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Thermodynamics of Black Holes and the Symmetric Generalized Uncertainty Principle

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Abstract In this paper, we have investigated the thermodynamics of Schwarzschild and Reissner-Nordström black holes using the symmetric generalised uncertainty principle which contains correction terms involving momentum and position uncertainty. The mass-temperature relationship and the heat capacity for these black holes have been computed using which the critical and remnant masses have been obtained. The entropy is found to satisfy the area law upto leading order logarithmic corrections and corrections of the form A^2 (which is a new finding in this paper) from the symmetric generalised uncertainty principle.

Keywords Generalized uncertainty principle · Black hole thermodynamics

1 Introduction

The understanding of the thermodynamic properties of black holes has been one of the most remarkable achievements in theoretical physics. Recently, the idea of a minimal length equal to the Planck length in various theories of quantum gravity [1, 2] have led to a bulk of investigations in black hole thermodynamics [3–7] and its quantum corrected entropy [8–18] and quantum gravity corrections in quantum systems, namely, particle in a box, Landau levels, simple harmonic oscillator, etc. [19–24].

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From rather general and model independent considerations [1, 8, 25, 26], the minimal length has been introduced by replacing the Heisenberg uncertainty principle by generalised uncertainty principle (GUP)

$$\delta x \delta p \geq \frac{\hbar}{2}\left\{1+\frac{\beta^2 l_p^2}{\hbar^2}(\delta p)^2\right\} \tag{1}$$

where l_p is the Planck length ($\sim 10^{-35}m$) and β is a dimensionless constant for which upperbounds have been estimated by studying the effects of GUP in quantum systems such as Landau levels and Lamb shift in hydrogen atom [19]. The above relation implies the existence of a minimum observable length $(\delta x)_{min} = \beta l_p$. The result indicates the necessity of replacing point like particles by extended objects in theories of quantum gravity [25, 26]. It can also be interpreted as a signal of the breaking of our concept of continuum spacetime at very small length scales [8].

The role of GUP in black hole thermodynamics is as follows. It prevents the total evaporation of black holes to photons or other stable quantum particles and is responsible for remnants to be present [8]. An interesting work in which an expression of a GUP was obtained is [25, 26]. This was done by analyzing a thought experiment for the measurement of the area of the horizon of a black hole. The main physical hypothesis of the experiment is that (Hawking) radiation is emitted from the black hole.

A natural extension of (1) is the symmetric generalised uncertainty principle (SGUP) [27–29]

$$\delta x \delta p \geq \frac{\hbar}{2}\left\{1+\frac{\gamma^2}{L^2}(\delta x)^2+\frac{\beta^2 l_p^2}{\hbar^2}(\delta p)^2\right\} \tag{2}$$

where γ is a dimensionless constant (for which upper bound estimates can be made by following the same line of analysis as done for β) and L is a new unknown fundamental length. The $L \to \infty$ corresponds to the GUP case. In [30], with $L = L_\Lambda$, where $L_\Lambda = (3/\Lambda)^{1/2}$ is the de Sitter horizon, the temperature of (anti) de Sitter black holes have been obtained. The motivation for the extension of the GUP to the above form comes from the fact that the above relation implies the existence of a minimum uncertainty in momentum (together with a minimum uncertainty in position) which on large scales may give rise to new possibilities to describe situations where momentum cannot be precisely determined, such as on curved space [31].

In this paper we will study the thermodynamic properties of Schwarzschild as well as Reissner-Nordström black holes using the SGUP. We shall first obtain the mass-temperature relationship from which we compute the heat capacities of these black holes. We then proceed to calculate the critical and remnant masses in terms of the Planck mass and the constants β and γ appearing in the SGUP (2). Finally we compute the entropy which is found to yield the well known area law with corrections from the SGUP. Interestingly, we find that apart from the usual logarithmic corrections, there are an additional corrections of the order of A^2 in both cases and inverse power of A corrections in case of the Reissner-Nordström black hole.

The paper is organized as follows. In Section 2, we present a brief review of black hole thermodynamics using the standard (Heisenberg) uncertainty principle. In Sections 3 and 4, we study the thermodynamics of Schwarzschild and Reissner-Nordström black holes taking into account the effect of SGUP. Finally, we conclude in Section 5.

2 A Brief Review of Black Hole Thermodynamics

Let us consider a Schwarzschild black hole of mass M. Then near the event horizon of the black hole, the momentum uncertainty and the temperature for a massless elementary particle are related as [8]

$$T = \frac{(\delta p)c}{k_B} \tag{3}$$

where c is the speed of light and k_B is the Boltzmann constant. At thermodynamic equilibrium, the temperature of the black hole will be equal to that of the particle. Also, near the horizon of the Schwarzschild black hole, the position uncertainty of a particle can be expressed in terms of the Schwarchild radius [8, 32]

$$\delta x = \epsilon r_s \ ; \ r_s = \frac{2GM}{c^2} \tag{4}$$

where ϵ is a calibration factor, r_s is the radius of Schwarzschild black hole and G is the Newton's universal gravitational constant. Using the saturated form of the standard (Heisenberg) uncertainty principle

$$\delta x \delta p = \frac{\hbar}{2} \tag{5}$$

the mass temperature relation can be expressed as

$$M = \frac{M_p^2 c^2}{4\epsilon k_B T}. \tag{6}$$

Comparing this with the semi-classical Hawking temperature $T = \frac{M_p^2 c^2}{8\pi M k_B}$ [3, 4], yields the value of $\epsilon = 2\pi$. Hence mass-temperature relation can be written as

$$M = \frac{M_p^2 c^2}{8\pi k_B T}. \tag{7}$$

Now by definition, the heat capacity of the black hole is given by

$$C = c^2 \frac{dM}{dT}. \tag{8}$$

Using (7), the heat capacity is therefore given by

$$C = -\frac{k_B}{8\pi} \left(\frac{M_P c^2}{k_B T} \right)^2. \tag{9}$$

Hence the entropy can be calculated by using the first law of black hole thermodynamics as

$$S = \int c^2 \frac{dM}{T} = \int C \frac{dT}{T}. \tag{10}$$

Using (9) and (7), this yields the famous area theorem [5–7]

$$\frac{S}{k_B} \equiv \frac{S_{BH}}{k_B} = \frac{A}{4 l_p^2} \tag{11}$$

where $\frac{S_{BH}}{k_B}$ is the semi-classical Bekenstein-Hawking entropy for the Schwarzschild black hole and $A = 4\pi r_s^2 = 16\pi \frac{G^2 M^2}{c^4} = \frac{4\pi M^2}{M_p^2}$ is the horizon area of the black hole.

3 Thermodynamics of Schwarzchild Black Holes

In this section, we shall study the thermodynamic properties of Schwarzschild black holes incorporating the effect of the SGUP (2).

To relate the temperature with the mass of the black hole in this case, once again the SGUP (2) has to be saturated

$$\delta x\delta p = \frac{\hbar}{2}\left\{1+\frac{\gamma^2}{L^2}(\delta x)^2+\frac{\beta^2 l_p^2}{\hbar^2}(\delta p)^2\right\}. \tag{12}$$

Using (3, 4), the above relation can be put in the following form

$$M = \frac{M_p^2 c^2}{4\epsilon}\left\{\frac{1}{k_B T}+\frac{1}{k_B T}\frac{\gamma^2}{L^2}\left(\frac{2\epsilon M}{M_p}\right)^2\left(\frac{\hbar}{M_p c}\right)^2+\frac{\beta^2}{(M_p c^2)^2}k_B T\right\} \tag{13}$$

where the relations $\frac{c\hbar}{l_p} = M_p c^2$ and $M_p = \frac{c^2 l_p}{G}$ (M_p being the Planck mass) has been used. Following the procedure in the previous section for determining the value of the calibration factor ϵ, leads to the mass-temperature relationship for the black hole

$$M = \frac{M_p^2 c^2}{8\pi}\left\{\frac{1}{k_B T}+\frac{1}{k_B T}\frac{\gamma^2}{4L^2}\left(\frac{8\pi M}{M_p}\right)^2\left(\frac{\hbar}{M_p c}\right)^2+\frac{\beta^2}{(M_p c^2)^2}k_B T\right\}. \tag{14}$$

Now by using (8), the heat capacity of the black hole (using (14)) will be

$$C = \frac{k_B}{8\pi}\left[\frac{-\left(\frac{M_p c^2}{k_B T}\right)^2+\beta^2-\frac{\gamma^2}{4L^2}\left(\frac{\hbar}{M_p c}\right)^2\left(\frac{8\pi M}{M_p}\right)^2\left(\frac{M_p c^2}{k_B T}\right)^2}{1-\frac{\gamma^2}{2L^2}\left(\frac{\hbar}{M_p c}\right)^2\left(\frac{8\pi M}{M_p}\right)\left(\frac{M_p c^2}{k_B T}\right)}\right]. \tag{15}$$

To get the remnant mass (where the radiation process stops), we set $C = 0$ and this leads to

$$M_{rem} = \frac{\beta}{4\pi}M_p\sqrt{\frac{1}{1-\frac{\beta^2\gamma^2\hbar^2}{L^2 M_p^2 c^4}}}\,. \tag{16}$$

The condition that remnant mass is real and does not diverge leads to the following inequality involving the constants γ and β

$$\frac{\beta^2\gamma^2\hbar^2}{L^2 M_p^2 c^4} < 1. \tag{17}$$

From the mass-temperature relation (14), we can express the temperature in terms of the mass as

$$T = \frac{8\pi}{k_B}\frac{Mc^2}{2\beta^2}\left[1-\sqrt{1-\frac{\beta^2\gamma^2}{L^2}\left(\frac{\hbar}{M_p c}\right)^2-\beta^2\frac{{M_p}^2}{16\pi^2 M^2}}\right] \tag{18}$$

where the negative sign before the square root has been taken to reproduce (7) in the $\gamma, \beta \to 0$ limit. The above relation easily leads to the existence of a critical mass below which the temperature becomes a complex quantity

$$M_{cr} = \frac{\beta}{4\pi}M_p\sqrt{\frac{1}{1-\frac{\beta^2\gamma^2\hbar^2}{L^2 M_p^2 c^4}}}. \tag{19}$$

Equations (16) and (19) imply that the remnant and critical masses are equal and for both the masses, the condition in (17) applies. Also, in the limit $\gamma \to 0$, both the results reduce to those found in [11].

We now move to calculate the entropy from the first law of black hole thermodynamics (10). Substituting (18) in (10) and carrying out the integration expansion keeping terms up to leading order in γ^2 and β^2 yields

$$\begin{aligned} \frac{S}{k_B} &= \frac{4\pi M^2}{M_p^2} - \frac{\beta^2}{8\pi}\ln\left(\frac{8\pi M}{M_p}\right) - \frac{\gamma^2\hbar^2}{128\pi M_p^2 c^2 L^2}\left(\frac{8\pi M}{M_p}\right)^4 \\ &= \frac{S_{BH}}{k_B} - \frac{\beta^2}{16\pi}\ln\left(\frac{S_{BH}}{k_B}\right) - \frac{\beta^2}{16\pi}\ln(16\pi) - \frac{2\pi\gamma^2 l_p^2}{L^2}\left(\frac{S_{BH}}{k_B}\right)^2 \end{aligned} \tag{20}$$

where $\frac{S_{BH}}{k_B} = \frac{4\pi M^2}{M_p^2}$ is the semi-classical Bekenstein-Hawking entropy for the Schwarzschild black holes, which is mentioned before. In terms of the area of the horizon A, (20) can be recast in the following form

$$\frac{S}{k_B} = \frac{A}{4l_p^2} - \frac{\beta^2}{16\pi}\ln\left(\frac{A}{4l_p^2}\right) - \frac{\beta^2}{16\pi}\ln(16\pi) - \frac{2\pi\gamma^2 l_p^2}{L^2}\left(\frac{A}{4l_p^2}\right)^2 \tag{21}$$

which is the famous area law [5–7] with corrections from the SGUP. Interestingly, we obtain corrections quadratic in the horizon area of the black hole apart from the well known logarithmic corrections [33, 34]. It can be easily seen that the quadratic corrections in the horizon area owes its origin to the position uncertainty term in the right hand side of the SGUP. Further, it is evident that this correction would be smaller than the logarithmic corrections (even for a large horizon area A) since it is accompanied by the square of the Planck length and square of L which is a number greater than unity.

It is to be noted that there are no inverse power of A corrections when computations are carried out upto $\mathcal{O}(\beta^2)$. However keeping terms upto $\mathcal{O}(\beta^4)$ leads to

$$\frac{S}{k_B} = \frac{\tilde{A}}{4l_p^2} - \frac{\beta^2}{16\pi}\ln\left(\frac{\tilde{A}}{4l_p^2}\right) - \frac{\beta^2}{16\pi}\ln(16\pi) + \frac{3\beta^4}{64\pi^2}\left(\frac{l_p^2}{\tilde{A}}\right) - \frac{2\pi\gamma^2 l_p^2}{L^2}\left(\frac{\tilde{A}}{4l_p^2}\right)^2 \tag{22}$$

where the new variable $\tilde{A}$ is given by

$$\tilde{A} = A - \frac{\beta^2}{2\pi}l_p^2. \tag{23}$$

With this we conclude our study of the thermodynamics of Schwarzschild black holes and in the subsequent section we shall investigate the same for Reissner-Nordström black holes.

4 Thermodynamics of Reissner-Nordström Black Holes

In this section, we consider a Reissner-Nordström black hole of mass M and charge Q and study the effect of the SGUP on the thermodynamics of this black hole. For RN black hole, the position uncertainty of a particle near the horizon can be written as

$$\begin{aligned} \delta x &= \epsilon r_h \\ r_h &= \frac{G r_0}{c^2} \\ r_0 &= M + \sqrt{M^2 - Q^2} \end{aligned} \tag{24}$$

where r_h is the radius of the horizon of the RN black hole. The momentum uncertainty for this remains the same as (3). Following the analysis in [11, 12] using (2), we get the relation between the mass, charge and temperature of this black hole to be

$$\left(\frac{\pi r_0^2}{r_0 - M}\right)\left(\frac{k_B T}{M_p c^2}\right) = \frac{M_p}{2}\left[1 + \frac{\gamma^2 l_p^2}{L^2 M_p^2}\left(\frac{\pi r_0^2}{r_0 - M}\right)^2 + \beta^2\left(\frac{k_B T}{M_p c^2}\right)^2\right] \tag{25}$$

where we have used the identity

$$\frac{r_0}{(M r_0 - Q^2)} = \frac{1}{(r_0 - M)}. \tag{26}$$

For the sake of simplicity we will write (25) as

$$u(r_0)\left(\frac{k_B T}{M_p c^2}\right) = \frac{M_p}{2}\left[1 + g^2\{u(r_0)\}^2 + \beta^2 T'^2\right] \tag{27}$$

where

$$u(r_0) = \frac{\pi r_0^2}{r_0 - M}\ ; \quad g^2 = \frac{\gamma^2 l_p^2}{L^2 M_p^2}. \tag{28}$$

The heat capacity for the black hole can be obtained using (8, 27):

$$C = k_B \frac{\left(\beta^2 \frac{k_B T}{M_p c^2} - \frac{1}{M_p}\frac{\pi r_0^2}{r_0 - M}\right)(\pi^2 r_0^4)}{\left(\frac{\pi r_0^2}{r_0 - M}\right)^3 \left(\frac{k_B T}{M_p c^2} - M_p g^2 \frac{\pi r_0^2}{r_0 - M}\right)(2r_0 - 3M)}. \tag{29}$$

From the mass-temperature relation (27), we can find out the solution for T as

$$T = \frac{c^2}{k_B \beta^2}\left[u(r_0) - \sqrt{\{u(r_0)\}^2 - M_p^2\beta^2[1 + g^2\{u(r_0)\}^2]}\right]. \tag{30}$$

Here negative sign has been taken to reproduce (18) in the $Q \to 0$ limit.

Now for this solution to be real

$$\{u(r_0)\}^2 - M_p^2\beta^2[1 + g^2\{u(r_0)\}^2] \geq 0. \tag{31}$$

Taking the equality sign in this condition leads to the following cubic equation for the critical mass below which the temperature becomes a complex quantity

$$4bM_{cr}^3 - b^2 M_{cr}^2 - 4bM_{cr}Q^2 + Q^4 + b^2 Q^2 = 0\ ;\ b = \frac{\beta M_p}{\pi\sqrt{1 - \frac{\gamma^2\beta^2 l_p^2}{L^2}}}. \tag{32}$$

Solving the above equation we get the expression for critical mass as

$$M_{cr} = \frac{b}{12}\left[1 + \frac{b^2 + 48Q^2}{D^{\frac{1}{3}}} + \frac{D^{\frac{1}{3}}}{b^2}\right] \tag{33}$$

where

$$D = b^6 - 144b^4Q^2 - 216b^2Q^4 + 12\sqrt{3}b^2\sqrt{-b^6Q^2 + 31b^4Q^4 - 112b^2Q^6 + 108Q^8}. \tag{34}$$

The above expression for the critical mass reduces to the critical mass for the Schwarzschild black hole (19) in the $Q \to 0$ limit.

To obtain the remnant mass at which the radiation process terminates, we set $C = 0$ which yields the same cubic equation for the remnant mass as that for the critical mass (32).

Hence, the remnant mass is once again equal to the critical mass (for the RN black hole) similar to the Schwarzschild black hole.

Finally we compute the entropy using the expression of temperature (30) and definition of entropy (10) keeping terms up to leading order in γ^2 and β^2 which yields

$$\begin{aligned}\frac{S}{k_B} &= \frac{\pi r_h^2}{l_p^2} - \frac{2\pi\gamma^2 l_p^2}{L^2}\left(\frac{\pi r_h^2}{l_p^2}\right)^2 - \frac{8\pi^2\gamma^2 l_p^2 Q^2}{M_p^2 L^2}\left(\frac{\pi r_h^2}{l_p^2}\right) - \frac{12\pi^3\gamma^2 l_p^2 Q^4}{M_p^4 L^2}\ln\left(\frac{r_h^2}{l_p^2} - \frac{Q^2}{M_p^2}\right) \\ &\quad - \frac{\beta^2}{8\pi}\ln\left(\frac{r_h}{l_p}\right) - \frac{\beta^2 Q^2}{8\pi}\frac{l_p^2}{M_p^2 r_h^2} - \frac{\beta^2 Q^4}{32\pi}\frac{l_p^4}{M_p^4 r_h^4} \\ &= \frac{S_{BH}}{k_B} - \frac{\beta^2}{16\pi}\ln\left(\frac{S_{BH}}{k_B}\right) - \frac{\beta^2}{16\pi}\ln(16\pi) - \frac{2\pi\gamma^2 l_p^2}{L^2}\left(\frac{S_{BH}}{k_B}\right)^2 - \frac{8\pi^2\gamma^2 l_p^2 Q^2}{M_p^2 L^2}\left(\frac{S_{BH}}{k_B}\right) \\ &\quad - \frac{12\pi^3\gamma^2 l_p^2 Q^4}{M_p^4 L^2}\ln\left(\frac{S_{BH}}{k_B} - \frac{\pi Q^2}{M_p^2}\right) - \frac{\beta^2 Q^2}{8M_p^2}\frac{k_B}{S_{BH}} - \frac{\pi\beta^2 Q^4}{32 M_p^4}\left(\frac{k_B}{S_{BH}}\right)^2 \end{aligned} \tag{35}$$

where $\frac{S_{BH}}{k_B} = \frac{\pi r_h^2}{l_p^2}$ is the semi-classical Bekenstein-Hawking entropy for the RN black hole. In terms of the area of the horizon $A = 4\pi r_h^2 = 4l_p^2 \frac{S_{BH}}{k_B}$, the above equation can be recast as

$$\begin{aligned}\frac{S}{k_B} &= \left(1 - \frac{8\pi^2\gamma^2 l_p^2 Q^2}{M_p^2 L^2}\right)\left(\frac{A}{4l_p^2}\right) - \frac{2\pi\gamma^2 l_p^2}{L^2}\left(\frac{A}{4l_p^2}\right)^2 - \frac{\beta^2}{16\pi}\ln\left(\frac{A}{4l_p^2}\right) - \frac{\beta^2}{16\pi}\ln(16\pi) \\ &\quad - \frac{12\pi^3\gamma^2 l_p^2 Q^4}{M_p^4 L^2}\ln\left(\frac{A}{4l_p^2} - \frac{\pi Q^2}{M_p^2}\right) - \frac{\beta^2 Q^2}{8M_p^2}\frac{4l_p^2}{A} - \frac{\pi\beta^2 Q^4}{32M_p^4}\left(\frac{4l_p^2}{A}\right)^2 . \end{aligned} \tag{36}$$

Now introducing a new variable A' defined as

$$A' = \left(1 - \frac{8\pi^2\gamma^2 l_p^2 Q^2}{M_p^2 L^2}\right) A \tag{37}$$

one can rewrite (36) as

$$\begin{aligned}\frac{S}{k_B} &= \frac{A'}{4l_p^2} - \frac{2\pi\gamma^2 l_p^2}{L^2}\left(\frac{A'}{4l_p^2}\right)^2 - \frac{\beta^2}{16\pi}\ln\left(\frac{A'}{4l_p^2}\right) - \frac{\beta^2}{16\pi}\ln(16\pi) \\ &\quad - \frac{12\pi^3\gamma^2 l_p^2 Q^4}{M_p^4 L^2}\ln\left(\frac{A'}{4l_p^2} - \frac{\pi Q^2}{M_p^2}\right) - \frac{\beta^2 Q^2}{8M_p^2}\frac{4l_p^2}{A'} - \frac{\pi\beta^2 Q^4}{32M_p^4}\left(\frac{4l_p^2}{A'}\right)^2 \end{aligned} \tag{38}$$

which is the area theorem with corrections for the RN black hole. This expression reduces to (21) in the $Q \to 0$ limit.

5 Conclusion

We finally conclude by summarizing our findings. In this paper, we study the effect of the SGUP in the thermodynamics of Schwarzschild and Reissner-Nordström black holes. The mass-temperature relation and the heat capacity for these black holes are obtained. These relations are then used to compute the critical and remnant masses which are found to be equal and are consistent with our earlier findings [11] in the $\gamma \to 0$ limit. From the expression for the critical mass of the Schwarzschild black hole, we also obtain an inequality involving the constants γ and β. Finally, we compute the entropy and obtain the area theorem [5–7] with the SGUP corrections. Interestingly we observe in both cases that apart from the logarithmic and inverse power of A corrections in the entropy, the SGUP leads to a correction term of the form A^2. This quadratic correction term in the horizon area has not been reported earlier in the literature and is a new finding in this paper.

To place our results in the proper perspective, we would like to mention that the existence of remnants in black hole thermodynamics have also been found in rainbow gravity [35–38]. Rainbow gravity owes its origin to a modified dispersion relation proposed in [39, 40] and the proposal that the spacetime for a test particle depends on its energy. So instead of a single metric describing spacetime, there is a rainbow of metrics depending upon the energy of the test particles. This model is also used in investigating the thermal stability of black holes [41–43], black hole information loss paradox [44–47], etc.

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