


# Friedmann Cosmology with Matter Creation in Modified $f(R, T)$ Gravity

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**Abstract** The theoretical and observational consequences of thermodynamics of open systems which allow matter creation, are investigated in modified  $f(R, T)$  ( $R$  is the Ricci scalar and  $T$  is the trace of energy-momentum tensor) theory of gravity within the framework of a flat Friedmann-Robertson-Walker line element. The simplest model  $f(R, T) = R + 2f(T)$  with “gamma-law” equation of state  $p = (\gamma - 1)\rho$  is assumed to obtain the exact solution. A power-law expansion model is proposed by considering the natural phenomenological particle creation rate  $\psi = 3\beta nH$ , where  $\beta$  is a pure number of the order of unity,  $n$  the particle number density and  $H$  is the Hubble parameter. A Big Rip singularity is observed for  $\gamma < 0$  describing phantom cosmology. The accelerated expansion of the Universe is driven by the particle creation. The density parameter shows the negative curvature of the Universe due to particle creation. The entropy increases with the evolution of the Universe. Some kinematics tests such as lookback time, luminosity distance, proper distance, angular diameter *versus* redshift are discussed in detail to observe the role of particle creation in early and late time evolution of the Universe.

**Keywords** Cosmology · Modified gravity theory · Matter creation

## 1 Introduction

Many theoretical and observational studies of the Universe, such as Type Ia supernovae [1, 2], Planck collaboration [3], cosmic microwave background (CMB) anisotropy [4], large

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scale structures (LSS) [5] etc., have shown that a pure Friedmann-Lamaitre-Robertson-Walker (FLRW) cosmology with matter and radiation does not explain all the large scale properties of our Universe. The late time cosmic acceleration is usually assumed to be driven by a cosmic fluid/field generically known as dark energy (DE) [6–8], which can be observed only by its gravitational effects [9–11]. Since the time of the discovery of accelerating Universe in 1998, a large number of probable candidates of DE have been proposed. The simplest DE model is the standard  $\Lambda$ CDM model [12] where  $\Lambda$  is known as cosmological constant assumed possibly to be the quantum vacuum energy. In the context of early Universe, the standard  $\Lambda$ CDM model presents several theoretical and observational difficulties, such as the singularity problem [13], flatness and horizon problem, reheating during the inflationary epoch [14], conffliction between the age of the Universe and the age of the oldest stars in globular clusters (age problem) [15], the entropy problem [16] etc.

The flatness and horizon problems together with the entropy problem have been resolved (up to a certain extent) by the inflationary model proposed by Guth [17]. The age conffliction [18] is not an isolated complication, it comes with another serious trouble that is structure formation through gravitational amplification of small primeval density perturbation. These issues opened the door of investigations of different alternative theories of gravity not only alleviate the issues related to early Universe [19–21] but the mystery of unpredicted late time cosmic acceleration as well. Among the ways to resolve the problems of early Universe, Dirac's large number hypothesis [22] inspired a class of new cosmology named particle creation [23]. The steady state model introduced by Bondi and Gold [24] on the foundation of perfect cosmological principle (PCP) also asserts the continuous generation of matter in the Universe. Hoyle [25] and Narlikar [26] independently proposed a creation field theory and studied the matter creation during the evolution of the Universe. Tryon [27] and Fomin [28] in their individual work proposed a theoretical concept of the creation of the Universe as a vacuum fluctuation. Brout et al. [29–31] proposed a strong foundation of simultaneous creation of matter and curvature from a quantum fluctuation of the Minkowskian space-time vacuum.

Fulling and collaborator [32] investigated the evolution of the Universe where the material content of the Universe may have had its origin in the continuous creation of radiation and matter from the gravitational field of the expanding cosmos acting on the quantum vacuum. Later on, Gunzig et al. [33] and Prigogine et al. [34] established the theoretical scenario of matter creation in the framework of cosmology. They showed that the second law of thermodynamics might be modified to accommodate the flow of energy from gravitational field to the matter field, resulting in the creation of particles and consequently entropy. The work might suggest that at the expense of the gravitational field, particle creation takes place as an irreversible process constrained by the usual requirements of the non-equilibrium thermodynamics, however, the reverse process (matter destruction) thermodynamically forbidden. Calvão et al. [35] extended this new theoretical concept of matter creation under adiabatic conditions. The further results were generalized by Lima and Germano [36] through a contravariant formulation allowing specific entropy variation as usually expected for non-equilibrium process in fluids. Lima and Alcaniz [37], and Alcaniz and Lima [38] investigated observational consequences of FRW models driven by adiabatic matter creation through some kinematical tests.

After the discovery of accelerating Universe, the matter creation theory was reconsidered to explain it and obtained unexpected results. The studies of matter creation rapidly recognize to explain dark energy. The negative pressure due to particle creation, might play the role of exotic matter component. Zimdahl et al. [39], Qiang et al. [40], Lima et al.

[41] and Steigman et al. [42] have presented a model with matter creation where a transition from decelerated to accelerated phase exists which are consistent with SNe Ia data. Cárdenas [43] studied a model which offers a possible way to understand the current accelerated expansion. Singh and Beesham [44, 45], and Singh [46, 47] studied early Universe in FRW cosmology with particle creation through some kinematical tests. Lima et al. [48] proposed a creation cold dark matter cosmology with one free parameter that is equivalent to the  $\Lambda$ CDM evolution. Recently, many authors [49–51] have paid attention on the cosmologies driven by gravitational “adiabatic” particle production where matter and entropy are generated but the specific entropy remains constant. Even a complete cosmology where the space-time matter evolves between an early and a late time de Sitter phase driven by particle production has been proposed and its predictions has been compared with the available astronomical data.

On the other hand, a number of pioneer concepts of modifying the general relativity (GR) have been proposed to reconcile the problems plagued in cosmology [52]. The simplest modification is replacing the Ricci scalar curvature  $R$  with the general function of  $R$  called  $f(R)$  theories of gravity, describe transition from decelerated to accelerated expansion of the Universe [53]. Bertolami et al. [54] have generalized  $f(R)$  theories by introducing an explicit coupling between arbitrary function of the Ricci scalar  $R$  and the matter Lagrangian density. Harko [55] has extended this new concept to the arbitrary coupling between matter and geometry. Harko et al. [56] have considered another extension of GR, where the gravitational Lagrangian is given by an arbitrary function of the Ricci scalar  $R$  and of the trace  $T$  of the stress-energy tensor, so called  $f(R, T)$  theory of gravity. The authors have argued that due to the coupling between matter and geometry the theory depends on a source term, representing the variation of the matter-stress-energy tensor with respect to the metric. Consequently, the cosmic acceleration in  $f(R, T)$  gravity results not only from geometrical effect but also from the matter contribution. These interesting features of  $f(R, T)$  gravity motivate many authors to study it for resolving several issues of current interest in cosmology and astrophysics [57–65]. This newly proposed concept of modified gravity motivates us to study the particle creation phenomena and its effects on early and late time evolution of the Universe in  $f(R, T)$  gravity.

In this paper, we investigate the theoretical and observational implication of particle creation in modified  $f(R, T)$  gravity theory in a flat Friedmann-Robertson-Walker model. Exact solutions of the field equations are obtained by assuming the suitable form of  $f(R, T) = R + 2f(T)$ , equation of state, and particle creation rate. We study some kinematical tests to explain the physical significance of particle creation during early and late time evolution of the Universe.

The paper is organized as follows. The thermodynamics of particle creation is presented in the Section 2. A brief review of modified  $f(R, T)$  gravity theory and its field equations are given in Section 3. In Section 4, the model and field equations are presented with particle creation. The exact solutions of the field equations with suitable assumptions are obtained in Section 5. Section 6 and its subsections are devoted to study some kinematical tests of the model. Finally, the outcomes are summarized in concluding Section 7.

## 2 Theory of Particle Creation

If we regard the whole Universe as a closed thermodynamical system in which the number of the particles in a given volume is constant, then the laws of thermodynamics have the

form [14]

$$d(\rho_m V) = dQ - p_m dV, \quad (1)$$

and

$$TdS = d(\rho_m V) + p_m dV, \quad (2)$$

where  $\rho_m$ ,  $V$ ,  $p_m$ ,  $T$  and  $S$  are the energy density, volume, thermodynamical pressure, temperature, and entropy, respectively. Here,  $dQ$  is the heat received by the system during time  $dt$ . It is observed from (1) and (2) that the entropy production is given by

$$TdS = dQ. \quad (3)$$

Equation (3) shows that the entropy remains stationary, *i.e.*,  $dS = 0$  for a closed adiabatic system for which  $dQ = 0$ . However, if we treat the Universe as an open thermodynamic system allowing irreversible matter creation from the energy of the gravitational field, we can account for entropy production right from the beginning, and the second law of thermodynamics is also incorporated into the evolutionary equations in a more meaningful way. Thus, the creation of matter acts as a source of internal energy. In such situation the number of particles  $N$  in a given volume  $V$  is not to be a constant but is time -dependent. Therefore, (1) modifies as

$$d(\rho_m V) = dQ - p_m dV + (h/n) d(nV), \quad (4)$$

where  $N = nV$ ,  $n$  is the particle number density, and  $h = (\rho_m + p_m)$  is the enthalpy per unit volume of the system. In case of adiabatic system where  $dQ = 0$ , (4) for an open system reduces to

$$d(\rho_m V) + p_m dV = (h/n) d(nV). \quad (5)$$

We see that in such a system the thermal energy is received due to the change of the number of particles. In cosmology, this change may be considered as a transformation of energy from gravitational field to the matter.

In the context of an open system, (5) can be rewritten as

$$d(\rho_m V) = -(p_m + p_c)dV, \quad (6)$$

where

$$p_c = -(h/n)(dN/dV). \quad (7)$$

Equation (6) suggests that the creation of matter in an open thermodynamic system corresponds to a supplementary pressure  $p_c$ , which must be considered as a part of the cosmological pressure entering into the Einstein field equations (decaying of matter leads to a positive decay pressure) and is equivalent to adding the term  $p_c$  given by (7) to the thermodynamic pressure  $p_m$ . It is to be noted that  $p_c$  is negative or zero depending on the presence or absence of particle creation.

Since the increment in entropy for an adiabatic system is only caused by creation of matter, the entropy is an extensive property of the system. In present scenario,  $S$  is proportional to the number of particles included in the system. Therefore, the entropy change  $dS$  for open systems from (2) and (5) becomes

$$TdS = (h/n) d(nV) - \mu d(nV) = (TS/N)dN \Rightarrow \frac{dS}{S} = \frac{dN}{N}, \quad (8)$$

where  $\mu$  is the chemical potential given by  $\mu = (h - Ts)/n$ ,  $s = S/V$ . Since the second law of thermodynamics is a fundamental law in physics, the presence or absence of particle creation can not affect it. This law basically requires  $dS \geq 0$ , consequently, (8) gives

$$dN \geq 0. \quad (9)$$

The above inequality implies that the space-time can produce matter, whereas the reverse process is thermodynamically not admissible.

The basic idea of this entire formulation is to modify the usual energy momentum conservation law in an open thermodynamical system, which leads to the explicit use of a balance equation for the number density of the particles created, in addition to Einstein’s field equations.

The particle flux vector is given by

$$N^\alpha = nu^\alpha, \tag{10}$$

and  $N^\alpha$  is assumed to satisfy the balance equation [34, 66]

$$N^\alpha_{;\alpha} = \psi, \tag{11}$$

where the function  $\psi$  denotes a source term of particle creation which is positive or negative depending on whether there is production or annihilation of particles. In standard cosmology  $\psi$  is usually assumed to be zero.

In the presence of a gravitational particle source, the balance equation (11) for the particle flux becomes

$$\dot{n} + 3nH = \psi, \tag{12}$$

where  $n$  and  $V = a^3$  are the particle number density and volume, respectively.

Thus, the pressure creation  $p_c$  depends on the particle creation rate, and for adiabatic matter creation, (7) takes the following form [36]

$$p_c = -\frac{(\rho_m + p_m)}{3nH} \psi. \tag{13}$$

Therefore, (13) shows that  $p_c$  is negative for  $\psi > 0$ , which can help to drive the era of accelerated cosmic expansion.

### 3 A Brief Review of Modified $f(R, T)$ Gravity Theory

The  $f(R, T)$  theory is a modified theory of gravity, in which the Einstein-Hilbert Lagrangian is modified by replacing Ricci scalar curvature  $R$  by an arbitrary function of  $R$  and trace  $T$  of energy-momentum tensor, *i.e.*,  $f(R, T)$ . So the gravitational action for  $f(R, T)$  modified theory of gravity [56] in the units  $G = 1 = c$  takes the following form.

$$S = \frac{1}{8\pi} \int d^4x \sqrt{-g} \left[ \frac{f(R, T)}{2} + \mathcal{L}_m \right], \tag{14}$$

where  $g$  is the determinant of the metric tensor  $g_{\mu\nu}$  and  $\mathcal{L}_m$  corresponds to matter Lagrangian. The energy-momentum tensor  $T_{\mu\nu}$ , defined by fluid Lagrangian density is given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_m)}{\delta g^{\mu\nu}}, \tag{15}$$

and its trace,  $T = g^{\mu\nu} T_{\mu\nu}$ . By assuming that the matter Lagrangian density  $\mathcal{L}_m$  depends only on the metric tensor components  $g_{\mu\nu}$ , not on its derivatives, we obtain

$$T_{\mu\nu} = g_{\mu\nu} \mathcal{L}_m - 2 \frac{\partial \mathcal{L}_m}{\partial g^{\mu\nu}}. \tag{16}$$

The equations of motion by varying the action (14) with respect to metric tensor become

$$f_R(R, T)R_{\mu\nu} - \frac{1}{2}f(R, T)g_{\mu\nu} + (g_{\mu\nu}\square - \nabla_\mu\nabla_\nu)f_R(R, T) = 8\pi T_{\mu\nu} - f_T(R, T)T_{\mu\nu} - f_T(R, T)\ominus_{\mu\nu}, \tag{17}$$

where  $f_R$  and  $f_T$  denote the derivatives of  $f(R, T)$  with respect to  $R$  and  $T$ , respectively. Here,  $\nabla_\mu$  is covariant derivative and  $\square \equiv \nabla_\mu\nabla^\mu$  is the d'Alembert operator and  $\ominus_{\mu\nu}$  is defined by

$$\ominus_{\mu\nu} \equiv g^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}. \tag{18}$$

Using (16) into (18), we get

$$\ominus_{\mu\nu} = -2T_{\mu\nu} + g_{\mu\nu}\mathcal{L}_m - 2g^{\alpha\beta} \frac{\partial^2 \mathcal{L}_m}{\partial g^{\mu\nu} \partial g^{\alpha\beta}}. \tag{19}$$

As the field equations of  $f(R, T)$  gravity depend on  $\ominus_{\mu\nu}$ , *i.e.*, on the physical nature of the matter. A number of models corresponding to different form of  $f(R, T)$ , e.g.,  $f(R, T) = R + 2f(T)$ ,  $f(R, T) = \mu f_1(R) + \nu f_2(T)$ , where  $f_1(R)$  and  $f_2(T)$  are arbitrary functions of  $R$  and  $T$ , and  $\mu$  and  $\nu$  are real constants, respectively [56–58], and  $f(R, T) = R f(T)$  [60], etc., may be assumed depending on the nature of the matter source to solve the field equations. In this paper we assume the following simplest form [56]:

$$f(R, T) = R + 2f(T), \tag{20}$$

where  $R$  is a function of cosmic time  $t$  and  $f(T)$  is an arbitrary function of the trace of energy-momentum tensor of matter. Equation (20) shows that the action is given by the same Einstein-Hilbert of General Relativity plus a function of  $T$ . The term  $2f(T)$  in the gravitational action modifies the gravitational interaction between matter and curvature.

Using (20), one can re-write the gravitational field equations defined in (17) as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} - 2(T_{\mu\nu} + \ominus_{\mu\nu})f'(T) + f(T)g_{\mu\nu}. \tag{21}$$

Here, a prime stands for derivative of  $f(T)$  with respect to  $T$ .

The main issue now arises on the matter contents of the Universe through the energy momentum tensor and consequently on the matter Lagrangian  $\mathcal{L}_m$  and the trace of the energy momentum tensor. We assume the universe is filled with a perfect fluid which is incorporated in the next section.

### 4 Model and Field Equations

We consider a homogenous and isotropic universe with spatially flat geometry described by flat Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2), \tag{22}$$

where  $a(t)$  is the scale factor, which is a function of cosmic time  $t$  only.

In the formalism of particle creation, the second law of thermodynamics leads naturally to a modification of the energy momentum tensor with an additional creation pressure depends on the rate of creation of particles. In the presence of particle creation, the energy momentum tensor of perfect fluid is given by

$$T_{\mu\nu} = (\rho_m + p_m + p_c)u_\mu u_\nu - (p_m + p_c)g_{\mu\nu}, \tag{23}$$

where  $u^\mu_\mu$  is the four velocity of the fluid such that  $u_\mu u^\mu = 1$ , and in comoving coordinates  $u^\mu = \delta^\mu_0$ .

The trace of energy momentum tensor (23), gives

$$T = \rho_m - 3(p_m + p_c). \tag{24}$$

We treat the scalar invariant  $\mathcal{L}_m$  as the effective pressure of the perfect fluid matter and pressure originated by creation of particles. Therefore, the matter Lagrangian may be assumed as  $\mathcal{L}_m = -(p_m + p_c)$ , therefore, (19) becomes

$$\ominus_{\mu\nu} = -2T_{\mu\nu} - (p_m + p_c)g_{\mu\nu}. \tag{25}$$

In view of (25), the field (21) can be rewritten as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} + 2[T_{\mu\nu} + g_{\mu\nu}(p_m + p_c)]f'(T) + g_{\mu\nu}f(T). \tag{26}$$

The field equations (26) for a fluid endowed with matter creation (23) in the background of metric (22), yield

$$3H^2 = 8\pi\rho_m + 2(\rho_m + p_m + p_c)f'(T) + f(T), \tag{27}$$

$$2\dot{H} + 3H^2 = -8\pi(p_m + p_c) + f(T), \tag{28}$$

where  $H(t) = \dot{a}/a$  is the Hubble parameter. A dot denotes derivative with respect to cosmic time  $t$ .

### 5 Solution of Field Equations

We have two field equations (27)–(28) and five unknowns variables, namely,  $H$ ,  $\rho_m$ ,  $p_m$ ,  $p_c$  and  $f(T)$ . Therefore, one needs three more relations in order to construct a definite cosmological scenario.

In first choice, we consider a particular function given by [56]

$$f(T) = \lambda T, \quad (\lambda \text{ is a constant}). \tag{29}$$

Using (29) and (24), the field (27) and (28) yield

$$3H^2 = (8\pi + 3\lambda)\rho_m - \lambda(p_m + p_c), \tag{30}$$

$$2\dot{H} + 3H^2 = -(8\pi + 3\lambda)(p_m + p_c) + \lambda\rho_m. \tag{31}$$

In order to obtain the exact solution of the field equations, we assume two more additional relations: the equation of state and the matter creation rate. In the cosmological domain, the former is usually given by the “gamma-law” form

$$p_m = (\gamma - 1)\rho_m, \tag{32}$$

where  $\gamma$  is a constant lies in the interval  $[0, 2]$  and known as the EoS parameter of the perfect fluid.

Using (32) into (30) and (31), and simplifying, we get a single evolution equation for  $H$ :

$$2\dot{H} + (8\pi + 2\lambda)(\gamma\rho_m + p_c) = 0. \tag{33}$$

We confine our attention to the simple phenomenological expression for the matter creation rate [16]

$$\psi(t) = 3\beta nH, \tag{34}$$

where the parameter  $\beta$  lies in the interval  $[0, 1]$  and is assumed to be a constant.

Using (32) and (34) into (13), we have

$$p_c = -\beta\gamma\rho_m, \tag{35}$$

Substituting (32) and (35) into (30), we obtain

$$\rho_m = \frac{3H^2}{8\pi + 4\lambda - \gamma\lambda(1 - \beta)}. \tag{36}$$

Using (35) and (36) into (33), we get

$$\dot{H} + \frac{3}{2} \frac{\gamma(8\pi + 2\lambda)(1 - \beta)}{[8\pi + 4\lambda - \gamma\lambda(1 - \beta)]} H^2 = 0. \tag{37}$$

The solution of (37) for  $\gamma \neq 0$  and for all values of  $\lambda$  and  $\beta$  is given by

$$H(t) = \left( C + \frac{3}{2} \frac{\gamma(8\pi + 2\lambda)(1 - \beta)}{[8\pi + 4\lambda - (1 - \beta)\gamma\lambda]} t \right)^{-1}, \tag{38}$$

where  $C$  is an integration constant. For  $\gamma = 0$ , the well known de-Sitter scale factor  $a(t) = a_0 e^{H_0 t}$  is obtained.

From (38) we find the following expression for the scale factor

$$a(t) = D \left( C + \frac{3}{2} A\gamma t \right)^{\frac{2}{3A\gamma}}, \tag{39}$$

where  $D$  is a new integration constant and  $A = (8\pi + 2\lambda)(1 - \beta)/8\pi + 4\lambda - \gamma\lambda(1 - \beta)$ .

The above scale factor may be rewritten as

$$a(t) = a_0 \left( 1 + \frac{3}{2} A\gamma H_0(t - t_0) \right)^{\frac{2}{3A\gamma}}, \tag{40}$$

where  $H = H_0 > 0$  at  $t = t_0$ . The subscript ‘0’ refers to the present values of parameters. Since  $0 \leq \gamma \leq 2$ , we must have  $A > 0$  for expansion of the Universe. Also,  $A > 0$  implies  $\lambda > 0$  as  $0 \leq \beta < 1$ .

If  $\gamma < 0$ , we have a Big -Rip singularity at a finite value of cosmic time  $t_{br} = t - t_0 = -2/3H_0A\gamma$ . Thus, we have a phantom cosmology for  $\gamma < 0$ . If one choose  $t_0 = 2H_0^{-1}/3A\gamma$ , Eq. (40) takes the familiar form of power-law expansion of the Universe, *i.e.*,

$$a(t) = a_0 \left( \frac{3}{2} A\gamma H_0 t \right)^{\frac{2}{3A\gamma}}. \tag{41}$$

If  $\lambda = 0 = \beta$ , (39) and (41) reduce to the well-known expressions of power-law expansion of scale factor for a flat FRW model in GR.

By use of (40) one may express the energy density of matter, particle creation pressure and the particle number density as functions of the scale factor  $a$ . These parameters respectively have the following forms

$$\rho_m = \rho_0 \left( \frac{a_0}{a} \right)^{3A\gamma}, \tag{42}$$

$$p_c = -\beta\gamma\rho_0 \left( \frac{a_0}{a} \right)^{3A\gamma}, \tag{43}$$

$$n = n_0 \left( \frac{a_0}{a} \right)^{3(1-\beta)}, \tag{44}$$

where  $\rho_0 = 3H_0^2/[8\pi + 4\lambda - \gamma\lambda(1 - \beta)]$  is the present value of energy density. Here,  $n_0$  is the present value of particle number density for any values of  $\beta$ . The above results show that



the transition from one phase to other phase, in the course of expansion, happens exactly as in the standard cosmological model.

The number of particles  $N$  in a given volume  $V$  is given by

$$N = N_0 \left( \frac{a}{a_0} \right)^{3\beta}, \tag{45}$$

which shows that  $N$  increases with time. If  $\beta = 0$ ,  $N$  would remain constant throughout the evolution of the Universe and we would recover the standard FRW model of the Universe in  $f(R, T)$  theory. Again, from (8)  $S = S_0(N/N_0)$ , the entropy in terms of scale factor is given by

$$S = S_0 \left( \frac{a}{a_0} \right)^{3\beta}. \tag{46}$$

The deceleration parameter defined as  $q = -a\ddot{a}/\dot{a}^2$ , gives

$$q = -1 + \frac{3\gamma A}{2} = \left[ \frac{3\gamma}{2} \frac{(8\pi + 2\lambda)(1 - \beta)}{[8\pi + 4\lambda - (1 - \beta)\gamma\lambda]} - 1 \right], \tag{47}$$

which shows that  $q$  is independent of cosmic time  $t$ . Therefore, there is no transition from one phase to other. The decelerated or accelerated phase is described according the sign of  $q$ . Here,  $q$  may be positive or negative for a given set of values of  $\beta$  and  $\lambda$ . We know that the model accelerates for  $q < 0$ , therefore, the value of  $A$  must be  $0 < A < 2/3\gamma$  for an accelerated universe.

As expected, the above solutions reduce to the standard FRW model of GR for  $\beta = 0$  and  $\lambda = 0$  and for all values of  $\gamma$ . In what follows, we study the role of  $f(R, T)$  gravity and particle creation in early and late time evolution of the Universe.

**Case 1**  $\gamma = \frac{2}{3}$

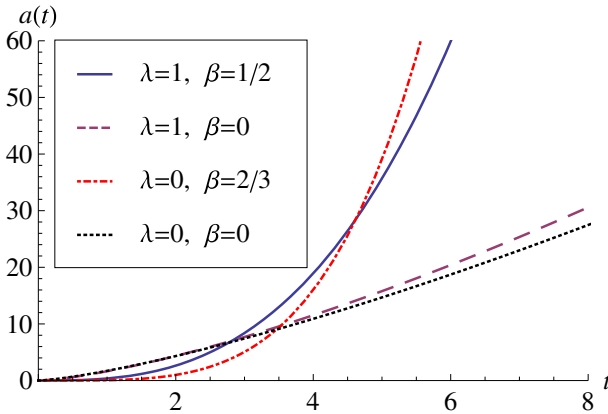
Figure 1 plots the scale factor as a function of time for  $\gamma = 2/3$  and some selected values of  $\lambda$  and  $\beta$ . We observe that if  $\beta = 0$ ,  $q < 0$  for all  $\lambda > 0$ , therefore, we find that the Universe accelerates in  $f(R, T)$  gravity without particle creation. Similarly, if  $\lambda = 0$ , *i.e.*, in the absence of  $f(T)$ ,  $q = -\beta < 0$  for any values of  $\beta > 0$ . Thus, the acceleration occurs due to particle creation. The rate of expansion increases more rapidly for non-zero values of  $\beta$  and  $\lambda$ . It is to be noted that if  $\lambda = 0 = \beta$  then the the marginal inflationary phase of GR is recovered, *i.e.*,  $a \sim t$  and  $q = 0$ .

**Case 2**  $\gamma = \frac{4}{3}$

In this case, if  $\beta = 0$  and  $\lambda > 0$ , we have  $q > 0$ . This shows that the Universe decelerates in the absence of particle creation. If  $\lambda = 0$  then  $q \geq 0$  for  $0 < \beta \leq 1/2$ , and  $q < 0$  for  $1/2 < \beta < 1$ . Therefore, the Universe decelerates or accelerates depending on the rate of particle creation. However, if  $\lambda \neq 0$  and  $\beta \neq 0$ , the deceleration or acceleration of the Universe depend on the following constrains, respectively:

$$0 < \beta \leq \frac{1}{4}, \lambda > 0 \text{ or } \frac{1}{4} < \beta < \frac{1}{2}, 0 < \lambda < \frac{6\pi - 12\pi\beta}{-1 + 4\beta}, \tag{48}$$

$$\frac{1}{4} < \beta \leq \frac{1}{2}, \lambda > \frac{6\pi - 12\pi\beta}{-1 + 4\beta} \text{ or } \frac{1}{2} < \beta < 1, \lambda > 0. \tag{49}$$



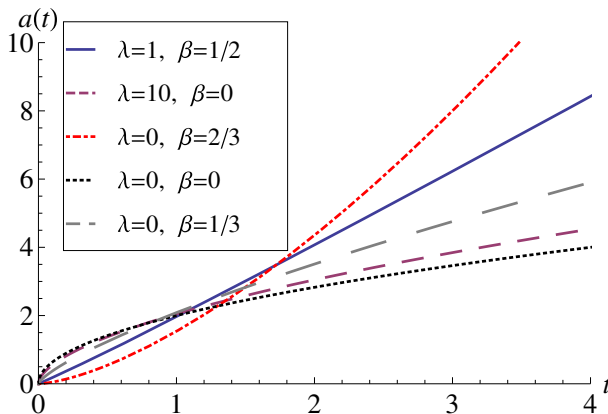
**Fig. 1** Scale factor as a function of time for  $\gamma = \frac{2}{3}$  and some selected values of  $\lambda$  and  $\beta$

The behavior of scale factor *versus* time is shown in Fig. 2 for some selected values of  $\lambda$  and  $\beta$ . The figure shows that the Universe accelerates fast due to higher particle creation rate. For  $\lambda = 0 = \beta$ ,  $a \sim t^{1/2}$  and  $q = 1$ , which is the radiation phase in GR.

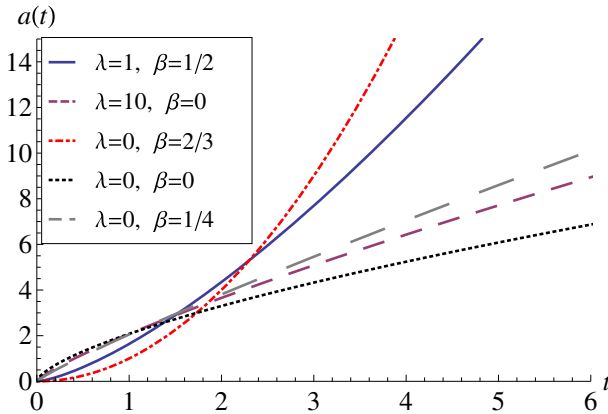
**Case 3**  $\gamma = 1$

In this case, the Universe expands with decelerated rate as  $q > 0$  for  $\beta = 0$  and  $\lambda > 0$ . Figure 3 plots graph between scale factor *versus* time for some selected values of  $\lambda$  and  $\beta$ . For  $\lambda = 0$ , we have  $q > 0$  for  $0 < \beta < 1/3$ , and  $q < 0$  for  $1/3 < \beta < 1$ . The critical case ( $\beta = 1/3, q = 0$ ), describes a coasting cosmology. For  $\lambda \neq 0$  and  $\beta \neq 0$ , the model decelerates or accelerates under the following constraints:

$$0 < \beta < \frac{1}{3}, 0 < \lambda < \frac{\pi - 3\pi\beta}{\beta}, \tag{50}$$



**Fig. 2** Scale factor as a function of time for  $\gamma = \frac{4}{3}$  and some selected values of  $\lambda$  and  $\beta$



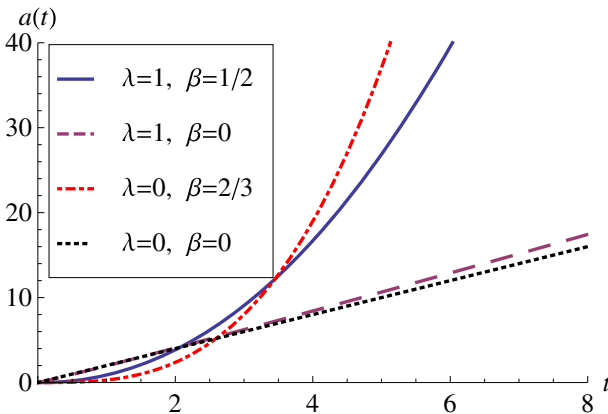
**Fig. 3** Scale factor as a function of time for  $\gamma = 1$  and some selected values of  $\lambda$  and  $\beta$

$$0 < \beta \leq \frac{1}{3}, \lambda > \frac{\pi - 3\pi\beta}{\beta} \text{ or } \frac{1}{3} < \beta < 1, \lambda > 0, \tag{51}$$

respectively. As expected, for  $\lambda = 0 = \beta$ , we have  $a \sim t^{2/3}$  and  $q = 1/2$ , i.e., the model reduces to standard matter-dominated era of GR.

**Case 4**  $\gamma = \frac{1}{2}$

In this case, if  $\lambda = 0 = \beta$ ,  $a \sim t^{4/3}$  and  $q = -1/4$ , which corresponds to the present accelerated phase of the Universe of the standard FRW universe in GR. Since the Universe accelerates even in absence of both  $f(T)$  and particle creation, therefore, the contribution of  $f(R, T)$  gravity or particle creation just enhance the rate of acceleration of the Universe. Figure 4 plots the dynamics of scale factor *versus*  $t$ , which is similar to case I.



**Fig. 4** Scale factor as a function of time for  $\gamma = \frac{1}{2}$  and some selected values of  $\lambda$  and  $\beta$

## 6 Kinematic Tests

Now, we derive some kinematical relations of the model as proposed in the preceding sections.

### 6.1 The Density Parameter

The density parameter, defined as  $\Omega_m = \rho_m/\rho_c$ , where  $\rho_c = 3H^2/8\pi$ , is given by

$$\Omega_m = \frac{8\pi}{8\pi + 4\lambda - (1 - \beta)\gamma\lambda}. \tag{52}$$

Therefore, it is clear that  $\Omega_m < 1$  for all values of  $0 \leq \gamma \leq 2$ ,  $0 < \beta < 1$  and  $\lambda > 0$ . Hence the Universe is negatively curved. In absence of both  $\lambda$  and  $\beta$ , we have  $\Omega_m = 1$  for all  $\gamma$ , i.e., the flat model of GR is recovered.

### 6.2 Lookback Time-Redshift

The lookback time  $\Delta t = t_0 - t(z)$ , is the difference between the age of the Universe at the present time  $z = 0$  and the age of the Universe when a particular right ray at redshift  $z$  was emitted.

For a given redshift  $z$ , the scale factor  $a(t_z)$  is related to  $a_0$  by

$$a(t_z) = a_0(1 + z)^{-1}. \tag{53}$$

From (41) and (53), the cosmic time in terms of redshift is given by

$$t(z) = \frac{2H_0^{-1}}{3\gamma A} (1 + z)^{-\frac{3\gamma A}{2}}, \tag{54}$$

where  $H_0$  is expressed in the usual practical observational units of  $km\ s^{-1}\ Mpc^{-1}$  and its value is believed to be somewhere between  $50\text{--}80\ km\ s^{-1}\ Mpc^{-1}$ . However,  $H_0$  is dimensionally similar to the reciprocal of time. The reciprocal of Hubble constant is called the Hubble time  $t_H$  :  $t_H = H_0^{-1}$ , where  $t_H$  is expressed in  $s$  and  $H_0$  in  $s^{-1}$ . If  $H_0$  is expressed in  $km\ s^{-1}\ Mpc^{-1}$  and  $t_H$  in *gigayears* ( $1\ gr = 1\ million\ years = 10^9\ years$ ) then  $t_H = 977.8/H_0$ .

Therefore, from (54) we have

$$t_0 - t(z) = \frac{2H_0^{-1}}{3\gamma A} \left[ 1 - (1 + z)^{-\frac{3\gamma A}{2}} \right]. \tag{55}$$

Figure 5 plots lookback time *versus* redshift for  $\gamma = 1$  and some selected values of  $\lambda$  and  $\beta$ . All models coincide for lower redshift since they follow the same behavior. The graph shows that the lookback time increases for higher values of  $\beta$ . Thus, the Universes with larger matter creation rate are older.

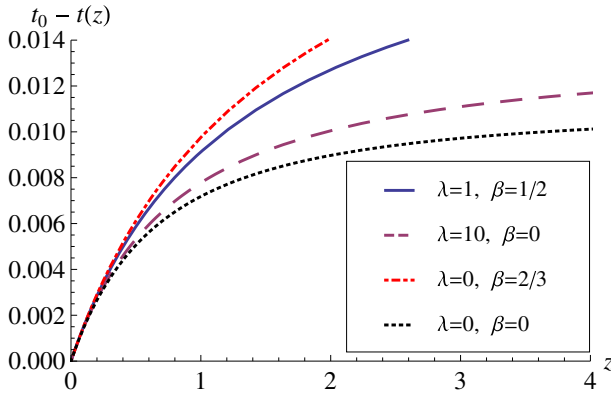
For small values of redshift, (55) becomes

$$H_0(t_0 - t(z)) = z - \left( 1 + \frac{3\gamma A}{2} \right) z^2 + \dots \tag{56}$$

Taking  $\lim\ z \rightarrow \infty$  in (55), the present age of the universe is

$$t_0 = \frac{2H_0^{-1}}{3\gamma A} = \frac{H_0^{-1}}{1 + q}. \tag{57}$$

Thus, the age of the Universe depends on both parameters  $\beta$  and  $\lambda$ .



**Fig. 5** Lookback time versus redshift for  $\gamma = 1, H_0 = 60$  and some selected values of  $\lambda$  and  $\beta$

### 6.3 Proper Distance-Redshift

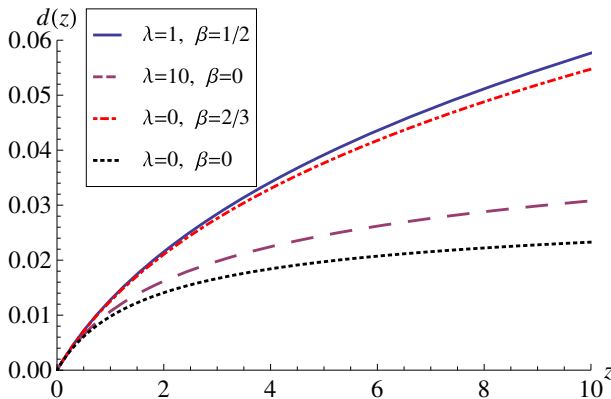
The proper distance between the source and observer is defined as  $d(z) = a_0 r(z)$ , where  $r(z)$  is the radial distance of the object at light emission in term of redshift given by

$$r(z) = \int_t^{t_0} \frac{dt}{a(t)} = \frac{H_0^{-1}}{a_0 \left( \frac{3\gamma A}{2} - 1 \right)} \left[ 1 - (1+z)^{-\left( \frac{3\gamma A}{2} - 1 \right)} \right]. \tag{58}$$

Consequently, the proper distance becomes

$$d(z) = \frac{H_0^{-1}}{\left( \frac{3\gamma A}{2} - 1 \right)} \left[ 1 - (1+z)^{-\left( \frac{3\gamma A}{2} - 1 \right)} \right]. \tag{59}$$

The proper distance as a function of redshift for some selected values of  $\beta$  and  $\lambda$  are displayed in Fig. 6. We observe that the  $f(T)$  contribution in  $f(R, T)$  and particle creation gives rise to proper distance.



**Fig. 6** Proper distance versus redshift for  $\gamma = 1, H_0 = 60$  and some selected values of  $\lambda$  and  $\beta$

Equation (59) can be rewritten as

$$H_0 d(z) = z - \frac{3\gamma A}{4} z^2 + \dots \tag{60}$$

From (59), it is observed that the distance  $d z$  is maximum at  $z \rightarrow \infty$ . Hence,

$$H_0 d(z \rightarrow \infty) = \frac{1}{\frac{3\gamma A}{2} - 1} = \frac{1}{q} \tag{61}$$

### 6.4 Luminosity Distance-Redshift

The best-known way to trace the evolution of the Universe observationally is to look into the redshift-luminosity distance relation. The luminosity distance  $d_l$  is defined by the relation  $d_l^2 = \frac{l}{4\pi L}$ , where  $l$  is the luminosity of the object and  $L$  is the measured flux from the object. In standard FRW cosmology it is defined in terms of redshift as

$$d_l = a_0(1+z)r(z) = (1+z)d(z) \tag{62}$$

From (59) and (62), we get

$$d_l H_0 = \frac{1}{\left(\frac{3\gamma A}{2} - 1\right)} \left[ (1+z) - (1+z)^{-\left(\frac{3\gamma A}{2} - 2\right)} \right] \tag{63}$$

The graph between Luminosity distance and redshift for some selected values of  $\beta$  and  $\lambda$  is plotted in Fig. 7. One may observe that the luminosity distance corresponding to any specific value of redshift rises due to  $f(R, T)$  gravity and particle creation.

Expanding (63) for small  $z$ , we find

$$H_0 d_l = z - \frac{1}{2} \left( \frac{3\gamma A}{2} - 2 \right) z^2 + \dots \tag{64}$$

As expected, we find the same behavior for different models at  $z \ll 1$  and the possible difference in behaviors for different models come at large redshift ( $z \gg 1$ ). In Fig. 7 we observe that all curves start off with the linear Hubble law ( $z = d_l H_0$ ) for small  $z$ , but then, only the curve for  $q = 1$ , i.e.,  $\beta = 0 = \lambda$  stays linear all the way. We also note that for the small redshift the luminosity distance is larger for lower values of  $q$ .

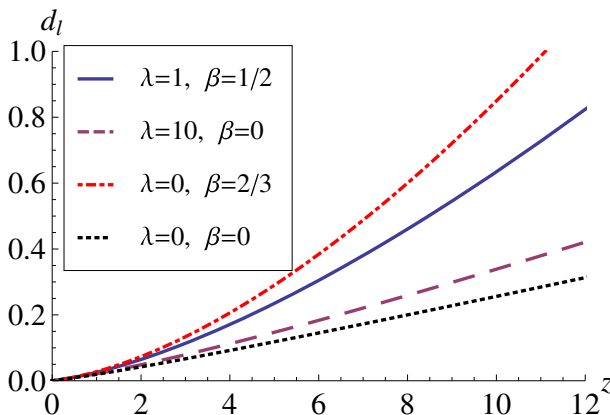
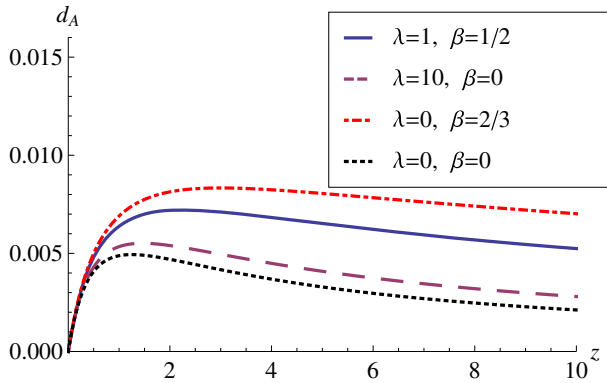


Fig. 7 Luminosity distance versus redshift for  $\gamma = 1, H_0 = 60$  and some selected values of  $\lambda$  and  $\beta$



**Fig. 8** Angular diameter distance versus redshift for  $\gamma = 1, H_0 = 60$  and some selected values of  $\lambda$  and  $\beta$

### 6.5 Angular Diameter Distance-Redshift

The angular diameter distance  $d_A$  is the ratio of physical transverse size of an object to its angular size (in radians). In terms of  $z$ , it is given by

$$d_A = \frac{d(z)}{1+z} = \frac{d_l}{(1+z)^2}. \tag{65}$$

Using (59), we have

$$H_0 d_A = \frac{1}{\left(\frac{3\gamma A}{2} - 1\right)} \left[ (1+z)^{-1} - (1+z)^{-\frac{3\gamma A}{2}} \right]. \tag{66}$$

In Fig. 8 we plot the angular diameter distance *versus* redshift for some selected values of  $\beta$  and  $\lambda$ . The graph shows that the  $f(R, T)$  and particle creation enhance the angular distance. The angular diameter distance initially increases with increasing  $z$  and eventually begins to decrease.

On expanding (66), we get

$$H_0 d_A = z + \left[ 1 - \frac{\left(\frac{3\gamma A}{2} + 1\right) \left(\frac{3\gamma A}{2} + 2\right)}{2 \left(\frac{3\gamma A}{2} - 1\right)} \right] z^2 + \dots \tag{67}$$

Thus, the angular diameter shows linear behavior up to first approximation whereas it has quadratic up to second approximation.

### 7 Conclusion

In this paper, we have studied a flat FRW cosmological model described by an open thermodynamic system including particle creation at the expense of gravitational field in  $f(R, T)$  theory of gravity. We have obtained exact solutions for the scale factor and various physical quantities by assuming a suitable form of  $f(R, T) = R + 2f(T)$  and “gamma-law” equation of state. The model exhibits non-singular power-law expansion of the Universe for  $0 \leq \gamma \leq 2$ . The model shows Big Rip singularity for  $\gamma < 0$ . The dynamics of the scale

factor and other physical quantities have been examined through some graphical representations in various phases of evolution of the Universe with some selected values of  $\lambda$  and  $\beta$ . The expressions of the energy densities and particle number density as function of scale factor have been established.

It has been observed that the scale factor always increases with decelerated and accelerated rate depending upon the contribution of particle creation and the parameter  $\lambda$ . It has been noticed that the Universe expands with slow rate in early time but fast during late time. The energy density and effective pressure always decrease with time and both tend to zero in late time for  $0 \leq \gamma \leq 2$ . The number of particles increase with time in all phases. The number of particles in the absence of particle production remain constant throughout the evolution of the Universe, which is quit obvious. The deceleration parameter has been found as a constant, which exhibits both decelerated and accelerated Universe under some constraints on different parameters. The density parameter shows that the model becomes open in the presence of particle creation in  $f(R, T)$  theory.

We have also discussed some observational consequences of the model through some kinematics tests such as lookback time, proper distance, luminosity distance and angular diameter distance with respect to redshift. The results for the cosmological tests are found to be compatible with the recent observations. These tests are found to be depend on  $\lambda$  and  $\beta$ . The Universe with particle creation is always older than the usual FRW model in absence of particle creation. The model of Lima et al. [16] may be recovered for  $\lambda = 0$ .

In summary, we have studied a cosmological model with particle creation in  $f(R, T)$  gravity theory to understand the current acceleration expansion of the Universe. We have found that the negative pressure due to the matter creation may play the role of dark energy and derive the accelerating expansion of the universe in  $f(R, T)$  theory. We may expect that the process of particle creation is also an ingredient which accounts this unexpected observational results. The changes introduced by the particle creation process, which is quantified by the parameter  $\beta$ , provide reasonable observational results. The new fact justifying the present work is that we have considered the thermodynamics approach for which particle creation is at the expense of the gravitational field. One may find that the particle creation changes the predictions of standard cosmology, thereby alleviating the problem of reconciling observations with the inflationary scenario. In future work, one may plan to constraint cosmological model with matter creation using complimentary astronomical observations.

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