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Geometric Global Quantum Discord of Two-qubit X States

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Abstract In this paper, we find that the geometric global quantum discord proposed by Xu and the total quantum correlations proposed by Hassan and Joag are *identical*. Moreover, we work out the analytical formulas of the geometric global quantum discord and geometric quantum discord both for two-qubit X states, respectively. We further illustrate how to use these formulas to deal with a few particular examples. We also compare the results achieved by using three kinds of geometric quantum discords. The geometric quantum discord is verified as a tight lower bound of the geometric global quantum discord for two-qubit X states.

Keywords Geometric global quantum discord · Two-qubit system · X states

1 Introduction

Quantum correlations, as a fundamental character of a multipartite quantum system and an essential resource for quantum information processing [1], was initially studied in the entanglement-versus-separability scenario [2–4]. Even though entanglement has attracted much attention to many authors, it is not a unique characteristic of a quantum system, and it has no any advantage for some quantum information tasks. In some cases [5–7], although there is no entanglement, certain quantum information processing tasks can still be done efficiently by using quantum discord [8–10], which is believed more workable than the entanglement. The quantum discord (QD), first introduced by Ollivier and Zurek [8] and

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by Henderson and Vedral [9], is a measure of quantum correlations, which extends beyond entanglement, and a quantum-versus-classical paradigm for correlations [11–13].

Since the calculation of quantum discord involves a difficult optimization procedure, generally it is not easy to obtain analytical results except for a few typical examples of two-qubit states [14–22]. Huang has proved that computing quantum discord is NP-complete: the running time of any algorithm for computing quantum discord is believed to grow exponentially with the dimensions of the Hilbert space. Therefore, computing quantum discord even with a moderate size is impossible in practice [23]. Recently, some authors have to restrict their researches to two-qubit X states, which are frequently encountered in condensed matter systems, quantum dynamics, etc. [20, 22, 24–27] with an interest in the dynamics of quantum discord [28]. Ali et al. first studied the quantum discord for two-qubit X states and derived an explicit expression for X states [14, 15], but Lu immediately found a counterexample to his results [29]. After that, Chen pointed out that Ali's algorithm is only valid for a class of X states. However, for some family of X states, Ali's algorithm is not correct because of the inequivalence between the minimization over positive operator-valued measures and that over von Neumann measurements [30]. Indeed, Ali's algorithm is not correct even if we only consider von Neumann measurements. The main reason for this error is that Ali did not find all extrema. Soon after, Rau and his co-authors [14, 15] extended the procedure of calculating discord of two-qubit X-states used in Ref. [14] to so-called extended X-states with N qubits. They also gave a formula to calculate the geometric measure of quantum discord for qubit-qudit systems [31]. In this aspect, Huang also found a counterexample to the analytical formula derived in [25], and proposed an analytical formula with a very small worst-case error [32].

Considering the difficulty in calculating the quantum discord, Dakić et al. [33] introduced a geometric measure of quantum discord[1] and obtained an analytic formula for two-qubit states. Very soon, Luo and Fu generalized GD to an arbitrary bipartite system and derived an explicit tight lower bound on it [34]. Rana et al. and Hassan et al. also obtained a rigorous lower bound to GD [35, 36] independently. Girolami et al. got another expression of GD for qubit-qubit states [37, 38]. Tufarelli et al. proposed an algorithm to calculate GD for any $2 \times d$ systems, which is valid for $d \to \infty$ case [39].

Because the original definitions of both QD and GD consider a set of local measurements only on one subsystem, it is not symmetric for two subsystems in the two partite case, Rulli et al. suggested a symmetric extension of QD named global quantum discord(GQD) [40], which has been extended to q-global quantum discord [41]. Some analytical expressions of GQD for some special quantum states have also been found [42]. On the other hand, inspired by Rulli's work, Xu generalized the geometric quantum discord to multipartite states and proposed a geometric global quantum discord (GGQD) [43], which is alternatively known as symmetric or two-side geometric measure of quantum discord for two-qubit system [44, 45]. Almost at the same time, Hassan and Joag proposed total quantum correlations (TQC) and presented an algorithm to calculate TQC for a N-partitle quantum state [46]. It is worth pointing out that GD has attracted considerable attention to many authors, but there existed some argument on the geometric measure of quantum discord [47]. Piani argued that the geometric measure of quantum discord is not a good measure for the quantum correlations. Tafarelli et al. analyzed GD and indicated that it has two fatal weaknesses, i.e., both of them are related to the Hilbert-Schmidt norm or distance. They further defined a Hilbert space metric based on the Hilbert-Schmidt norm and proposed the so-called rescaled geometric

[1] It is also named geometric discord (GD).

discord (RGD) [48]. A detailed discussion about this issue is out of the scope of this paper. Nevertheless, we will compare the GGQD, GD and RGD of the same states. Compared with QD and GD, obviously, the study of GQD and GGQD as well as TQC is not yet enough. Hence, in this paper we first prove that GGQD and TQC are identical and then restrict ourselves to the study of GGQD. We attempt to derive an explicit analytical expression of GGQD for two-qubit X states.

This paper is organized as follows. In the next section, we give a brief review of GD, GGQD and TQC as well as RGD. We will prove that GGQD and TQC are identical. We derive the analytical formulas of GGQD and GD of two-qubit X states in Section 3. Some particular examples are given in Section 4. A related discussion is presented in Section 5 and we give concluding remarks in the last section.

2 Brief Review of Geometric Measure of Quantum Discord and Geometric Global Quantum Discord

We start with a brief review of QD, GD, GGQD and TQC as well as RGD. The QD of a bipartite state ρ on a system $H^a \otimes H^b$ with marginals ρ^a and ρ^b can be expressed as

$$Q(\rho) = \min_{\Pi^a} \left\{ I(\rho) - I(\Pi^a(\rho)) \right\}. \tag{1}$$

Here the minimum is over von Neumann measurements $\Pi^a = \{\Pi_k^a\}$ on subsystem a, and

$$\Pi^a(\rho) = \sum_k \left(\Pi_k^a \otimes I^b \right) \rho \left(\Pi_k^a \otimes I^b \right) \tag{2}$$

is the resulting state after the measurement. $I(\rho) = S(\rho^a) + S(\rho^b) - S(\rho)$ is the quantum mutual information, $S(\rho) = -\mathrm{tr}\rho \ln \rho$ is the von Neumann entropy, and I^b is the identity operator on H^b. The GD for a state ρ is defined as [33]:

$$D(\rho) = \min_{\chi} \|\rho - \chi\|^2, \tag{3}$$

where the minimum is over the set of zero-discord states (i.e., $Q(\chi) = 0$) and $\|\rho - \chi\|^2 := \mathrm{tr}(\rho - \chi)^2$ is the square of Hilbert-Schmidt norm of Hermitian operators. The GD of any two-qubit state is evaluated as

$$D(\rho) = \frac{1}{4} \left(\|\mathbf{x}\|^2 + \|\mathbf{T}\|^2 - k_{max} \right), \tag{4}$$

where $\mathbf{x} := (x_1, x_2, x_3)^t$ is a column vector, $\|\mathbf{x}\|^2 := \sum_i x_i^2$, $x_i = \mathrm{tr}(\rho(\sigma_i \otimes \mathbf{I}^b))$, $T := (t_{ij})$ is a matrix and $t_{ij} = \mathrm{tr}(\rho(\sigma_i \otimes \sigma_j))$, k_{max} is the largest eigenvalue of matrix $\mathbf{x}\mathbf{x}^t + \mathbf{T}\mathbf{T}^t$.

Since Dakić et al. proposed the GD, many authors extended Dakić's results to the general bipartite states. Luo and Fu evaluated the GD for an arbitrary state ρ and obtained an explicit formula

$$D(\rho) = \mathrm{tr}(\mathbf{C}\mathbf{C}^t) - \max_A \mathrm{tr}(\mathbf{A}\mathbf{C}\mathbf{C}^t\mathbf{A}^t), \tag{5}$$

where $\mathbf{C} = (c_{ij})$ is a $m^2 \times n^2$ matrix, given by the expansion $\rho = \sum c_{ij} X_i \otimes Y_j$ in terms of orthonormal operators $X_i \in L(H^a)$, $Y_j \in L(H^b)$ and $A = (a_{ki})$ is an $m \times m^2$ matrix given by $a_{ki} = \mathrm{tr}|k\rangle\langle k|X_i = \langle k|X_i|k\rangle$ for any orthonormal basis $|k\rangle$ of H^a. They also gave a tight lower bound for GD of arbitrary bipartite states [34]. Recently, a different tight lower bound for GD of arbitrary bipartite states was given by Rana et al. [35], and Hassan et al. [36] independently. Other explicit expressions of GD for two-qubit system and $2 \otimes d$ systems are also found [37–39].

On the other hand, in order to overcome the weaknesses of the GD, Tafarelli et al. defined the distance of two density matrices ρ_1 and ρ_2 as

$$d_T(\rho_1, \rho_2) = \left\| \frac{\rho_1}{\|\rho_1\|} - \frac{\rho_2}{\|\rho_2\|} \right\|, \tag{6}$$

where $\|\cdot\|$ stands for the Hibert-Schmidt norm as usual. Then, they proposed the rescaled geometric discord $D_T(\rho)$ for a state ρ as [48]

$$D_T(\rho) = \beta_A \min_{\Pi^A} d_T(\rho, \Pi^A[\rho])^2, \tag{7}$$

where β_A is a normalization constant and depends on the dimension of H_A. If the convention $\alpha_A = d_A/(d_A - 1)$ with $d_A = \dim\{H_A\}$ was adopted,

$$\beta_A = \frac{\sqrt{d_A}}{2\left(\sqrt{d_A} - 1\right)}. \tag{8}$$

Finally, they derived the RGD as

$$D_T(\rho) = \beta_A \left[2 - 2\sqrt{1 - \frac{D_G(\rho)}{\alpha_A \mathrm{Tr}\{\rho^2\}}} \right], \tag{9}$$

where $D_G(\rho)$ is the GD defined as (3).

The QD and GD have been revealed as useful measurements, but they are originally not symmetric for its all subsystems. As an extension of QD, Rulli proposed a global quantum discord (GQD) for an arbitrary multipartite state $\rho_{A_1\cdots A_N}$ as [40]:

$$D(\rho_{A_1\cdots A_N}) = \min_{\{\Pi_k\}}[S(\rho_{A_1\cdots A_N}\|\Phi(\rho_{A_1\cdots A_N})) - \sum_{j=1}^{N} S(\rho_{A_j}\|\Phi_j(\rho_{A_j}))], \tag{10}$$

where $\Phi_j(\rho_{A_j}) = \sum_{j'} \Pi_{A_j}^{j'} \rho_{A_j} \Pi_{A_j}^{j'}$ and $\Phi(\rho_{A_1\cdots A_N}) = \sum_k \Pi_k \rho_{A_1\cdots A_N} \Pi_k$, with $\Pi_k = \Pi_{A_1}^{j_1} \otimes \cdots \otimes \Pi_{A_N}^{j_N}$ and k denoting the index string $(j_1 \cdots j_N)$.

To calculate $D(\rho_{A_1\cdots A_N})$ conveniently, Xu has given an equivalent expression of (10)[42]

$$D(\rho_{A_1\cdots A_N}) = \sum_{k=1}^{N} S(\rho_{A_k}) - S(\rho_{A_1A_2\cdots A_N}) - \max_{\Pi} \left[\sum_{k=1}^{N} S(\Pi_{A_k}(\rho_{A_k})) - S(\Pi(\rho_{A_1\cdots A_N})) \right], \tag{11}$$

where $\Pi = \Pi_{A_1A_2\cdots A_N}$ is any locally projective measurement performed on $A_1A_2\cdots A_n$.

The definition of GGQD for state $\rho_{A_1A_2\cdots A_N}$ is

$$D^G(\rho_{A_1A_2\ldots A_N}) = \min_{\sigma_{A_1A_2\cdots A_N}} \{\mathrm{tr}[\rho_{A_1A_2\cdots A_N} - \sigma_{A_1A_2\cdots A_N}]^2 \quad : D(\sigma_{A_1A_2\cdots A_N}) = 0\}, \tag{12}$$

where $D(\sigma_{A_1A_2\cdots A_N})$ is defined by (10). To simplify the calculation of (12), Xu derived two equivalent formulas of GGQD. The first one is:

$$\begin{aligned} D^G(\rho_{A_1A_2\ldots A_N}) &= \min_{\Pi} \left\{ \mathrm{tr}\left[\rho_{A_1A_2\cdots A_n} - \Pi(\rho_{A_1A_2\cdots A_N})\right]^2 \right\} \\ &= \mathrm{tr}\left[\rho_{A_1A_2\cdots A_N}\right]^2 - \max_{\Pi} \left\{ \mathrm{tr}\left[\Pi(\rho_{A_1A_2\cdots A_N})\right]^2 \right\}, \end{aligned} \tag{13}$$

where Π is the same as the one in (11).

The second formula of GGQD can be expressed as

$$D^G(\rho_{A_1A_2\ldots A_N}) = \sum_{\alpha_1,\alpha_2,\cdots\alpha_N} \left(C_{\alpha_1\alpha_2\cdots\alpha_N}\right)^2 - \max_{\Pi} \sum_{i_1i_2\cdots i_N} \left(\sum_{\alpha_1,\alpha_2,\cdots\alpha_N} A_{\alpha_1 i_1} A_{\alpha_2 i_2} \cdots A_{\alpha_N i_N} C_{\alpha_1\alpha_2\cdots\alpha_N} \right)^2, \tag{14}$$

where $C_{\alpha_1\alpha_2\cdots\alpha_N}$ and $A_{\alpha_k i_k}$ are determined by

$$\rho_{A_1A_2\cdots A_N} = \sum_{\alpha_1\alpha_2\cdots\alpha_N} C_{\alpha_1\alpha_2\cdots\alpha_N} X_{\alpha_1}\otimes X_{\alpha_2}\otimes\cdots\otimes X_{\alpha_N}, \tag{15}$$

$$A_{\alpha_k i_k} = \langle i_k|X_{\alpha_k}|i_k\rangle \tag{16}$$

and $\{X_{\alpha_k}\}_{\alpha_k=1}^{n_k^2}$ are orthonormal bases of $L(H_k)$, which were constituted by all Hermitian operators on H_k; $\{|i_k\rangle\}_{i_k=1}^{n_k^2}$ are orthonormal bases of H_k. For any two-qubit state ρ_{AB}, (14–16) are reduced to:

$$D^G(\rho_{AB}) = \sum_{\alpha\beta}\left(C_{\alpha\beta}\right)^2 - \max_{AB}\sum_{i,j}\left(\sum_{\alpha\beta}A_{i\alpha}B_{j\beta}C_{\alpha\beta}\right)^2, \tag{17}$$

$$C_{\alpha\beta} = \mathrm{tr}\left(\rho_{AB}X_\alpha Y_\beta\right), \tag{18}$$

$$A_{i\alpha} = \langle i|X_\alpha|i\rangle,\quad B_{j\beta} = \langle j|Y_\beta|j\rangle,\quad i,j = 1,2;\quad \alpha,\beta = 0,1,2,3. \tag{19}$$

Here, for consistency with other literatures, such as [34, 36], we have exchanged the indexes of A and B in (19), which do not affect the following results. On the other hand, In (18) and (19), $X_0 = \mathbf{I}^A/\sqrt{2}$, $X_i = \sigma_i^A/\sqrt{2}, i = 1,2,3$; $Y_0 = \mathbf{I}^B/\sqrt{2}$, $Y_j = \sigma_j^B/\sqrt{2}, j = 1,2,3$, where $\mathbf{I}^A$ and σ_i^A are 2×2 unitary matrix and Pauli matrix for qubit A, $\mathbf{I}^B$ and σ_j^B are the same for qubit B, respectively. We can further express (17) in the matrix form,

$$D^G(\rho_{AB}) = \mathrm{tr}(CC^t) - \max_{AB}\mathrm{tr}(ACB^tBC^tA^t), \tag{20}$$

where X^t denote the transpose of matrix X, $A = \{A_{i\alpha}\}$, $B = \{B_{j\beta}\}$ and $C = \{C_{\alpha\beta}\}$. Equation (20) is obviously the generalization of (5) in [34] to the case of GGQD.

Now, we turn our attention to TQC. Hassan and Joang introduced total quantum correlations in a state $\rho_{12\cdots N}$ [46]. They assumed that the non-selective von Neumann projective measurements $\widetilde{\Pi}^{(1)}, \widetilde{\Pi}^{(2)}, \cdots, \widetilde{\Pi}^{(N)}$ are acted on N parts $12\cdots N$ of the system successively. The corresponding post-measurement states are expressed as

$$\widetilde{\Pi}^{(1)}(\rho_{12\cdots N}),\ \widetilde{\Pi}^{(2)}(\widetilde{\Pi}^{(1)}(\rho_{12\cdots N})),\cdots,\ \widetilde{\Pi}^{(N)}(\cdots(\widetilde{\Pi}^{(1)}(\rho_{12\cdots N})\cdots)).$$

The GDs of these successive measurement states are given by

$$D_1(\rho_{12\cdots N}),\ D_2(\widetilde{\Pi}^{(1)}(\rho_{12\cdots N})),\cdots,\ D_N(\widetilde{\Pi}^{(N-1)}(\cdots(\widetilde{\Pi}^{(1)}(\rho_{12\cdots N}))\cdots)).$$

Then, the geometric measure of total quantum correlations of a N-partite quantum state $\rho_{12\cdots N}$ is defined as

$$Q(\rho_{12\cdots N}) = D_1(\rho_{12\cdots N}) + D_2(\widetilde{\Pi}^{(1)}(\rho_{12\cdots N})) + \cdots + D_N(\widetilde{\Pi}^{(N-1)}(\cdots(\widetilde{\Pi}^{(1)}(\rho_{12\cdots N}))\cdots)). \tag{21}$$

In the following, we shall see that the definitions of GGQD and TQC are different in form, but they are identical to each other. To this end, we recall that 1) (13) is also obviously valid for GD with $\Pi = \Pi^k, k = 1,2,\cdots,N$, which only performed on kth part of the system; 2) the projector $\widetilde{\Pi}^{(1)}$ is defined as the von Neumann measurement minimizing the quantity $\|\rho - \Pi^{(1)}(\rho)\|^2$ [46], which implies that

$$\mathrm{tr}[\widetilde{\Pi}^{(1)}(\rho)]^2 = \max_{\Pi^{(1)}}\left\{\mathrm{tr}\left[\Pi^{(1)}(\rho)\right]^2\right\}. \tag{22}$$

Keeping these in mind, we can rewrite (21) for $N = 2$ as

$$\begin{aligned} Q(\rho) &= D_1(\rho) + D_2(\widetilde{\Pi}^{(1)}(\rho)) = \mathrm{tr}[\rho]^2 - \max_{\Pi^{(1)}} \left\{ \mathrm{tr}\left[\Pi^{(1)}(\rho)\right]^2 \right\} + \mathrm{tr}\left[\widetilde{\Pi}^{(1)}(\rho)\right]^2 - \max_{\Pi^{(2)}} \left\{ \mathrm{tr}[\Pi^{(2)}(\widetilde{\Pi}^{(1)}(\rho))]^2 \right\} \\ &= \mathrm{tr}[\rho]^2 - \max_{\Pi^{(2)}} \left\{ \mathrm{tr}\left[\Pi^{(2)}(\widetilde{\Pi}^{(1)}(\rho))\right]^2 \right\} = \mathrm{tr}[\rho]^2 - \max_{\Pi} \left\{ \mathrm{tr}\left[\Pi(\rho)\right]^2 \right\} = D^G(\rho). \end{aligned} \tag{23}$$

There are two key points to be emphasized. First, the terms $\max_{\Pi^{(1)}}\{\mathrm{tr}[\Pi^{(1)}(\rho)]^2\}$ and $\mathrm{tr}[\widetilde{\Pi}^{(1)}(\rho)]^2$ after the second equal sign naturally canceled out each other because of (22). Second, since $\widetilde{\Pi}^{(1)}$ has been determined by (22), therefore, we can write $\max_{\Pi^{(2)}}\{\mathrm{tr}[\Pi^{(2)}(\widetilde{\Pi}^{(1)}(\rho))]^2\}$ as $\max_{\Pi}\{\mathrm{tr}[\Pi(\rho)]^2$ with $\Pi = \Pi^{(2)}\widetilde{\Pi}^{(1)}$. The proof of $Q(\rho) = D^G(\rho)$ for $N \geq 3$ cases is similar and straightforward. The identity of GGQD with TQC is not surprising, because both of them use the original definition of the geometric measure of quantum discord to every individual of the system. This can be further manifested by observing that (64) for TQC in Ref. [46] and the (14) for GGQD are the same. Due to this identity, therefore, hereafter we use the name 'geometric global quantum discord (GGQD)', which also stands for 'total quantum correlations (TQC)'. In the next section, we are going to use (20) to calculate the GGQD of X state.

3 GGQD of Two-qubit X State

The two-qubit X state usually arises as the two-particle reduced density matrix in many physical systems. In the computational basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, the visual appearance of its density matrix resembles the letter X, so it is commonly known as X state in literatures. The density matrix of a two-qubit X state

$$\rho_{AB} = \begin{pmatrix} \varrho_{00} & 0 & 0 & \varrho_{03} \\ 0 & \varrho_{11} & \varrho_{12} & 0 \\ 0 & \varrho_{12}^* & \varrho_{22} & 0 \\ \varrho_{03}^* & 0 & 0 & \varrho_{33} \end{pmatrix} \tag{24}$$

has nonzero elements only on the diagonal and the antidiagonal, where $\varrho_{00}, \varrho_{11}, \varrho_{22}, \varrho_{33} \geq 0$ satisfy $\varrho_{00} + \varrho_{11} + \varrho_{22} + \varrho_{33} = 1$. The antidiagonal elements $\varrho_{03}, \varrho_{12}$ are generally complex numbers, but can be made real and nonnegative by the local unitary transformation $e^{-i\theta_1\sigma_z} \otimes e^{-i\theta_2\sigma_z}$ with $\theta_1 = -(\arg\varrho_{03} + \arg\varrho_{12})/4$, $\theta_2 = -(\arg\varrho_{03} - \arg\varrho_{12})/4$, where σ is the Pauli matrix. Hereafter we assume $\varrho_{03}, \varrho_{12} \geq 0$. Recall that the matrix C in (20) can be written as [34, 35]

$$C = (C_{ij}) = \frac{1}{2} \begin{pmatrix} 1 & \mathbf{y}^t \\ \mathbf{x} & \mathbf{T} \end{pmatrix}, \tag{25}$$

Matrix A and B in (20) can be expressed as [34]

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathbf{a} \\ 1 & -\mathbf{a} \end{pmatrix}, \tag{26}$$

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathbf{b} \\ 1 & -\mathbf{b} \end{pmatrix}, \tag{27}$$

where $\mathbf{a} = \{a_1, a_2, a_3\} = \sqrt{2}(A_{11}, A_{12}, A_{13})$, $\mathbf{b} = \{b_1, b_2, b_3\} = \sqrt{2}(B_{11}, B_{12}, B_{13})$ and $\|\mathbf{a}\| = \|\mathbf{b}\| = 1$. Using (25–27), we can easily get the first term in (20)

$$\mathrm{tr}(CC^t) = \frac{1}{4}\left(1 + \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + \|\mathbf{T}\|^2\right) \tag{28}$$

and the second term

$$\mathrm{tr}(AC B^t BC^t A^t) = \frac{1}{4}\left[1 + \mathbf{y}^t\mathbf{b}^t\mathbf{b}\mathbf{y} + \mathbf{a}(\mathbf{x}\mathbf{x}^t + \mathbf{T}\mathbf{b}^t\mathbf{b}\mathbf{T}^t)\mathbf{a}^t\right]. \tag{29}$$

The maximization over matrixes A and B in (20) can be done by two steps. First, we maximize $\mathbf{a}(\mathbf{x}\mathbf{x}^t + \mathbf{T}\mathbf{b}^t\mathbf{b}\mathbf{T}^t)\mathbf{a}^t$ on matrix A. The maximum of this term is the largest eigenvalue λ_{max-A} of matrix $\mathbf{x}\mathbf{x}^t + \mathbf{T}\mathbf{b}^t\mathbf{b}\mathbf{T}^t$. According to the Lemma 1 of Ref. [45], which states that for any two vectors $|a\rangle$ and $|b\rangle$ (not necessarily normalized) in $\mathbb{R}^3$, the largest eigenvalue of the matrix $|a\rangle\langle a| + |b\rangle\langle b|$ is $\lambda = [a^2 + b^2 + \sqrt{(a^2 - b^2)^2 + 4\langle a|b\rangle^2}]/2$ with $a^2 = \langle a|a\rangle$ and $b^2 = \langle b|b\rangle$, we get

$$\lambda_{max-A} = \frac{1}{2}[\|\mathbf{x}\|^2 + \|\mathbf{T}\mathbf{b}^t\|^2 + \sqrt{(\|\mathbf{x}\|^2 - \|\mathbf{T}\mathbf{b}^t\|^2)^2 + 4(\mathbf{x}^t(\mathbf{T}\mathbf{b}^t))^2}]. \tag{30}$$

Substituting (28 – 30) into (20), we obtain the GGQD of any two-qubit systems

$$\begin{aligned} D^G(\rho_{AB}) = \frac{1}{4}\Bigg\{ &\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + \|\mathbf{T}\|^2 - \frac{1}{2}\max_{\mathbf{b}}\Big[\|\mathbf{x}\|^2 + \|\mathbf{T}\mathbf{b}^t\|^2 \\ &+ \sqrt{(\|\mathbf{x}\|^2 - \|\mathbf{T}\mathbf{b}^t\|^2)^2 + 4(\mathbf{x}^t(\mathbf{T}\mathbf{b}^t))^2} + 2\|\mathbf{b}\mathbf{y}\|^2\Big]\Bigg\}. \end{aligned} \tag{31}$$

The second step to maximize $\mathrm{tr}(AC B^t BC^t A^t)$ in (20) is reduced to maximize $\|\mathbf{x}\|^2 + \|\mathbf{T}\mathbf{b}^t\|^2 + \sqrt{(\|\mathbf{x}\|^2 - \|\mathbf{T}\mathbf{b}^t\|^2)^2 + 4(\mathbf{x}^t(\mathbf{T}\mathbf{b}^t))^2} + 2\|\mathbf{b}\mathbf{y}\|^2$ in above equation on $\mathbf{b} = \{b_1, b_2, b_3\}$. For X state (24),

$$\mathbf{x}^t = \{x_1, x_2, x_3\} = \{0, 0, \varrho_{00} + \varrho_{11} - \varrho_{22} - \varrho_{33}\}, \tag{32}$$

$$\mathbf{y}^t = \{y_1, y_2, y_3\} = \{0, 0, \varrho_{00} - \varrho_{11} + \varrho_{22} - \varrho_{33}\}, \tag{33}$$

$$\mathbf{T} = \begin{pmatrix} T_{11} & 0 & 0 \\ 0 & T_{22} & 0 \\ 0 & 0 & T_{33} \end{pmatrix} = \begin{pmatrix} 2(\varrho_{12} + \varrho_{03}) & 0 & 0 \\ 0 & 2(\varrho_{12} - \varrho_{03}) & 0 \\ 0 & 0 & \varrho_{00} - \varrho_{11} - \varrho_{22} + \varrho_{33} \end{pmatrix}, \tag{34}$$

$$\begin{aligned} &\|\mathbf{x}\|^2 + \|\mathbf{T}\mathbf{b}^t\|^2 + 2\|\mathbf{b}\mathbf{y}\|^2 + \sqrt{(\|\mathbf{x}\|^2 - \|\mathbf{T}\mathbf{b}^t\|^2)^2 + 4(\mathbf{x}^t(\mathbf{T}\mathbf{b}^t))^2} \\ &= x_3^2 + V + 2b_3^2y_3^2 + \sqrt{\left(x_3^2 - V\right)^2 + 4(V - W)x_3^2}, \end{aligned} \tag{35}$$

where

$$W = b_1^2T_{11}^2 + b_2^2T_{22}^2, \qquad V = b_3^2T_{33}^2 + W. \tag{36}$$

To complete the maximization in (35), let

$$b_1 = \sin\theta\cos\phi,\ b_2 = \sin\theta\sin\phi,\ b_3 = \cos\theta.$$

The half of (35) becomes

$$\begin{aligned} f(\theta,\phi) &= \frac{1}{2}\left[x_3^2 + 2y_3^2\cos^2\theta + \gamma(\theta,\phi) + \sqrt{(\gamma(\theta,\phi) - x_3^2)^2 + 4T_{33}^2x_3^2\cos^2\theta)}\right], \\ \gamma(\theta,\phi) &= T_{33}^2\cos^2\theta + \mu(\phi)\sin^2\theta, \\ \mu(\phi) &= T_{11}^2\cos^2\phi + T_{22}^2\sin^2\phi. \end{aligned} \tag{37}$$

We find

$$\begin{array}{ll}
\left\{\frac{\partial f(\theta,\phi)}{\partial\theta},\frac{\partial f(\theta,\phi)}{\partial\phi}\right\}|_{\theta=0}=0, & f(0,\phi)=x_3^2+y_3^2+T_{33}^2,\\
\left\{\frac{\partial f(\theta,\phi)}{\partial\theta},\frac{\partial f(\theta,\phi)}{\partial\phi}\right\}|_{\theta=\frac{\pi}{2},\phi=0\bigvee\pi}=0, & f(\frac{\pi}{2},0\bigvee\pi)=\left\{\begin{array}{l}T_{11}^2,\ \text{for}\ T_{11}^2\geq x_3^2;\\ x_3^2\ \ \text{for}\ T_{11}^2<x_3^2,\end{array}\right.\\
\left\{\frac{\partial f(\theta,\phi)}{\partial\theta},\frac{\partial f(\theta,\phi)}{\partial\phi}\right\}|_{\theta=\frac{\pi}{2},\phi=\frac{\pi}{2}\bigvee\frac{3\pi}{2}}=0, & f(\frac{\pi}{2},\frac{\pi}{2}\bigvee\frac{3\pi}{2})=\left\{\begin{array}{l}T_{22}^2,\ \text{for}\ T_{22}^2\geq x_3^2;\\ x_3^2\ \ \text{for}\ T_{22}^2<x_3^2.\end{array}\right.
\end{array}\tag{38}$$

Obviously, $x_3^2+y_3^2+T_{33}^2\geq x_3^2$ and $T_{11}^2\geq T_{22}^2$, therefore, $\max[f(\theta,\phi)]=\max[x_3^2+y_3^2+T_{33}^2,T_{11}^2]$. Substituting expressions of x_3, y_3, T_{11} and T_{33} into this expression, we obtain the maximum value of $f(\theta,\phi)$,

$$\max[f(\theta,\phi)]=\max\left[\varrho_{00}^2+\varrho_{11}^2+\varrho_{22}^2+\varrho_{33}^2-1/4,(\varrho_{12}+\varrho_{03})^2\right]$$

and the GGQD of X states

$$D^G(\rho_X)=\varrho_{00}^2+\varrho_{11}^2+\varrho_{22}^2+\varrho_{33}^2-\frac{1}{4}+2\left(\varrho_{12}^2+\varrho_{03}^2\right)-\max\left[\varrho_{00}^2+\varrho_{11}^2+\varrho_{22}^2+\varrho_{33}^2-\frac{1}{4},(\varrho_{12}+\varrho_{03})^2\right].\tag{39}$$

For comparing GGQD with GD for some X states in the next section, we also calculated the GD of X state according to Ref. [34] and got the following formula:

$$\begin{aligned}D(\rho_X)\ =\ &\frac{1}{2}\left(\varrho_{00}^2+\varrho_{11}^2+\varrho_{22}^2+\varrho_{33}^2\right)-\varrho_{00}\varrho_{22}-\varrho_{11}\varrho_{33}+2\left(\varrho_{12}^2+\varrho_{03}^2\right)\\ &-\ \max\left[\frac{1}{2}\left(\varrho_{00}^2+\varrho_{11}^2+\varrho_{22}^2+\varrho_{33}^2\right)-\varrho_{00}\varrho_{22}-\varrho_{11}\varrho_{33},(\varrho_{12}+\varrho_{03})^2\right].\end{aligned}\tag{40}$$

This formula can also be derived by (23) of Ref. [31]. In simplifying (39, 40) the condition $\varrho_{00}+\varrho_{11}+\varrho_{22}+\varrho_{33}=1$ has been used repeatedly. It is easy to obtain the corresponding expression of D_T by substituting (40) into (9).

4 Illustrative Examples

In this section, we give some concrete examples to illustrate how to use these formulas obtained above.

(1) The first example is to consider the initial state $\rho=a|\phi^+\rangle\langle\phi^+|+(1-a)|1_A,1_B\rangle\langle 1_A,1_B|(0<a\leq 1)$, where $|\phi^+\rangle=(|0_A,0_B\rangle+|1_A,1_B\rangle)/\sqrt{2}$ is a maximally entangled state [14]. The density matrix of this state is given by

$$\rho_X=\begin{pmatrix}\frac{a}{2}&0&0&\frac{a}{2}\\0&0&0&0\\0&0&0&0\\\frac{a}{2}&0&0&1-\frac{a}{2}\end{pmatrix}.\tag{41}$$

The corresponding GGQD, GD and D_T are

$$D^G(\rho_X)=D(\rho_X)=\frac{a^2}{2}.\tag{42}$$

$$D_T(\rho_X)=\frac{2+\sqrt{2}}{2}\left(2-\sqrt{\frac{3a^2-4a+4}{a^2-a+1}}\right).\tag{43}$$

We plot $D^G(\rho_X)$, $D(\rho_X)$ and $D_T(\rho_X)$ in Fig. 1, which shows that $D^G(\rho_X)$ and $D(\rho_X)$ are completely coincident, but $D_T(\rho_X)$ is unequal to $D^G(\rho_X)$ and $D(\rho_X)$ in this state.

(2) We take a class of states defined as $\rho = a|\psi^+\rangle\langle\psi^+| + (1-a)|1_A, 1_B\rangle\langle 1_A, 1_B|(0 \leq a \leq 1)$,where $|\psi^+\rangle = (|0_A, 1_B\rangle + |1_A, 0_B\rangle)/\sqrt{2}$ is a maximally entangled state [14]. The density matrix of this state is:

$$\rho_X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{a}{2} & \frac{a}{2} & 0 \\ 0 & \frac{a}{2} & \frac{a}{2} & 0 \\ 0 & 0 & 0 & 1-a \end{pmatrix}. \tag{44}$$

The corresponding GGQD and GD are

$$D^G(\rho_X) = \begin{cases} \frac{a^2}{2}, & 0 \leq a \leq \frac{3}{5} \\ \frac{1}{4}(3 - 8a + 7a^2), & \frac{3}{5} < a \leq 1. \end{cases} \tag{45}$$

$$D(\rho_X) = \begin{cases} \frac{a^2}{2}, & 0 \leq a \leq \frac{1}{2} \\ \frac{1}{2}(1 - 3a + 3a^2), & \frac{1}{2} < a \leq 1. \end{cases} \tag{46}$$

We plot $D^G(\rho_X)$, $D(\rho_X)$ and $D_T(\rho_X)$ for the state (44) in Fig. 2. We see that $D^G(\rho_X) = D(\rho_X)$, for $0 \leq a \leq \frac{1}{2}$ and $D^G(\rho_X) \geq D(\rho_X)$, for $\frac{1}{2} < a \leq 1$. Finally $D^G(\rho_X) = D(\rho_X)$ when $a = 1$. The $D^G(\rho_X)$, $D(\rho_X)$ and $D_T(\rho_X)$ have the similar trend, but $D_T(\rho_X)$ does not always greater or less than $D^G(\rho_X)$, $D(\rho_X)$.

(3) We take a class of states defined as $\rho = \frac{1}{3}\{(1-a)|0_A, 0_B\rangle\langle 0_A, 0_B| + 2|\psi^+\rangle\langle\psi^+| + a|1_A, 1_B\rangle\langle 1_A, 1_B|\}(0 \leq a \leq 1)$, where $|\psi^+\rangle$ is the same as that in example (2) [14]. The density matrix of this state is:

$$\rho_X = \begin{pmatrix} \frac{1-a}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{a}{3} \end{pmatrix}. \tag{47}$$

The corresponding GGQD and GD are

$$D^G(\rho_{AB}) = \frac{2}{9}\left(a - \frac{1}{2}\right)^2 + \frac{5}{36}, \tag{48}$$

$$D(\rho_{AB}) = \frac{1}{9}\left(a - \frac{1}{2}\right)^2 + \frac{5}{36}. \tag{49}$$

We plot $D^G(\rho_X)$, $D(\rho_X)$ and $D_T(\rho_X)$ for the state (47) in Fig. 3. We see that $D^G(\rho_X)$ and $D(\rho_X)$ have the same minimum values $\frac{5}{36} \approx 0.14$ at $a = \frac{1}{2}$, but D_T has the minimum

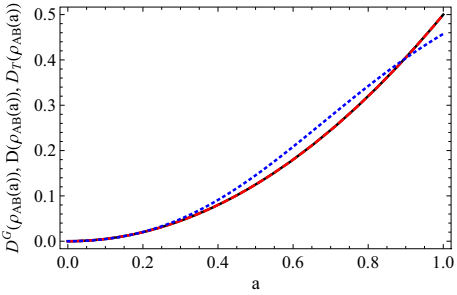

Fig. 1 (*Color online*) Graphs of $D^G(\rho_X)$(black line), $D(\rho_X)$(*red dashed line*) and $D_T(\rho_X)$ (*blue dotted line*) as functions of the parameter a for the class of states in (41)

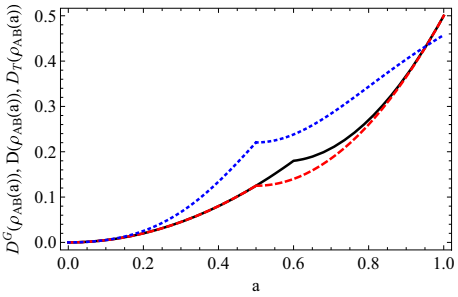

Fig. 2 (*Color online*) Graphs of $D^G(\rho_X)$(*black line*), $D(\rho_X)$(*red dashed line*) and $D_T(\rho_X)$(*blue dotted line*) as functions of the parameter a for the class of states in (44)

value $\frac{1}{2}(2+\sqrt{2})(2-\frac{\sqrt{31}}{3}) \approx 0.25$ at the same point. The three curves are symmetric about $a = \frac{1}{2}$.

(4) Two atoms in the Tavis-Cumming model [49]. We consider two atoms (A and B), each of which interacts resonantly with a single quantized cavity field (system C) in a Fock state. This system is described by the two-atom Tavis-Cummings (TC) Hamiltonian: $H = \hbar g[(\sigma_A + \sigma_B)a_C^\dagger + (\sigma_A^\dagger + \sigma_B^\dagger)a_C]$, where σ_j and $\sigma_j^\dagger$ denote the Pauli ladder operators for the jth atom, $a(a^\dagger)$ stands for the annihilation (creation) operator of photons in cavity C, and g is the coupling constant. We consider that the system is initially in the state $|\psi(0)\rangle = (\alpha|0_A 0_B\rangle + \beta|1_A 1_B\rangle)|n_C\rangle$. Because the total number of excitations is conserved by TC Hamiltonian, the cavity mode will develop within a five-dimensional Hilbert space spanned by $\{|(n-2)_C\rangle, |(n-1)_C\rangle, |n_C\rangle, |(n+1)_C\rangle, |(n+2)_C\rangle\}$ for $n \geq 2$. When $n = 0, 1$ the dimension is 3 and 4, respectively. On the other hand, since the atomic system evolves within the subspace $\{|0_A 0_B\rangle, |+\rangle, |1_A 1_B\rangle\}$ with $|+\rangle = (|1_A 0_B\rangle + |0_A 1_B\rangle)/\sqrt{2}$ independently of n, for our purpose, we only consider $n = 0$ case. By solving the Schrödinger equation, we obtain the state of the system at time t,

$$|\psi(t)\rangle = c_1(t)|0_A 0_B\rangle|2_C\rangle + c_2(t)|+\rangle|1_C\rangle + c_3(t)|1_A 1_B\rangle|0_C\rangle + c_4(t)|0_A 0_B\rangle|0_C\rangle \quad (50)$$

with the following probability amplitudes

$$\begin{aligned} c_1(t) &= -\frac{\sqrt{2}}{3}\beta\left[1 - \cos(\sqrt{6}gt)\right], \\ c_2(t) &= -\frac{i\beta}{\sqrt{3}}\sin(\sqrt{6}gt), \\ c_3(t) &= \beta\left\{1 + \frac{1}{3}[\cos(\sqrt{6}gt) - 1]\right\}, \\ c_4(t) &= \alpha. \end{aligned} \quad (51)$$

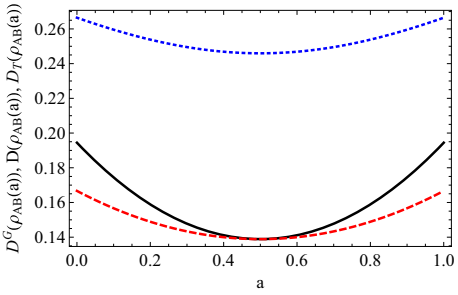

Fig. 3 (*Color online*) Graphs of $D^G(\rho_X)$(*black line*), $D(\rho_X)$(*red dashed line*) and $D_T(\rho_X)$(*dotted blue line*) as functions of the parameter a for the class of states in (47)

Now, we take trace of the density operator $\rho = |\psi(t)\rangle\langle\psi(t)|$ over cavity C resulting in the reduced density matrix of the qubit-qubit system

$$\rho_{AB} = \begin{pmatrix} |c_1|^2 + |c_4|^2 & 0 & 0 & |c_3c_4| \\ 0 & \frac{|c_2|^2}{2} & \frac{|c_2|^2}{2} & 0 \\ 0 & \frac{|c_2|^2}{2} & \frac{|c_2|^2}{2} & 0 \\ |c_3c_4| & 0 & 0 & |c_3|^2 \end{pmatrix}. \tag{52}$$

Using (39) and (40), we obtain

$$D^G(\rho_{AB}) = \left(|c_1|^2 + |c_4|^2\right)^2 + |c_2|^4 + |c_3|^4 + 2|c_3c_4|^2 - \frac{1}{4} - \max\left[\frac{1}{2}|c_2|^4 + |c_3|^4 + \left(|c_1|^2 + |c_4|^2\right)^2 - \frac{1}{4}, \left(\frac{1}{2}|c_2|^2 + |c_3c_4|\right)^2\right], \tag{53}$$

$$D(\rho_{AB}) = \frac{1}{2}\left(|c_1|^4 + 4|c_2|^4 + |c_3|^4 + |c_4|^4 - |c_2|^2\right) + \left(|c_1|^2 + 2|c_3|^2\right)|c_4|^2 - \max\left[\frac{1}{2}\left(\frac{1}{2}|c_2|^4 - 1 - |c_2|^2\right) + (1 - |c_2|^2)|c_3|^2, \left(\frac{1}{2}|c_2|^2 + |c_3c_4|\right)^2\right] \tag{54}$$

In this case, $D^G(\rho_{AB})$, $D(\rho_{AB})$ and $D_T(\rho_{AB})$ as functions of dimensionless time $\tau = \sqrt{6}gt/(6\pi)$ are plotted in Fig. 4, which shows that three curves change periodically with a period $T_\tau = 1$. In addition, they simultaneously arrive to their maximums and minimums. Furthermore, the practical calculation shows the results for $n \geq 1$ are the same as those in Fig. 4, which enhances that the evolution of two atomic system is independent of n, as pointed out earlier.

(5) As a final example, let us consider two atoms A and B in a common reservoir C [49]. We suppose that the initial state of this system was $|\Psi(0)\rangle = (|g_Ag_B\rangle + |e_Ae_B\rangle)|\bar{0}\rangle_C$, where $|\bar{0}\rangle = \prod_k |0\rangle_k$ is the reservoir vacuum state. The overall state of the system at time t can be written as

$$|\Psi(t)\rangle = \alpha|g_Ag_B\rangle|\bar{0}_C\rangle + c_1(t)|e_Ae_B\rangle|\bar{0}_C\rangle + c_2(t)|+\rangle_{AB}|\bar{1}_C\rangle + c_3(t)|g_Ag_B\rangle|\bar{2}_C\rangle, \tag{55}$$

where $|+\rangle_{AB} = (|e_Ag_B\rangle + |g_Ae_B\rangle)/\sqrt{2}$ and $|\bar{k}\rangle$ denotes the collective states of the reservoir in k excitations. The probability amplitudes for this case are

$$c_1(t) = \beta e^{-\gamma t}, \quad c_2(t) = \beta\sqrt{2\gamma t}e^{-\gamma t}, \quad c_3(t) = \sqrt{1 - \alpha^2 - c_1^2(t) - c_2^2(t)}. \tag{56}$$

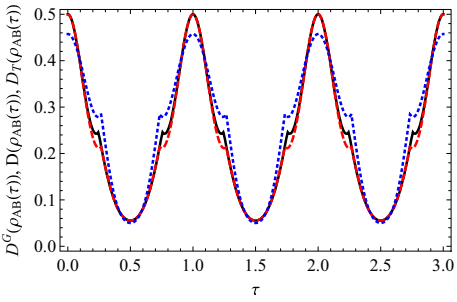

Fig. 4 (*Color online*) The evolution of $D^G(\rho_{AB})$, $D(\rho_{AB})$ and $D_T(\rho_{AB})$ as functions of the dimensionless time $\tau = \sqrt{6}gt/(2\pi)$ for the initial state $|\psi(0)\rangle = (\alpha|0_A0_B\rangle + \beta|1_A1_B\rangle)|n_C\rangle$ with $\alpha = \beta = 1/\sqrt{2}$. The black solid line corresponds to $D^G(\rho_{AB})$, the red dashed line to $D(\rho_{AB})$ and the blue dotted line to $D_T(\rho_{AB})$

Tracing out the reservoir, we obtain the density matrix of atomic subsystem

$$\rho_{AB} = \begin{pmatrix} \alpha^2 + c_3^2 & 0 & 0 & \alpha c_1 \\ 0 & \frac{c_2^2}{2} & \frac{c_2^2}{2} & 0 \\ 0 & \frac{c_2^2}{2} & \frac{c_2^2}{2} & 0 \\ \alpha c_1 & 0 & 0 & c_1^2 \end{pmatrix}, \tag{57}$$

which is just an X state. The corresponding GGQD and GD are

$$D^G(\rho_{AB}) = 3/4 - 2\left[c_1^2\left(c_2^2 + c_3^2\right) + c_2^2\left(c_3^2 + \alpha^2\right)\right] - \max\left[c_1^4 + c_2^4/2 + \left(\alpha^2 + c_3^2\right)^2 - 1/4, \left(c_2^2/2 + \alpha c_1\right)^2\right], \tag{58}$$

$$D(\rho_{AB}) = \frac{1}{4}\left[2 + 7c_2^4 - 6c_2^2 - 4c_1^2\left(c_3^2 - \alpha^2\right)\right] - \max\left\{\frac{1}{2}\left[1 - 3c_2^2\left(1 - c_2^2\right) - 2c_1^2\left(c_3^2 + \alpha^2\right) - c_2^4/2\right], \left(c_2^2/2 + \alpha c_1\right)^2\right\}. \tag{59}$$

In deducing above two equations, $c_1^2 + c_2^2 + c_3^2 + \alpha^2 = 1$ has been used. We plot $D^G(\rho_{AB})$, $D(\rho_{AB})$ and $D_T(\rho_{AB})$ as functions of the dimensionless time γt in Fig. 5. $D^G(\rho_{AB})$, $D(\rho_{AB})$ and $D_T(\rho_{AB})$ have the similar behavior with γt: they have two relative maximums as well as one relative minimum, respectively. Their corresponding relative maximums and relative minimums are close to each other. In addition, $D^G(\rho_{AB})$ and $D(\rho_{AB})$ have the same initial values $2\alpha^2(1-\alpha^2)$, which is greater than the initial values of $D_T(\rho_{AB})$. Finally, when $t \to \infty$, $D^G(\rho_{AB})$, $D(\rho_{AB})$ and $D_T(\rho_{AB})$ simultaneously approach zero.

5 Discussion

We have derived analytical formulas of GGQD and GD for two-qubit X states. Here we give some useful remarks. First, it should be pointed out that (20, 31), from which (39) was derived, are applicable not only to two-qubit X states, but also to any two-qubit states. Second, because of $\mathrm{tr}(ACB^tBC^tA^t) = \mathrm{tr}(BC^tA^tACB^t)$, we can alternatively first optimize system B, then system A. This is equivalent to exchange subsystems A and B, and transpose matrix C. Of course, the two procedures give the same results. Third, more importantly, we find that GGQD are always greater than or equal to GD in five examples given in Section 4. In fact, this is true for any X state. We give a proof below.

First, using $\mathrm{tr}(\rho_X) = \varrho_{00} + \varrho_{11} + \varrho_{22} + \varrho_{33} = 1$ we easily obtain

$$\left(\varrho_{00}^2 + \varrho_{11}^2 + \varrho_{22}^2 + \varrho_{33}^2 - 1/4\right) - \left[\left(\varrho_{00}^2 + \varrho_{11}^2 + \varrho_{22}^2 + \varrho_{33}^2\right)/2 - \varrho_{00}\varrho_{22} - \varrho_{11}\varrho_{33}\right]$$
$$= [2(\varrho_{00} + \varrho_{22}) - 1]^2/4 = [2(\varrho_{11} + \varrho_{33}) - 1]^2/4 \geq 0, \tag{60}$$

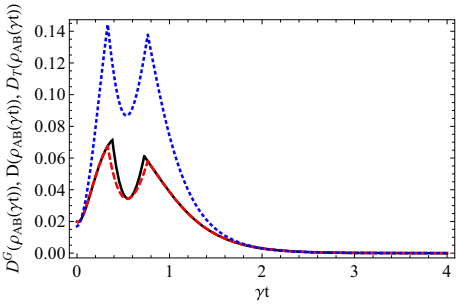

Fig. 5 (*Color online*) The evolution of $D^G(\rho_{AB})$, $D(\rho_{AB})$ and $D_T(\rho_{AB})$ as functions of the dimensionless time γt for the initial state $|\psi(0)\rangle = (\alpha|0_A 0_B\rangle + \beta|1_A 1_B\rangle)|\bar{0}_C\rangle$ with $\alpha = 0.1$ and $\beta = \sqrt{1-\alpha^2}$. The black solid line corresponds to $D^G(\rho_{AB})$, the red dashed line to $D(\rho_{AB})$ and the blue dotted line to $D_T(\rho_{AB})$

which means $\varrho_{00}^2+\varrho_{11}^2+\varrho_{22}^2+\varrho_{33}^2-1/4 \geq (\varrho_{00}^2+\varrho_{11}^2+\varrho_{22}^2+\varrho_{33}^2)/2-\varrho_{00}\varrho_{22}-\varrho_{11}\varrho_{33}$. Therefore, there are only three cases need to be considered.

(1) $(\varrho_{12}+\varrho_{03})^2 \geq \varrho_{00}^2+\varrho_{11}^2+\varrho_{22}^2+\varrho_{33}^2-1/4 \geq \left(\varrho_{00}^2+\varrho_{11}^2+\varrho_{22}^2+\varrho_{33}^2\right)/2-\varrho_{00}\varrho_{22}-\varrho_{11}\varrho_{33}$:

$$D^G(\rho_{AB}) = \varrho_{00}^2+\varrho_{11}^2+\varrho_{22}^2+\varrho_{33}^2-\frac{1}{4}+(\varrho_{12}-\varrho_{03})^2, \tag{61}$$

$$D(\rho_{AB}) = \frac{1}{2}\left[(\varrho_{00}-\varrho_{22})^2+(\varrho_{11}-\varrho_{33})^2\right]+(\varrho_{12}-\varrho_{03})^2, \tag{62}$$

$$D^G(\rho_{AB})-D(\rho_{AB}) = \frac{1}{4}[2(\varrho_{00}+\varrho_{22})-1]^2 = \frac{1}{4}[2(\varrho_{11}+\varrho_{33})-1]^2 \geq 0. \tag{63}$$

(2) $\varrho_{00}^2+\varrho_{11}^2+\varrho_{22}^2+\varrho_{33}^2-1/4 \geq (\varrho_{12}+\varrho_{03})^2 \geq \left(\varrho_{00}^2+\varrho_{11}^2+\varrho_{22}^2+\varrho_{33}^2\right)/2-\varrho_{00}\varrho_{22}-\varrho_{11}\varrho_{33}$:

$$D^G(\rho_{AB}) = 2\left(\varrho_{12}^2+\varrho_{03}^2\right), \tag{64}$$

$$D(\rho_{AB}) = \frac{1}{2}\left[(\varrho_{00}-\varrho_{22})^2+(\varrho_{11}-\varrho_{33})^2\right]+(\varrho_{12}-\varrho_{03})^2, \tag{65}$$

$$D^G(\rho_{AB})-D(\rho_{AB}) = (\varrho_{12}+\varrho_{03})^2-\frac{1}{2}\left[(\varrho_{00}-\varrho_{22})^2+(\varrho_{11}-\varrho_{33})^2\right] \geq$$
$$\frac{1}{2}(\varrho_{00}^2+\varrho_{11}^2+\varrho_{22}^2+\varrho_{33}^2)-\varrho_{00}\varrho_{22}-\varrho_{11}\varrho_{33}-\frac{1}{2}\left[(\varrho_{00}-\varrho_{22})^2+(\varrho_{11}-\varrho_{33})^2\right] = 0. \tag{66}$$

(3) $(\varrho_{12}+\varrho_{03})^2 \leq (\varrho_{00}^2+\varrho_{11}^2+\varrho_{22}^2+\varrho_{33}^2)/2-\varrho_{00}\varrho_{22}-\varrho_{11}\varrho_{33} \leq \varrho_{00}^2+\varrho_{11}^2+\varrho_{22}^2+\varrho_{33}^2-1/4$:

$$D^G(\rho_{AB}) = D(\rho_{AB}) = 2(\varrho_{12}^2+\varrho_{03}^2). \tag{67}$$

We conclude that $D^G(\rho_{AB}) \geq D(\rho_{AB})$ for any X state from (60, 63, 66, 67). However, our examples show that D_T is not always greater than $D^G(\rho_{AB})$ or $D(\rho_{AB})$.

6 Summary

In summary, we have first proven GGQD and TQC are the same and then obtained analytical formulas of GGQD and GD for two-qubit X states. We have also compared GGQD, TQC and RGD by five concrete examples. We have further found that GD is the tight lower bound of GGQD, which means that GD is a good approximation for GGQD at least for X states. There are still some interesting opening problems to be studied in this aspect, such as, are there any analytical expressions of GGQD for qubit-qudit system? Can GD be a tight lower bound of GGQD for any bipartite system? We shall report our research results on these issues later.

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