mathcal{U}} = \Big(|s_z=1\rangle\langle s_z=1|,\ |s_z=0\rangle\langle s_z=0|,\ |s_z=-1\rangle\langle s_z=-1|,$$
$$|s_x=1\rangle\langle s_x=1|,\ |s_x=0\rangle\langle s_x=0|,\ |s_{xy}=1\rangle\langle s_{xy}=1|,$$
$$|s_{xy}=0\rangle\langle s_{xy}=0|,\ |s_{yz}=0\rangle\langle s_{yz}=0|,\ |s_{xz}=0\rangle\langle s_{xz}=0| \Big), \tag{18}$$

where $|s_j=\pm1,\,0\rangle$ is an eigenfunction of the projection of spin operator to the direction j corresponding to the eigenvalue ±1 or 0, and $|s_{xy}\rangle$, $|s_{yz}\rangle$, $|s_{xz}\rangle$ are eigenfunctions of projections of spin operator to directions $\vec{e}_{xy}=(1/\sqrt{2},\,1/\sqrt{2},\,0)$, $\vec{e}_{yz}=(0,\,1/\sqrt{2},\,1/\sqrt{2})$, $\vec{e}_{xz}=(1/\sqrt{2},\,0,\,1/\sqrt{2})$ respectively.

Choose the spin representation, in which components of spin operator are defined as follows:

$$\hat{s}_x = \frac{1}{\sqrt{2}}\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix}, \quad \hat{s}_y = \frac{i}{\sqrt{2}}\begin{bmatrix}0&-1&0\\1&0&-1\\0&1&0\end{bmatrix}, \quad \hat{s}_z = \begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}.$$

After calculations in matrix notations for dequantizer $\hat{\boldsymbol{\mathcal{U}}}$ we have

$$\hat{\boldsymbol{\mathcal{U}}} = \left\{\hat{\mathcal{U}}_{\beta(kl)}\right\} = \left(\begin{bmatrix} 1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix} 0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&0\\0&0&1\end{bmatrix}, \frac{1}{4}\begin{bmatrix} 1&\sqrt{2}&1\\ \sqrt{2}&2&\sqrt{2}\\1&\sqrt{2}&1\end{bmatrix},\right.$$
$$\frac{1}{2}\begin{bmatrix} 1&0&-1\\0&0&0\\-1&0&1\end{bmatrix}, \frac{1}{4}\begin{bmatrix} 1&1-i&-i\\ i+1&2&1-i\\ i&1+i&1\end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1&0&i\\0&0&0\\-i&0&1\end{bmatrix},$$
$$\left.\frac{1}{4}\begin{bmatrix} 1&i\sqrt{2}&1\\ -i\sqrt{2}&2&-i\sqrt{2}\\ 1&i\sqrt{2}&1\end{bmatrix}, \frac{1}{4}\begin{bmatrix} 1&-\sqrt{2}&-1\\ -\sqrt{2}&2&\sqrt{2}\\ -1&\sqrt{2}&1\end{bmatrix}\right). \tag{19}$$

From duality relation (14) after some calculations we obtain spin quantizer $\hat{\boldsymbol{\mathcal{D}}}$, which is a 3×3 matrix of nine-component vectors

$$\hat{\boldsymbol{\mathcal{D}}} = \left\{\hat{\mathcal{D}}_{(jk)\beta}\right\} = \begin{bmatrix} \hat{\boldsymbol{\mathcal{D}}}_{(11)} & \hat{\boldsymbol{\mathcal{D}}}_{(12)} & \hat{\boldsymbol{\mathcal{D}}}_{(13)}\\ \hat{\boldsymbol{\mathcal{D}}}_{(21)} & \hat{\boldsymbol{\mathcal{D}}}_{(22)} & \hat{\boldsymbol{\mathcal{D}}}_{(23)}\\ \hat{\boldsymbol{\mathcal{D}}}_{(31)} & \hat{\boldsymbol{\mathcal{D}}}_{(32)} & \hat{\boldsymbol{\mathcal{D}}}_{(33)}\end{bmatrix}, \tag{20}$$

where (jk) are the indexes of 3×3 matrix and $\beta = 1, 2, ..., 9$ is the index of the component of nine-component vector

$$\begin{aligned}
\hat{\boldsymbol{\mathcal{D}}}_{(11)} &= \left(1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\right),\\
\hat{\boldsymbol{\mathcal{D}}}_{(12)} &= \Big(-\frac{1}{2\sqrt{2}} + i\frac{1-\sqrt{2}}{2},\ i\frac{1-\sqrt{2}}{2},\ -\frac{1}{2\sqrt{2}} + i\frac{1-\sqrt{2}}{2},\ \frac{1+i}{\sqrt{2}},\\
&\qquad \frac{1+i}{\sqrt{2}},\ -i,\ -\frac{i}{2},\ \frac{i}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\Big),\\
\hat{\boldsymbol{\mathcal{D}}}_{(13)} &= \Big(\frac{1-i}{2},\ 0,\ \frac{1-i}{2},\ 0,\ -1,\ 0,\ i,\ 0,\ 0\Big),\\
\hat{\boldsymbol{\mathcal{D}}}_{(22)} &= \left(0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\right),\\
\hat{\boldsymbol{\mathcal{D}}}_{(23)} &= \Big(-\frac{1}{2\sqrt{2}} + \frac{i}{2},\ -\frac{1}{\sqrt{2}} + \frac{i}{2},\ -\frac{1}{2\sqrt{2}} + \frac{i}{2},\ \frac{1+i}{\sqrt{2}},\ 0,\ -i,\ -\frac{i}{2},\ -\frac{i}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\Big),\\
\hat{\boldsymbol{\mathcal{D}}}_{(33)} &= \left(0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\right),
\end{aligned}$$

$$\hat{\boldsymbol{\mathcal{D}}}_{(21)} = \hat{\boldsymbol{\mathcal{D}}}^*_{(12)}, \quad \hat{\boldsymbol{\mathcal{D}}}_{(31)} = \hat{\boldsymbol{\mathcal{D}}}^*_{(13)}, \quad \hat{\boldsymbol{\mathcal{D}}}_{(32)} = \hat{\boldsymbol{\mathcal{D}}}^*_{(23)}.$$

Obviously that for such of definition (18) of dequantizer $\hat{\boldsymbol{\mathcal{U}}}$, three components of the vector $\mathbf{w}(x, \eta, t)$ are normalized by the condition

$$\int w_1(x, \eta, t)\mathrm{d}x + \int w_2(x, \eta, t)\mathrm{d}x + \int w_3(x, \eta, t)\mathrm{d}x = 1. \tag{21}$$

For Hamiltonian (13) the evolution (17) of the optical vector tomogram is written as follows (see [7]):

$$\partial_t \mathbf{w}(\mathbf{X},\boldsymbol{\theta},t) = \hat{\mathcal{M}}_w(\mathbf{X},\boldsymbol{\theta},t)\,\mathbf{w}(\mathbf{X},\boldsymbol{\theta},t) + \hat{\mathbf{S}}_w(\mathbf{X},\boldsymbol{\theta},t)\,\mathbf{w}(\mathbf{X},\boldsymbol{\theta},t), \tag{22}$$

where

$$\hat{\mathcal{M}}_w(\mathbf{X},\boldsymbol{\theta},t) = \frac{2}{\hbar}\mathrm{Im}\,\hat{H}_0\left([\hat{\mathbf{q}}]_w(\mathbf{X},\boldsymbol{\theta}),[\hat{\mathbf{p}}]_w(\mathbf{X},\boldsymbol{\theta}),t\right) \tag{23}$$

is a real operator depending on position $[\hat{\mathbf{q}}]_w$ and momentum $[\hat{\mathbf{p}}]_w$ operators in the optical tomographic representation [22]

$$[\hat{q}_\sigma]_w(\mathbf{X},\boldsymbol{\theta}) = \sin\theta_\sigma\frac{\partial}{\partial\theta_\sigma}\left[\frac{\partial}{\partial X_\sigma}\right]^{-1} + X_\sigma\cos\theta_\sigma + i\frac{\hbar\sin\theta_\sigma}{2m_\sigma\omega_\sigma}\frac{\partial}{\partial X_\sigma}, \tag{24}$$

$$[\hat{p}_\sigma]_w(\mathbf{X},\boldsymbol{\theta}) = m\omega_\sigma\left(-\cos\theta_\sigma\left[\frac{\partial}{\partial X_\sigma}\right]^{-1}\frac{\partial}{\partial\theta_\sigma} + X_\sigma\sin\theta_\sigma\right) - \frac{i\hbar}{2}\cos\theta_\sigma\frac{\partial}{\partial X_\sigma}, \tag{25}$$

and $\hat{\mathbf{S}}_w(\mathbf{X},\boldsymbol{\theta},t)$ is a real 9×9 matrix operator, responsible for the interaction of spin with the magnetic field

$$\hat{\mathbf{S}}_w(\mathbf{X},\boldsymbol{\theta},t) = -\frac{2\varkappa}{\hbar s}\mathrm{Im}\left\{\sum_{l,m,m'=1}^{2s+1}\mathcal{U}_{\beta(lm)}\left[\hat{\mathbf{s}}\,\mathbf{H}\left([\hat{\mathbf{q}}]_w(\mathbf{X},\boldsymbol{\theta})\right)\right]_{(mm')}\mathcal{D}_{(m'l)\beta'}\right\}. \tag{26}$$

With omitted arguments and introduced designations

$$[\hat{A}_j]_w = A_j\left([\hat{\mathbf{q}}]_w(\mathbf{X},\boldsymbol{\theta}),t\right),\quad \tilde{\mathrm{H}}_j = [\hat{\mathrm{H}}_j]_w = \mathrm{H}_j\left([\hat{\mathbf{q}}]_w(\mathbf{X},\boldsymbol{\theta}),t\right),$$
$$\left[\nabla_{\mathbf{q}}\hat{\mathbf{A}}\right]_w = \nabla_{\mathbf{q}}\mathbf{A}\left(\mathbf{q}\to[\hat{\mathbf{q}}]_w(\mathbf{X},\boldsymbol{\theta}),t\right)$$

the explicit form of $\hat{\mathcal{M}}_w$ in general case of time-dependent and non-homogeneous electromagnetic field is written as

$$\begin{aligned}\hat{\mathcal{M}}_w(\mathbf{X},\boldsymbol{\theta},t) &= \sum_{n=1}^{3}\omega_n\left[\cos^2\theta_n\frac{\partial}{\partial\theta_n} - \frac{1}{2}\sin 2\theta_n\left\{1 + X_n\frac{\partial}{\partial X_n}\right\}\right] + \frac{2e}{\hbar}\mathrm{Im}\,[\hat{\varphi}]_w\\ &+ \frac{e^2}{mc^2\hbar}\mathrm{Im}[\hat{\mathbf{A}}]_w^2 - \frac{2e}{mc\hbar}\mathrm{Im}\left[\hat{\mathbf{A}}\hat{\mathbf{p}}\right]_w + \frac{e}{mc}\mathrm{Re}\left[\nabla_{\mathbf{q}}\mathbf{A}\right]_w,\end{aligned} \tag{27}$$

where the designation “Re” signifies the real part of the subsequent expression.

For symplectic vector tomography we can find the evolution equation

$$\partial_t\mathbf{M}(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu},t) = \hat{\mathcal{M}}_M(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu},t)\,\mathbf{M}(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu},t) + \hat{\mathbf{S}}_M(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu},t)\,\mathbf{M}(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu},t), \tag{28}$$

where the real operator $\hat{\mathcal{M}}_M(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu},t)$ corresponds to spinless part $\hat{H}_0$ of the Hamiltonian (13)

$$\begin{aligned}\hat{\mathcal{M}}_M(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu},t) &= \frac{2}{\hbar}\mathrm{Im}\,\hat{H}_0\left([\hat{\mathbf{p}}]_M(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu}),[\hat{\mathbf{q}}]_M(\mathbf{X}\boldsymbol{\mu},\boldsymbol{\nu}),t\right) = \frac{\boldsymbol{\mu}}{m}\frac{\partial}{\partial\vec{\nu}} + \frac{2e}{\hbar}\mathrm{Im}\,[\hat{\varphi}]_M\\ &+ \frac{e^2}{mc^2\hbar}\mathrm{Im}[\hat{\mathbf{A}}]_M^2 - \frac{2e}{mc\hbar}\mathrm{Im}\left[\hat{\mathbf{A}}\hat{\mathbf{p}}\right]_M + \frac{e}{mc}\mathrm{Re}\left[\nabla_{\mathbf{q}}\mathbf{A}\right]_M,\end{aligned} \tag{29}$$

where

$$[\hat{A}_j]_M = A_j\left([\hat{\mathbf{q}}]_M(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu}),t\right),\quad [\hat{\varphi}]_M = \varphi\left([\hat{\mathbf{q}}]_M(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu}),t\right),$$
$$[\nabla_{\mathbf{q}}\mathbf{A}]_M = \nabla_{\mathbf{q}}\mathbf{A}\left(\mathbf{q}\to[\hat{\mathbf{q}}]_M(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu}),t\right),$$

and $[\hat{\mathbf{q}}]_M$, $[\hat{\mathbf{p}}]_M$ are position and momentum operators (30) in the symplectic representation (see [6])

$$[\hat{p}_\sigma]_M = \left(-\left[\frac{\partial}{\partial X_\sigma}\right]^{-1}\frac{\partial}{\partial \nu_\sigma} - i\frac{\mu_\sigma \hbar}{2}\frac{\partial}{\partial X_\sigma}\right),$$
$$[\hat{q}_\sigma]_M = \left(-\left[\frac{\partial}{\partial X_\sigma}\right]^{-1}\frac{\partial}{\partial \mu_\sigma} + i\frac{\nu_\sigma \hbar}{2}\frac{\partial}{\partial X_\sigma}\right). \tag{30}$$

The 9×9 real matrix operator $\hat{\mathbf{S}}_M(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}, t)$ is defined by the similar formula (26), where the operators of components of the magnetic field $\tilde{\mathrm{H}}_j$ must be replaced with corresponding operators in the symplectic tomography representation $[\hat{\mathrm{H}}_j]_M = \mathrm{H}_j\left([\hat{\mathbf{q}}]_M(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}), t\right)$.

Making similar calculation we can obtain such evolution equation for our vector Wigner function, which is a generalization of the Moyal equation [28]

$$\frac{\partial}{\partial t}\mathbf{W}(\mathbf{q}, \mathbf{p}, t) = \Big[-\frac{\mathbf{p}}{m}\frac{\partial}{\partial \mathbf{q}} + \frac{2e}{\hbar}\,\mathrm{Im}\,\varphi\left(\mathbf{q} + \frac{i\hbar}{2}\frac{\partial}{\partial \mathbf{p}}, t\right) + \frac{e^2}{mc^2\hbar}\,\mathrm{Im}\mathbf{A}^2\left(\mathbf{q} + \frac{i\hbar}{2}\frac{\partial}{\partial \mathbf{p}}, t\right)$$
$$+ -\frac{2e}{mc\hbar}\,\mathrm{Im}\left\{\mathbf{A}\left(\mathbf{q} + \frac{i\hbar}{2}\frac{\partial}{\partial \mathbf{p}}, t\right)\left(\mathbf{p} - \frac{i\hbar}{2}\frac{\partial}{\partial \mathbf{q}}\right)\right\}$$
$$+ \frac{e}{mc}\,\mathrm{Re}\nabla_{\mathbf{q}}\mathbf{A}\left(\mathbf{q} \to \mathbf{q} + \frac{i\hbar}{2}\frac{\partial}{\partial \mathbf{p}}, t\right) + \hat{\mathbf{S}}_W(\mathbf{q}, \mathbf{p}, t)\Big]\mathbf{W}(\mathbf{q}, \mathbf{p}, t), \tag{31}$$

where 9×9 real matrix operator $\hat{\mathbf{S}}_W(\mathbf{q}, \mathbf{p}, t)$ is defined by the same formula (26), where the operators of components of the magnetic field $\tilde{\mathrm{H}}_j$ must be replaced with corresponding operators in the Wigner representation $\mathrm{H}_j\left(\mathbf{q} + \frac{i\hbar}{2}\frac{\partial}{\partial \mathbf{p}}, t\right)$.

The corresponding generalization of the evolution equation of the Husimi function [36] to the case of vector quasidistribution has the form (for simplicity we choose the system of measurements so that $m = \omega = \hbar = 1$):

$$\frac{\partial}{\partial t}\mathbf{Q}(\mathbf{q}, \mathbf{p}, t) = \Big[-\mathbf{p}\frac{\partial}{\partial \mathbf{q}} - \frac{1}{2}\frac{\partial}{\partial \mathbf{q}}\frac{\partial}{\partial \mathbf{p}} + \frac{2e}{\hbar}\,\mathrm{Im}\,\varphi\left(\mathbf{q} + \frac{1}{2}\frac{\partial}{\partial \mathbf{q}} + \frac{i}{2}\frac{\partial}{\partial \mathbf{p}}, t\right)$$
$$+ \frac{e^2}{c^2}\,\mathrm{Im}\mathbf{A}^2\left(\mathbf{q} + \frac{1}{2}\frac{\partial}{\partial \mathbf{q}} + \frac{i}{2}\frac{\partial}{\partial \mathbf{p}}, t\right)$$
$$- \frac{2e}{c}\,\mathrm{Im}\left\{\mathbf{A}\left(\mathbf{q} + \frac{1}{2}\frac{\partial}{\partial \mathbf{q}} + \frac{i}{2}\frac{\partial}{\partial \mathbf{p}}, t\right)\left(\mathbf{p} + \frac{1}{2}\frac{\partial}{\partial \mathbf{p}} - \frac{i}{2}\frac{\partial}{\partial \mathbf{q}}\right)\right\}$$
$$+ \frac{e}{c}\,\mathrm{Re}\nabla_{\mathbf{q}}\mathbf{A}\left(\mathbf{q} \to \mathbf{q} + \frac{1}{2}\frac{\partial}{\partial \mathbf{q}} + \frac{i}{2}\frac{\partial}{\partial \mathbf{p}}, t\right) + \hat{\mathbf{S}}_Q(\mathbf{q}, \mathbf{p}, t)\Big]\mathbf{Q}(\mathbf{q}, \mathbf{p}, t), \tag{32}$$

where 9×9 matrix operator $\hat{\mathbf{S}}_Q(\mathbf{q}, \mathbf{p}, t)$ is defined by (26) in which components of the magnetic field $\tilde{\mathrm{H}}_j$ are replaced with $\mathrm{H}_j\left(\mathbf{q} + \frac{1}{2}\frac{\partial}{\partial \mathbf{q}} + \frac{i}{2}\frac{\partial}{\partial \mathbf{p}}, t\right)$.

5 Conclusion

To resume we point out the main results of our paper. We have constructed the nine-component positive vector optical tomographic probability portrait of quantum state of spin-1 particles, which contains full spatial and spin information about state without redundancy. We have expanded suggested approach to symplectic tomography representation and to representations with quasidistributions. All of the components of the constructed vector Wigner function are real, and all of the components of the vector Husimi function are non-negative.

We found the real-valued evolution equations for such vector optical and symplectic tomograms and vector quasidistributions for arbitrary Hamiltonian and obtained these equations in special case of the quantum system of charged spin-1 particle in arbitrary electro-magnetic field, which are analogs of non-relativistic Proca equation in appropriate representations. Also we discussed the generalization of our approach to the cases of arbitrary spin.

The general equations obtained are relatively complicated, but in many special cases they are much simpler and could allow for the possibility of analytical and numerical solutions.

The results of the paper explicitly demonstrate the possibility of formulation of quantum mechanics of the systems with spins in terms of joint probability distributions without application of wave functions or density matrices.

Note that in relativistic (contrary to non-relativistic) quantum mechanics due to the speed limit c there are additional uncertainties in the measurement of the momentum and position [37]

$$\triangle p \triangle t \sim \hbar/c, \quad \triangle q \sim \hbar/mc.$$

This fact, in general, leads to the impossibility of constructing in this theory of time-dependent dynamic function of the probability distribution of positions or momentums (and observables, which are functions of positions and / or momentums). Therefore, the question of a possible extension of tomographic probability representation to the relativistic case demands supplementary investigations.

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