

Axially Symmetric Cosmological Model with Bulk Stress in Saez-Ballester Theory of Gravitation

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Abstract An Axially symmetric non-static space time is considered in presence of bulk stress in scalar tensor theory formulated by Saez and Ballester (Phys. Lett. **A113**, 467 1985). For solving the field equations, relation between metric potential and shear velocity is proportional to scale expansion are used. Also various physical and geometrical properties of the model have been discussed.

Keywords Scalar tensor theory · Viscous fluid

1 Introduction

Einstein's general theory of relativity has been successful in describing gravitational phenomena. It has also served as a basis for models of the universe. However since Einstein first published his theory of gravitation, there have been many criticisms of general relativity because of the lack of certain desirable features in the theory. For example Einstein himself pointed out that general relativity does not account satisfactorily for inertial properties of matter, i.e. Mach's principle is not substantiated by general relativity. So in recent years there has been lot of interest in several alternative theories of gravitation.

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The most important among them are scalar tensor theories of gravitation formulated by [5, 10, 15]. All version of the scalar tensor theories are based on the introduction of a scalar field φ into the formulation of general relativity, this scalar field together with the metric tensor field then forms a scalar tensor field representing the gravitational field.

In Saez-Ballester theory the metric is coupled with a dimensionless scalar field in a simple manner. This coupling gives a satisfactory description of weak fields and suggest a possible way to solve missing matter problem in non-flat FRW cosmologies.

The [15] field equations are

$$G_{ij} - \omega\varphi^n \left(\varphi_{,i}\varphi_{,j} - \frac{1}{2}g_{ij}\varphi_{,k}\varphi^{,k} \right) = -8\pi T_{ij} \tag{1}$$

$$2\varphi^n \varphi^i_{,i} + n\varphi^{n-1} \varphi_{,k}\varphi^{,k} = 0 \tag{2}$$

where $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$ is the Einstein tensor, T_{ij} is the stress energy tensor of the matter, ω and n are constant, comma (,) and semicolon (;) denotes partial and co-variant differentiation respectively.

Also energy conservation equation

$$T^i_{j,j} = 0 \tag{3}$$

Is the consequence of field equations (1) and (2).

A detailed discussion of Saez-Ballester cosmological models is given in the work of [14, 15, 17, 18]. Adhav et al. [2] have studied Axially symmetric non-static domain walls in scalar-tensor theories formulated by [5, 15].

Also several aspects of viscous fluid cosmological model in early universe have been extensively investigated by many authors [1, 3, 19]. Anirudha Pradhan et al [13]. have studied Accelerating Bianchi Type-I universe with time varying G and Λ -term in general relativity. Bulk viscous Kantowski-Sachs cosmological model with time dependent Λ -term in general relativity have been investigated by [8]. Recently [9] have presented Bianchi type-II cosmological model in presence of bulk stress with varying Λ -term in general relativity.

In this paper we have obtain non-static axially symmetric bulk stress cosmological model in scalar tensor theory of gravitation proposed by [15]. Our paper is organized as follows:

In Section 2, we derive the field equations. In Section 3, we deal with the solution of the field equations in presence of viscous fluid. Section 4 includes the solution for particular cases. Section 5 is mainly concerned with the physical and kinematical properties. The last section contains conclusion.

2 The Metric and Field Equation

We consider the axially symmetric metric [4] in the form

$$ds^2 = dt^2 - A^2(t) \left(d\chi^2 + f^2(\chi)d\varphi^2 \right) - B^2(t)dz^2 \tag{4}$$

with the convention $x^1 = \chi, x^2 = \varphi, x^3 = z, x^4 = t$ and A, B are functions of the proper time t alone while f is a function of coordinate χ alone.

The stress energy tensor in the presence of bulk stress given by Landau and Lifshitz is

$$T_i^j = (\rho + p)v_i v^j + p g_i^j - \eta(v_{;i}^j + v_{;i}^j + v^j v^l v_{i;l} + v_i v^l v_{;l}^j) - \left(\xi - \frac{2}{3}\eta\right)\theta \left(g_i^j + v_i v^j\right) \quad (5)$$

where ρ , p , η and ξ are the energy density, isotropic pressure, coefficient of shear velocity and bulk viscous coefficient respectively and v_i is the flow vector satisfying the relations

$$g_{ij} v^i v^j = -1 \quad (6)$$

We choose the co ordinates to be commoving, so that

$$v^1 = 0 = v^2 = v^3, v^4 = 1 \quad (7)$$

The field equations (1), (2) and (3) for the metric (4) with the help of Eqs. (5) and (6) can be written as

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} - \frac{\omega}{2} \varphi^n \varphi_4^2 = 8\pi \left[p - 2\eta \frac{A_4}{A} - \left(\xi - \frac{2}{3}\eta\right)\theta \right] \quad (8)$$

$$2 \frac{A_{44}}{A} + \left(\frac{A_4}{A}\right)^2 - \frac{f_{11}}{A^2 f} - \frac{\omega}{2} \varphi^n \varphi_4^2 = 8\pi \left[p - 2\eta \frac{B_4}{B} - \left(\xi - \frac{2}{3}\eta\right)\theta \right] \quad (9)$$

$$\frac{f_{11}}{A^2 f} - \left(\frac{A_4}{A}\right)^2 - 2 \frac{A_4 B_4}{AB} - \frac{\omega}{2} \varphi^n \varphi_4^2 = 8\pi \rho \quad (10)$$

$$\varphi_{44} + \varphi_4 \left(\frac{2A_4}{A} + \frac{B_4}{B}\right) + \frac{n}{2} \left(\frac{\varphi_4^2}{\varphi}\right) = 0 \quad (11)$$

and

$$\rho_4 + \left[p + \rho - \left(\xi - \frac{2}{3}\eta\right)\theta \right] \left(\frac{2A_4}{A} + \frac{B_4}{B}\right) - 2\eta \left(2 \left(\frac{A_4}{A}\right)^2 + \left(\frac{B_4}{B}\right)^2\right) = 0 \quad (12)$$

where suffix 4 at the symbols A , B , φ and ρ denotes ordinary differentiation with respect to t and θ is the shear expansion given by

$$\theta = v_{;i}^i \quad (13)$$

The functional dependence of the metric together with the equations (9) and (10) imply

$$\left(\frac{f_{11}}{f}\right) = k^2, \quad k^2 \text{ is the constant.} \quad (14)$$

If $k = 0$ then $f(\chi) = \text{constant } \chi$, $0 < \chi < \infty$

This constant can be made equal to 1 by suitably choosing units for φ . Thus we shall have $f(\chi) = \chi$ resulting in the flat model of the universe [6].

3 Solution of the Field Equations

Using equation (14), the set of equations (8) to (12) reduces to

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} - \frac{\omega}{2} \varphi^n \varphi_4^2 = 8\pi \left[p - 2\eta \frac{A_4}{A} - \left(\xi - \frac{2}{3} \eta \right) \theta \right] \tag{15}$$

$$2 \frac{A_{44}}{A} + \left(\frac{A_4}{A} \right)^2 - \frac{\omega}{2} \varphi^n \varphi_4^2 = 8\pi \left[p - 2\eta \frac{B_4}{B} - \left(\xi - \frac{2}{3} \eta \right) \theta \right] \tag{16}$$

$$\left(\frac{A_4}{A} \right)^2 + 2 \frac{A_4 B_4}{AB} + \frac{\omega}{2} \varphi^n \varphi_4^2 = -8\pi \rho \tag{17}$$

$$\varphi_{44} + \varphi_4 \left(\frac{2A_4}{A} + \frac{B_4}{B} \right) + \frac{n}{2} \left(\frac{\varphi_4^2}{\varphi} \right) = 0 \tag{18}$$

and

$$\rho_4 + \left[p + \rho - \left(\xi - \frac{2}{3} \eta \right) \theta \right] \left(\frac{2A_4}{A} + \frac{B_4}{B} \right) - 2\eta \left(2 \left(\frac{A_4}{A} \right)^2 + \left(\frac{B_4}{B} \right)^2 \right) = 0 \tag{19}$$

Equations (15)–(18) are four independent equations in seven unknowns A, B, ρ, p, η, ξ and φ . For the complete determinacy of the system, we need extra conditions.

Firstly we assume a relation in metric potential as

$$A = B^m \tag{20}$$

where m is real number.

and secondly we assume that the coefficient of shear velocity is proportional to the scale of expansion,

i.e.

$$\eta \propto \theta \tag{21}$$

equations (15) and (16) leads to

$$\frac{A_{44}}{A} + \left(\frac{A_4}{A} \right)^2 - \frac{B_{44}}{B} - \frac{A_4 B_4}{AB} = 16\pi \eta \left(\frac{A_4}{A} - \frac{B_4}{B} \right) \tag{22}$$

Condition (21) leads to

$$\eta = l \left(2 \frac{A_4}{A} + \frac{B_4}{B} \right) \tag{23}$$

where l is proportionality constant.

Equation (22) together with Eqs. (20) and (21) leads to

$$B B_{44} + \alpha B_4^2 = 0 \tag{24}$$

which can be rewritten as

$$\frac{d}{dB} (f^2) + \frac{2\alpha}{B} (f^2) = 0 \tag{25}$$

where

$$\alpha = 2m - 16\pi l(2m + 1) \tag{26}$$

and

$$B_4 = f(B) \tag{27}$$

From Eq. (25) we obtain

$$\left(\frac{dB}{dt}\right)^2 = \frac{\beta}{B^{2\alpha}} \tag{28}$$

where β is the constant of integration. After a suitable transformation of co ordinates, the metric (4) reduces to the form

$$ds^2 = \left[\frac{T^{2\alpha}}{\beta}\right]dT^2 - T^{2m} \left[d\chi^2 + f^2(\chi)d\varphi^2\right] - T^2dz^2 \tag{29}$$

where $B = T$.

Furthermore, to obtain the expression for Saez-Ballester scalar field φ , we rewrite the equation (18) as

$$\frac{\varphi_{44}}{\varphi_4} + \frac{2A_4}{A} + \frac{B_4}{B} + \frac{n}{2} \frac{\varphi_4}{\varphi} = 0 \tag{30}$$

after simplifying, we obtain

$$A^2 B \varphi^{\frac{n}{2}} d\varphi = c_3 dt \tag{31}$$

We now substitute the values of A and B , we obtain

$$\varphi^{\frac{n}{2}} d\varphi = \frac{c_3}{T^{2m+1}} dt \tag{32}$$

Integrating, we obtain

$$\varphi = K_1 \left(\frac{1}{T^{2m+\alpha}}\right)^{\frac{2}{n+2}} + \varphi_0 \tag{33}$$

where φ_0 is integrating constant and $K_1 = \left[-\left(\frac{n+2}{2}\right) \left(\frac{c_3}{2m}\right) \sqrt{\beta}\right]^{\frac{2}{n+2}}$

The pressure and density for the model (29) are given by

$$8\pi p = K_2 \frac{\beta}{T^{2(\alpha+1)}} + \frac{8\pi \xi (2m+1) \sqrt{\beta}}{T^{(\alpha+1)}} - \frac{\omega}{2} \varphi^n \varphi_4^2 \tag{34}$$

and

$$8\pi \rho = \frac{-m(m+2)\beta}{T^{2(\alpha+1)}} - \frac{\omega}{2} \varphi^n \varphi_4^2, \tag{35}$$

where $K_2 = \frac{[m^2(32\pi l+3)-(m+1)(16\pi l+3\alpha)]}{3}$

For the specification of ξ , we assume that the fluid obeys an equation of state of the form

$$p = \gamma \rho \tag{36}$$

where $\gamma(0 \leq \gamma \leq 1)$ is constant.

Thus, given $\xi(t)$ we can solve for the cosmological parameters. In most of the investigation involving bulk viscosity is assume to be a simple power function of the energy density [7, 11, 22]

$$\xi(t) = \xi_0 \rho^q \tag{37}$$

where ξ_0 and q are constant. If $q = 1$ equation (36) may correspond to a relative fluid [20]. However, more realistic models [16] are based on q lying in the regime $0 \leq q \leq \frac{1}{2}$.

Using equation (37) in (34), we obtain

$$8\pi p = K_2 \frac{\beta}{T^{2(\alpha+1)}} + \frac{8\pi \xi_0 \rho^q (2m + 1) \sqrt{\beta}}{T^{(\alpha+1)}} - \frac{\omega}{2} \varphi^n \varphi_4^2 \tag{38}$$

4 Particular Cases

Case-I: Solution for $\xi(t) = \xi_0$

When $q = 0$, equation (37) reduces to $\xi(t) = \xi_0 = \text{constant}$. Hence in this case equation (38) with the use of Eqs. (35) and (36), leads to

$$8\pi\rho(1 + \gamma) = \frac{[K_2 - m(m + 2)] \beta}{T^{2(\alpha+1)}} + \frac{8\pi \xi_0 (2m + 1) \sqrt{\beta}}{T^{(\alpha+1)}} - \omega \varphi^n \varphi_4^2 \tag{39}$$

Using equation (33) above equation leads to

$$8\pi\rho(1 + \gamma) = \frac{[K_2 - m(m + 2)] \beta}{T^{2(\alpha+1)}} + \frac{8\pi \xi_0 (2m + 1) \sqrt{\beta}}{T^{(\alpha+1)}} - \omega c_3^2 \left(\frac{2m + \alpha}{2m} \right)^2 \frac{\beta}{T^{2(2m+2\alpha+1)}} \tag{40}$$

From the above relations, we can obtain three types of physical relevant models.

Sub-case (i): When $\gamma = 0$ and $\rho > 0$, equation (36) yields

$$p = 0 \text{ (dust distribution) thus we obtain,}$$

$$8\pi\rho = \frac{[K_2 - m(m + 2)] \beta}{T^{2(\alpha+1)}} + \frac{8\pi \xi_0 (2m + 1) \sqrt{\beta}}{T^{(\alpha+1)}} - \omega c_3^2 \left(\frac{2m + \alpha}{2m} \right)^2 \frac{\beta}{T^{2(2m+2\alpha+1)}} \tag{41}$$

Sub-case (ii): When $\gamma = 1$, equation (36) yields

$$p = \rho$$

which is known as Zeldovich fluid or stiff fluid model [21] Using this value of p , equation (40) yields,

$$8\pi\rho = \frac{[K_2 - m(m + 2)] \beta}{2T^{2(\alpha+1)}} + \frac{4\pi \xi_0 (2m + 1) \sqrt{\beta}}{T^{(\alpha+1)}} - \omega c_3^2 \left(\frac{2m + \alpha}{8m} \right)^2 \frac{\beta}{T^{2(2m+2\alpha+1)}} \tag{42}$$

Sub-case (iii): When $\gamma = \frac{1}{3}$, equation (36) yields

$$\rho = 3p$$

known as radiating dominated model. thus equation (40) yields,

$$8\pi\rho = \frac{3[K_2 - m(m + 2)] \beta}{4T^{2(\alpha+1)}} + \frac{6\pi \xi_0 (2m + 1) \sqrt{\beta}}{T^{(\alpha+1)}} - \omega c_3^2 \left(\frac{2m + \alpha}{2m} \right)^2 \frac{3\beta}{4T^{2(2m+2\alpha+1)}} \tag{43}$$

Case-II: Solution for $\xi(t) = \xi_0 \rho$

When $q = 1$, equation (37) reduces to $\xi(t) = \xi_0\rho$, hence in this case equation (38) with the use of Eqs. (35) and (36), leads to

$$8\pi\rho = \left[\frac{1}{1 + \gamma - \frac{\xi_0(2m+1)\sqrt{\beta}}{T^{(\alpha+1)}}} \right] \frac{[K_2 - m(m+2)]\beta}{T^{2(\alpha+1)}} - \left[\frac{1}{1 + \gamma - \frac{\xi_0(2m+1)\sqrt{\beta}}{T^{(\alpha+1)}}} \right] \times \omega\varphi^n\varphi_4^2 \tag{44}$$

Using equation (33), above equation leads to

$$8\pi\rho = \left[\frac{1}{1 + \gamma - \frac{\xi_0(2m+1)\sqrt{\beta}}{T^{(\alpha+1)}}} \right] \frac{[K_2 - m(m+2)]\beta}{T^{2(\alpha+1)}} - \left[\frac{1}{1 + \gamma - \frac{\xi_0(2m+1)\sqrt{\beta}}{T^{(\alpha+1)}}} \right] \times \omega c_3^2 \left(\frac{2m+\alpha}{2m} \right)^2 \frac{\beta}{T^{2(2m+2\alpha+1)}} \tag{45}$$

From the above relations, we can obtain three types of physical relevant models:

Sub-case (i): When $\gamma = 0$ and $\rho > 0$, equation (36) yields

$$p = 0 \text{ (dust distribution)}$$

thus we obtain

$$8\pi\rho = \left[\frac{1}{1 - \frac{\xi_0(2m+1)\sqrt{\beta}}{T^{(\alpha+1)}}} \right] \frac{[K_2 - m(m+2)]\beta}{T^{2(\alpha+1)}} - \left[\frac{1}{1 - \frac{\xi_0(2m+1)\sqrt{\beta}}{T^{(\alpha+1)}}} \right] \times \omega c_3^2 \left(\frac{2m+\alpha}{2m} \right)^2 \frac{\beta}{T^{2(2m+2\alpha+1)}} \tag{46}$$

Sub-case (ii): When $\gamma = 1$, equation (36) yields

$$p = \rho$$

which is known as Zeldovich fluid or stiff fluid model [21]

Using this value of p , we get

$$8\pi\rho = \left[\frac{1}{2 - \frac{\xi_0(2m+1)\sqrt{\beta}}{T^{(\alpha+1)}}} \right] \frac{[K_2 - m(m+2)]\beta}{T^{2(\alpha+1)}} - \left[\frac{1}{2 - \frac{\xi_0(2m+1)\sqrt{\beta}}{T^{(\alpha+1)}}} \right] \times \omega c_3^2 \left(\frac{2m+\alpha}{2m} \right)^2 \frac{\beta}{T^{2(2m+2\alpha+1)}} \tag{47}$$

Sub-case (iii): When $\gamma = \frac{1}{3}$, equation (36) yield

$$\rho = 3p$$

known as radiating dominated model.

Using this value, we get

$$8\pi\rho = \left[\frac{1}{\frac{4}{3} - \frac{\xi_0(2m+1)\sqrt{\beta}}{T^{(\alpha+1)}}} \right] \frac{[K_2 - m(m+2)]\beta}{T^{2(\alpha+1)}} - \left[\frac{1}{\frac{4}{3} - \frac{\xi_0(2m+1)\sqrt{\beta}}{T^{(\alpha+1)}}} \right] \times \omega c_3^2 \left(\frac{2m+\alpha}{2m} \right)^2 \frac{\beta}{T^{2(2m+2\alpha+1)}} \tag{48}$$

5 Some Physical and Kinematical Properties

In this section we discuss some physical and kinematical properties of the velocity vector v^i of the metric (29), the spatial volume (V), the scalar expansion (θ), the shear scalar (σ) and deceleration parameter (q) of the fluid are given by

$$V = \sqrt{-g} = T^{2m+1} \quad (49)$$

$$\theta = (2m + 1) \frac{\beta_1}{T^{(\alpha+1)}} \quad (50)$$

$$\sigma = \sqrt{\frac{7}{18}} (2m + 1) \frac{\beta_1}{T^{(\alpha+1)}} \quad \because \beta_1 = \sqrt{\beta} \quad (51)$$

and

$$q = \frac{2 + 3\alpha - 2m}{1 + 2m} \quad (52)$$

The spatial volume of the model given by Eq. (49) shows the anisotropic expansion of the universe with time. For the model, expansion scalar θ , and shear scalar σ tends to zero as $T \rightarrow \infty$. The positive value of deceleration parameter indicates the model decelerates in the standard way.

6 Conclusion

In this paper, we have considered [15] field equations in the presence of an axially symmetric bulk stress source. To get a determinate solution of the field equations, we have assumed the relation between metric potential and shear viscosity is proportional to the scale expansion.

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