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The Measurement Process in the Generalized Contexts Formalism for Quantum Histories

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Abstract In the interpretations of quantum mechanics involving quantum histories there is no collapse postulate and the measurement is considered as a quantum interaction between the measured system and the measured instrument. For two consecutive non ideal measurements on the same system, we prove that both pointer indications at the end of each measurement are compatible properties in our generalized context formalism for quantum histories. Inmediately after the first measurement an effective state for the measured system is deduced from the formalism, generalizing the state that would be obtained by applying the state collapse postulate.

Keywords Quantum histories · Consistent histories · Generalized contexts

1 Introduction

The quantum histories approach was developed in order to give an interpretation of quantum mechanics potentially useful for quantum cosmology, i.e. an observer-independent

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formulation of quantum theory. A new theoretical framework was necessary for this purpose since, while the standard Copenhagen interpretation works well in the laboratory, it cannot be applied to closed systems.

In order to reach that goal, the quantum histories formalism was proposed, in which the fundamental objects are sets of quantum histories. R. Griffiths [1, 2], R. Omnès [3, 4], M. Gell-Mann and J. Hartle [5] developed the theory of consistent histories. More recently, we introduced the formalism of generalized contexts for quantum histories [6, 7]. By contrast to some interpretations of quantum mechanics, particularly the Copenhagen interpretation, in quantum histories formalisms there is no collapse postulate and the measurement is considered as a quantum interaction between the measured system and the measurement instrument.

In the consistent histories formulation of quantum theory, the allowed sets of quantum histories that can be included in a valid description of the system must satisfy some consistency conditions. Since the consistency conditions depend on the state of the system, the properties that can be included in a valid description also depend on the state. This is an odd situation compared with the standard formalism of quantum mechanics, where the allowed contexts of properties are the possible distributive sublattices of the Hilbert space, which do not depend on the state. Moreover, it was shown that it is possible to retrodict contrary properties in different consistent sets of histories. This fact is considered by some authors as a serious failure of the theory [8–10].

In our formalism of generalized contexts, the allowed sets of quantum histories must satisfy a compatibility condition, defined by the commutation of the corresponding projectors translated to a common time [6, 7]. In this formalism the allowed sets of histories are state independent and free from the problem of retrodiction of contrary properties [11].

The generalized-contexts formalism is an alternative to the theory of consistent histories, which has proved to be useful for the time dependent description of the logic of quantum measurements [12], the decay processes [13] and the double slit experiment with and without measurement instruments [7]. More recently, we have discussed the relation of our formalism with the theory of consistent histories [14].

In this paper we show that a generalized form of state collapse can be obtained from the formalism of generalized contexts when it is applied to the measurement process. For two consecutive non-ideal measurements on the same system, we prove that both pointer indications at the end of each measurement are compatible properties in our formalism. Moreover, immediately after the first measurement, we deduce, applying the generalized context formalism, an effective state for the measured system, generalizing the state that would be obtained by applying the state collapse postulate.

The paper is organized in the following way. In Section 2, we summarize the notion of context in ordinary quantum mechanics. In Section 3, we present the main ideas of the formalism of generalized contexts and its application to the description of a measurement process. In Section 4, we apply this formalism to the description of two consecutive measurements, in such a way that an effective state of the system can be defined immediately after the first measurement. This effective state is compared with the state obtained using the collapse postulate. The main conclusions are given in Section 5.

2 Quantum Contexts

In quantum mechanics, each isolated physical system is associated with a Hilbert space $\mathcal{H}$ and a Hamiltonian operator H. Each observable of the system is represented by a self-adjoint operator. The eigenvalues of this operator are the possible values of the observable. Using the spectral theorem, the possible values of the observable can be represented by the corresponding projection operators. More generally, each *property* p of the quantum system can be represented by a projection operator $\Pi_p : \mathcal{H} \longrightarrow \mathcal{H}$.

A *state* of the system at time t is represented by a statistical operator ρ_t. In the Schrödinger representation, the state operator evolves in time according to the Liouville – von Neumann equation $i\hbar\frac{d}{dt}\rho_t = [H; \rho_t]$.

Quantum mechanics does not give an operational meaning to the joint probability distribution of observables represented by non commuting operators. It operationally deals only with sets of properties belonging to a single *context*, i.e. a Boolean sublattice of properties. A context of properties $\mathcal{C}_i$ at the time t_i is obtained starting from a set of atomic properties $p_i^{k_i}$ $(k_i \in \sigma_i)$ represented by projectors $\Pi_i^{k_i}$ corresponding to a projective decomposition of the Hilbert space $\mathcal{H}$, i.e. verifying

$$\sum\nolimits_{k_i \in \sigma_i} \Pi_i^{k_i} = I, \qquad \Pi_i^{k_i} \Pi_i^{k_i'} = \delta_{k_i k_i'} \Pi_i^{k_i}. \tag{1}$$

Any property p of the context $\mathcal{C}_i$ is represented by a sum of some of the projectors of the projective decomposition

$$\Pi_p = \sum\nolimits_{k_i \in \sigma_p} \Pi_i^{k_i}, \tag{2}$$

where σ_p is a subset of the set of indexes σ_i.

The context $\mathcal{C}_i$ is an orthocomplemented distributive lattice, with the order relation $p \leq p'$ defined by $\Pi_p \mathcal{H} \subseteq \Pi_{p'} \mathcal{H}$ and the complement $\overline{p}$ of a property p defined by $\Pi_{\overline{p}} \equiv I - \Pi_p$. A well defined probability, i.e. additive, non negative and normalized, is defined by the Born rule $\Pr_{t_i}(p) \equiv \mathrm{Tr}(\rho_{t_i} \Pi_p)$ on the context $\mathcal{C}_i$.

In the Heisenberg representation the probability of a property p at time t_i is written in terms of the state at a reference time t_0, i.e.

$$\Pr_{t_i}(p) = Tr(\rho_{t_0} \Pi_{p,0}), \qquad \Pi_{p,0} \equiv U(t_0, t_i) \Pi_p U(t_i, t_0), \qquad U(t_i, t_0) = e^{-\frac{i}{\hbar} H(t_i - t_0)}. \tag{3}$$

Taking into account (2) and (3), the Heisenberg representation of the property p of the context $\mathcal{C}_i$ at time t_i is given by

$$\Pi_{p,0} = \sum\nolimits_{k_i \in \sigma_p} \Pi_{i,0}^{k_i}, \tag{4}$$

where the projectors $\Pi_{i,0}^{k_i} = U(t_0, t_i) \Pi_i^{k_i} U(t_i, t_0)$ represent the time translation of the atomic properties $p_i^{k_i}$ from time t_i to the time t_0. The projectors $\Pi_{i,0}^{k_i}$ also satisfy (1).

3 Generalized Contexts and Quantum Measurements

In this section we present a brief summary of our formalism of generalized contexts for quantum histories [6, 7] and its application to the measurement process [12]. We refer the reader to our previous papers for the detailed presentation.

3.1 Generalized Contexts

The Heisenberg representation of the context $\mathcal{C}_i$ at time t_i suggests a generalization of quantum mechanics for including the joint probability of properties belonging to different contexts $\mathcal{C}_1, ..., \mathcal{C}_i, ..., \mathcal{C}_n$ corresponding to n different times $t_1 < ... < t_i < ... < t_n$.

By extending what is a common assumption in ordinary quantum mechanics, we propose to give theoretical meaning to the joint probabilities of properties at different times only if they correspond to commuting projectors in the Heisenberg representation. This will be the case if the atomic properties that generate each of the n contexts are represented by projectors satisfying the *compatibility conditions* given by

$$\left[\Pi_{i,0}^{k_i};\Pi_{j,0}^{k_j}\right]=0,\qquad i,j=1,...,n,\qquad k_i\in\sigma_i,\qquad k_j\in\sigma_j. \tag{5}$$

If these projectors commute, the projectors $\Pi_0^{\mathbf{k}}\equiv\Pi_{10}^{k_1}...\Pi_{i0}^{k_i}...\Pi_{n0}^{k_n}$, with $\mathbf{k}=(k_1,...,k_n)$ and $k_i\in\sigma_i$, form a projective decomposition of the Hilbert space $\mathcal{H}$, as they satisfy

$$\sum_{\mathbf{k}}\Pi_0^{\mathbf{k}}=I,\qquad \Pi_0^{\mathbf{k}}\Pi_0^{\mathbf{k}'}=\delta_{\mathbf{k}\mathbf{k}'}\Pi_0^{\mathbf{k}},\qquad \mathbf{k},\mathbf{k}'\in\sigma_1\times...\times\sigma_n$$

In our formalism we postulate that an expression of the form "property $p_1^{k_1}$ at time t_1 and ... and property $p_n^{k_n}$ at time t_n" is an atomic generalized property $\mathbf{p^k}$ with the Heisenberg representation given by the projector $\Pi_0^{\mathbf{k}}$. A *generalized context* is defined by all the generalized properties $\mathbf{p}$ having a Heisenberg representation given by a partial sum of the projectors $\Pi_0^{\mathbf{k}}$, i.e.

$$\Pi_{\mathbf{p}}=\sum_{\mathbf{k}\in\sigma_{\mathbf{p}}}\Pi_0^{\mathbf{k}}, \tag{6}$$

where $\sigma_{\mathbf{p}}$ is a subset of $\sigma_1\times...\times\sigma_n$. The generalized context is an orthocomplemented distributive lattice, with the complement $\overline{\mathbf{p}}$ of $\mathbf{p}$ defined by $\Pi_{\overline{\mathbf{p}}}=I-\Pi_{\mathbf{p}}$, and the order relation $\mathbf{p}\leq\mathbf{p}'$ defined by the inclusion of the corresponding Hilbert subspaces ($\Pi_{\mathbf{p}}\mathcal{H}\subseteq\Pi_{\mathbf{p}'}\mathcal{H}$).

An extension of the Born rule provides a definition of an additive, non negative and normalized probability on the generalized context, given by

$$\Pr(\mathbf{p})\equiv\mathrm{Tr}(\rho_{t_0}\Pi_{\mathbf{p}}). \tag{7}$$

3.2 Quantum Measurements

The formalism of generalized contexts can be applied to the description of the measurement process. Let us consider a system S, on which a non ideal measurement is performed by an instrument A in the time interval (t_0,t_1). In the Hilbert space $\mathcal{H}_S\otimes\mathcal{H}_A$ the quantum measurement is represented by the unitary transformation U_{SA} satisfying

$$|q_j\rangle|a_0\rangle\xrightarrow{U_{SA}}|\phi_j\rangle|a_j\rangle, \tag{8}$$

where the vectors $|q_j\rangle\in\mathcal{H}_S$ are a complete set of eigenstates of an observable Q of the system S, $|a_0\rangle$ is the reference state of the measurement instrument A and $|a_j\rangle\in\mathcal{H}_A$ is the state corresponding to the pointer indication of the instrument correlated with the initial state $|q_j\rangle$ of the measured system S. As we assume a non ideal measurement, we consider $|\phi_j\rangle\neq|q_j\rangle$. The state $|q_j\rangle$ of the measured system S is changed into $|\phi_j\rangle$ by the non-ideal measurement process.

Equation (8) is a "toy model" of a real measurement, since we do not take into account the additional degrees of freedom of the measurement instrument which are different from the pointer variable. However, the results can be easily generalized to consider more realistic measurement processes [12].

We have proved in [12] the compatibility of the properties corresponding to the possible values of the observable Q of the system S at time t_0 and the properties corresponding to the possible values of the pointer observable of instrument A at time t_1. These properties are represented by commuting projectors when they are translated to a common time.

Therefore, a generalized context involving these properties is well defined. In this generalized context we deduced the conditional probability $\Pr(q_j \text{ at } t_0 | a_j \text{ at } t_1) = 1$. The interpretation of this result is that the value a_j of the pointer variable after the measurement process *implies* that the observable Q had the value q_j before the measurement.

4 Consecutive Measurements and State Collapse

We are now going to consider that the measurement by instrument A in the time interval (t_0, t_1) is followed by a measurement by an instrument B in the time interval (t_1, t_2), represented in $\mathcal{H}_S \otimes \mathcal{H}_B$ by the unitary transformation U_{SB} satisfying

$$|p_\mu\rangle|b_0\rangle \xrightarrow{U_{SB}} |\chi_\mu\rangle|b_\mu\rangle. \tag{9}$$

In the previous expression the vectors $|p_\mu\rangle \in \mathcal{H}_S$ form a complete orthonormal set of eigenvectors of an observable P of the system S, $|b_0\rangle$ is the reference state of instrument B and $|b_\mu\rangle$ corresponds to the pointer indication of the instrument B correlated with $|p_\mu\rangle$.

4.1 Compatibility of the Pointer Variables

We are going to prove that the pointer indications of both instruments at the end of each measurement process are compatible properties of the system composed by the measured system S and both instruments A and B.

The possible pointer indications a_j of instrument A at time t_1 are represented by the projectors

$$\Pi_1^j \equiv I_S \otimes |a_j\rangle\langle a_j| \otimes I_B. \tag{10}$$

These projectors form a projective decomposition of the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_B$.

Another projective decomposition of the Hilbert space is given by the projectors

$$\Pi_2^\mu \equiv I_S \otimes I_A \otimes |b_\mu\rangle\langle b_\mu|, \tag{11}$$

representing the possible pointer indications of the instrument B at time t_2.

Based on the fact that the commutation relations are invariant under unitary transformations, any reference time can be chosen to verify the compatibility conditions given in (5). The compatibility conditions for the pointer variables of both measurement instruments are easily verified by choosing t_1 as the common time to translate the projectors given in (10) and (11).

The projectors Π_1^j represent properties already defined at time t_1, while the time translation of the projectors Π_2^μ from time t_2 to time t_1 are given by $(U_{SB}^{-1} \otimes I_A)\Pi_2^\mu(U_{SB} \otimes I_A)$. Taking into account (10) and (11) we obtain the commutation relations

$$\left[(U_{SB}^{-1} \otimes I_A)\Pi_2^\mu(U_{SB} \otimes I_A);\, \Pi_1^j\right] = 0,$$

and therefore the possible values of both indication variables at the end of each measurement process are compatible properties. The pointer values a_j of instrument A at t_1 and the pointer values b_μ of instrument B at t_2 generate a generalized context of properties for a valid description of the composed quantum system.

4.2 Effective State After the First Measurement and State Collapse

At time t_0, before both measurements take place, we consider the state $|\Psi_{t_0}\rangle = |\varphi_0\rangle|a_0\rangle|b_0\rangle \in \mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_B$, corresponding to an arbitrary state $|\varphi_0\rangle$ of the system S and both instruments in their reference states.

In the previous subsection we proved that the properties corresponding to the instrument A indicating a_j at time t_1, and the instrument B indicating b_μ at time t_2, are represented by commuting projectors when they are translated to a common time. Therefore they are compatible properties and we can compute the probability for their conjunction using our formalism of generalized contexts.

For the state $|\Psi_{t_0}\rangle$, and using (6) and (7), this probability is given by

$$\Pr_{\Psi_{t_0}}(a_j \text{ at } t_1 \text{ and } b_\mu \text{ at } t_2) = \langle\Psi_{t_0}|\Pi^j_{10}\Pi^\mu_{20}|\Psi_{t_0}\rangle,$$

where

$$\Pi^j_{10} = (U^{-1}_{SA} \otimes I_B)\Pi^j_1(U_{SA} \otimes I_B),$$
$$\Pi^\mu_{20} = (U^{-1}_{SA} \otimes I_B)(U^{-1}_{SB} \otimes I_A)\Pi^\mu_{20}(U_{SB} \otimes I_A)(U_{SA} \otimes I_B).$$

Taking into account (8) and (9) we obtain

$$\Pr_{\Psi_{t_0}}(a_j \text{ at } t_1 \text{ and } b_\mu \text{ at } t_2) = |\langle q_j|\varphi_0\rangle|^2|\langle p_\mu|\phi_j\rangle|^2,$$

and also

$$\Pr_{\Psi_{t_0}}(a_j \text{ at } t_1) = \langle\Psi_{t_0}|\Pi^j_{10}|\Psi_{t_0}\rangle = |\langle q_j|\varphi_0\rangle|^2.$$

Therefore the probability for the second measurement to give b_μ conditional to the first measurement to have given a_j is

$$\Pr_{\Psi_{t_0}}(b_\mu \text{ at } t_2|a_j \text{ at } t_1) = \frac{\Pr_{\Psi_{t_0}}(a_j \text{ at } t_1 \text{ and } b_\mu \text{ at } t_2)}{\Pr_{\Psi_{t_0}}(a_j \text{ at } t_1)} = |\langle p_\mu|\phi_j\rangle|^2. \tag{12}$$

This result can be compared with the measurement process by the single instrument B on the system S, for an initial state at t_1 given by $|\Phi_{t_1}\rangle = |\phi_j\rangle|b_0\rangle \in \mathcal{H}_S \otimes \mathcal{H}_B$, where we obtain

$$\Pr_{\Phi_{t_1}}(b_\mu \text{ at } t_2) = \langle\Phi_{t_1}|\Pi^\mu_{21}|\Phi_{t_1}\rangle = |\langle p_\mu|\phi_j\rangle|^2. \tag{13}$$

with $\Pi^\mu_{21} = U^{-1}_{SB}(I_S \otimes |b_\mu\rangle\langle b_\mu|)U_{SB}$.

For the system S prepared at time t_0 in the state $|\varphi_0\rangle \in \mathcal{H}_S$, (12) gives the probability to obtain at time t_2 the pointer value b_μ for the measurement with instrument B, if a previous measurement with instrument A have given the pointer value a_j at time $t_1 < t_2$.

Equation (13) gives the probability to obtain at time t_2 the pointer value b_μ for a single measurement of system S with instrument B, on a system prepared at time t_1 in the state $|\phi_j\rangle \in \mathcal{H}_S$.

Different choices of the instrument B would give different vectors $|p_\mu\rangle$, but in any case the probabilities given in (12) and (13) would have the same values ($\Pr_{\Psi_{t_0}}(b_\mu$ at $t_2|a_j$ at $t_1) = \Pr_{\Phi_{t_1}}(b_\mu$ at $t_2)$). Therefore we can conclude that the preparation of the system S in any state $|\varphi_0\rangle$ followed by a result a_j of a measurement with an instrument A and the preparation of the system S in the state $|\phi_j\rangle$ defined in (8) *produce the same results* for a future measurement with *any* instrument B.

The state $|\phi_j\rangle$ can be considered as the *effective state* of the system S at time t_1, after a measurement process by instrument A has given the value a_j for the pointer variable. We can say that the preparation of system S in any state $|\varphi_0\rangle$ at time t_0, followed by a measurement by instrument A with the value a_j of the pointer variable is mathematically equivalent to consider the system S prepared in the state $|\phi_j\rangle$ at time t_1.

It should be stressed that from the perspective of the collapse hypothesis, the effective state is the real state of the system after the first measurement. However, from the generalized context formalism, it is not the real state of the system, but a computational artifact which supplies the same result as would be obtained with the complete calculation.

In general, the effective state after a measurement obtained with our formalism do not coincide with the state that would be obtained by applying the state collapse postulate. However, for the special case of an ideal measurement (8) is replaced by $|q_j\rangle|a_0\rangle \xrightarrow{U_{SA}} |q_j\rangle|a_j\rangle$ and our formalism gives the effective state $|\phi_j\rangle = |q_j\rangle$, i.e. the same result obtained by means of the state collapse postulate.

5 Conclusions

The quantum histories approach was developed in order to give an interpretation of quantum mechanics in which there is no collapse postulate and the measurement is considered as a quantum interaction between the measured system and the measurement instrument.

Our formalism of generalized contexts is one of these formalisms of quantum histories, which allows to define expressions of conjunctions and disjunctions of properties at different times and enables to organize them in a valid quantum history if they satisfy the *compatibility condition*, i.e. if the properties at different times are represented by commuting projectors when translated to a common time.

In this paper we applied this formalism to two consecutive non-ideal measurements on the same system. We proved that the possible values of the pointer variables, corresponding to two consecutive measurements on a system S are always compatible properties. The measured system S and both measurement instruments A and B form a composed quantum system with a valid description, involving the possible pointer values of the instruments immediately after each of the two successive measurements. Therefore, it is possible to use the formalism of generalized contexts to compute the probability for each result of the second measurement with instrument B, conditional to a given previous result of the first measurement with instrument A. We proved that the value of this conditional probability is the same that would be obtained performing only a measurement with instrument B on the system S in an effective state. This effective state depends on the result of the first measurement. Only in the case of an ideal measurement the effective state coincides with the state obtained applying the state collapse postulate.

It is interesting to note that, by contrast to the case of consistent histories, it was not necessary to use decoherence to deduce the effective state that generalizes the collapse postulate (see [4], chapters 19 and 21). The decoherence involving the many degrees of freedom of a real measurement instrument was not considered, as these degrees of freedom were absent in our toy model. This fact supports the position that, although decoherence appears to be essential for explaining the classical limit, it is not necessary for obtaining the state collapse postulate.

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