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The Join of the Variety of MV-Algebras and the Variety of Orthomodular Lattices

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Abstract We axiomatize the smallest variety that contains both the variety of MV-algebras and the variety (term equivalent to the variety) of orthomodular lattices.

Keywords MV-algebra · Orthomodular lattice · Lattice effect algebra

Effect algebras, which were introduced by Foulis and Bennett [8], are partially ordered partial algebras closely related to the logical foundations of quantum mechanics. The standard example is the structure of self-adjoint operators between zero and identity on a Hilbert space, the so-called effects. Also some commonly known (total) algebras, such as orthomodular lattices and MV-algebras, may be regarded as particular cases of effect algebras. We were not the first who observed that lattice-ordered effect algebras can in a natural way be made into total algebras $\langle A; \oplus, ', 0, 1\rangle$ of type $\langle 2, 1, 0, 0\rangle$; see [2, 3]. We proved that the class $\mathcal{E}$ of such algebras is a variety containing both the variety $\mathcal{MV}$ of MV-algebras and the variety $\mathcal{OM}$ (term equivalent to the variety) of orthomodular lattices; relative to $\mathcal{E}$, $\mathcal{MV}$ and $\mathcal{OM}$ can be axiomatized by the identities $x \oplus y = y \oplus x$ and $x \oplus x = x$, respectively. We also observed that $\mathcal{E}$ is not the join of $\mathcal{MV}$ and $\mathcal{OM}$ in the lattice of subvarieties of $\mathcal{E}$. The aim of the present paper is to axiomatize the join of the varieties $\mathcal{MV}$ and $\mathcal{OM}$, but we actually axiomatize the join of $\mathcal{MV}$ with any finitely based subvariety of $\mathcal{E}$.

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First, we recall some basic facts about lattice effect algebras; for more information we refer the reader to the book [7]. An *effect algebra* is a structure $\langle E; +, 0, 1\rangle$, where $+$ is a partial binary operation and 0,1 are two constants, satisfying the following conditions:

(i) $x + y = y + x$ if one side is defined;
(ii) $(x + y) + z = x + (y + z)$ if one side is defined;
(iii) for every $x \in E$ there exists a unique $x' \in E$ such that $x' + x = 1$;
(iv) if $x + 1$ is defined, then $x = 0$.

This is the definition by Foulis and Bennett [8], but we should mention that effect algebras are essentially the same as *weak orthoalgebras* introduced by Giuntini and Greuling [9] who, however, attribute their introduction to Foulis and Randall. Besides, effect algebras are equivalent to *D-posets* which were introduced by Kôpka and Chovanec [14].

Every effect algebra is naturally ordered by stipulating that $x \leq y$ iff $y = x + z$ for some $z \in E$; such an element z is unique and we may denote it by $y - x$. Thus, if $x \leq y$, then $y - x$ is the only element such that $y = x + (y - x)$, or more directly, $y - x = (x + y')'$. The structure $\langle E; \leq, -, 0, 1\rangle$ so obtained is a *D-poset* in the sense of [14].

It is worth noticing that $x + y$ is defined iff $x \leq y'$, in which case $x + y = (y' - x)'$.

An effect algebra (or D-poset) which is a lattice with respect to the natural order $\leq$ is called a *lattice effect algebra* (or *D-lattice*).

Two examples of lattice effect algebras that are of particular interest to us are orthomodular lattices and MV-algebras. We refer the reader to [13] for orthomodular lattices, and to [6] for MV-algebras.

Orthomodular lattices are equivalent to lattice effect algebras satisfying the condition that $x + x$ is defined only if $x = 0$. Indeed, if $\langle L; \vee, \wedge, ', 0, 1\rangle$ is an orthomodular lattice, then the structure $\langle L; +, 0, 1\rangle$, where $x + y$ is defined and equals $x \vee y$ iff $x \leq y'$, is a lattice effect algebra satisfying the condition; and conversely, if $\langle E; +, 0, 1\rangle$ is such a lattice effect algebra with induced lattice operations $\vee$ and $\wedge$, then $x + y = x \vee y$ if $x + y$ is defined, and $\langle E; \vee, \wedge, ', 0, 1\rangle$ is an orthomodular lattice.

MV-algebras are equivalent to *MV-effect algebras*, i.e., lattice effect algebras satisfying $(x \vee y) - y = x - (x \wedge y)$ for all x, y. Though we do not want to go into details here, we need to mention that every MV-algebra $\langle A; \oplus, ', 0, 1\rangle$ bears a lattice order which is given by $x \leq y$ iff $x' \oplus y = 1$ iff $y = x \oplus z$ for some $z \in A$. Now, if $\langle A; \oplus, ', 0, 1\rangle$ is an MV-algebra, then $\langle A; +, 0, 1\rangle$, where $x + y$ is defined and equals $x \oplus y$ iff $x \leq y'$, is an MV-effect algebra; and conversely, if $\langle E; +, 0, 1\rangle$ is an MV-effect algebra, then by setting

$$x \oplus y = (x \wedge y') + y \tag{1}$$

we obtain an MV-algebra $\langle E; \oplus, ', 0, 1\rangle$ with the same lattice order as $\langle E; +, 0, 1\rangle$.

Using (1), an arbitrary lattice effect algebra can be made into a total algebra. So let $\mathcal{E}$ be the class of all (total) algebras $\boldsymbol{A} = \langle A; \oplus, ', 0, 1\rangle$ of type $\langle 2, 1, 0, 0\rangle$ that arise from lattice effect algebras by means of (1). It is useful to define two more total operations, as follows:

$$x \oslash y = (x' \oplus y)' \quad \text{and} \quad x \ominus y = (y \oplus x')'. \tag{2}$$

In the language of lattice effect algebras we have

$$x \oslash y = (x \vee y) - y \quad \text{and} \quad x \ominus y = x - (x \wedge y), \tag{3}$$

and it is straightforward to show that all algebras in $\mathcal{E}$ satisfy the identities

$$x \oplus 0 = x, \tag{4}$$
$$x'' = x, \tag{5}$$
$$(x \oslash y) \oplus y = (y \oslash x) \oplus x, \tag{6}$$
$$(((x \oplus y) \oslash y) \oplus z)' \oplus (x \oplus z) = 1, \tag{7}$$

and the quasi-identity

$$x \oplus y \leq z' \quad \Rightarrow \quad (x \oplus y) \oplus z = x \oplus (z \oplus y). \tag{8}$$

On the other hand, we proved in [3] that if an algebra $\boldsymbol{A} = \langle A; \oplus, ', 0, 1\rangle$ satisfies the identities (4)–(7), then the rule $x \leq y$ iff $x' \oplus y = 1$ (equivalently, $x \oslash y = 0$ or $x \ominus y = 0$) defines a bounded lattice, with bounds 0 and 1, where

$$x \vee y = (x \oslash y) \oplus y \quad \text{and} \quad x \wedge y = (x' \vee y')' = x \ominus (x \ominus y).$$

In [3], as well as in some other papers, we called these algebras "basic algebras". Moreover, if $\boldsymbol{A}$ satisfies also (8), then the partial algebra $\langle A; +, 0, 1\rangle$ obtained by restricting $\oplus$ as in the case of MV-algebras, i.e., $x + y$ is defined and equals $x \oplus y$ iff $x \leq y'$, is a lattice effect algebra with the same natural order as $\boldsymbol{A}$, which entails that the total algebra associated with $\langle A; +, 0, 1\rangle$ via (1) is the initial algebra $\boldsymbol{A}$.

The quasi-identity (8) may be replaced with an identity; for example, it suffices to write $(x \oplus y)' \wedge z$ instead of z. Hence the class $\mathcal{E}$ is a variety.

Recalling that MV-algebras correspond to MV-effect algebras, it is evident by (3) that the variety of MV-algebras $\mathcal{MV}$ is the subvariety of $\mathcal{E}$ axiomatized by the identity $x \oslash y = x \ominus y$, which is in view of (2) equivalent to $x \oplus y = y \oplus x$. MV-algebras are usually defined as algebras $\boldsymbol{A} = \langle A; \oplus, ', 0, 1\rangle$ such that $\langle A; \oplus, 0\rangle$ is a commutative monoid, satisfying the identities (5), (6) and $1 \oplus x = 1 = 0'$; see [6].

We have seen that orthomodular lattices correspond to lattice effect algebras satisfying the condition that $x \leq x'$ implies $x = 0$. In the language of $\mathcal{E}$, this is equivalent to the identity $x \wedge x' = 0$, and in turn to $x \oplus x = x$. We let $\mathcal{OM}$ denote the subvariety of $\mathcal{E}$ defined by $x \wedge x' = 0$ or $x \oplus x = x$. Note that the total addition in orthomodular lattices is given by $x \oplus y = (x \wedge y') \vee y$, which is not the same as $x \vee y$.

The variety $\mathcal{E}$ is congruence distributive, because its members are lattice based algebras. In fact, $\mathcal{E}$ is an arithmetical variety; see [3]. Congruence distributivity implies that for any two subvarieties $\mathcal{V}_1, \mathcal{V}_2$ of $\mathcal{E}$ one has $\mathrm{Si}(\mathcal{V}_1 \vee \mathcal{V}_2) = \mathrm{Si}(\mathcal{V}_1) \cup \mathrm{Si}(\mathcal{V}_2)$, where $\mathrm{Si}(\mathcal{K})$ denotes the class of subdirectly irreducible algebras in the respective class $\mathcal{K}$. Since there exist subdirectly irreducible algebras in $\mathcal{E}$ which are neither in $\mathcal{MV}$ nor in $\mathcal{OM}$, it follows that $\mathcal{E}$ is not the join $\mathcal{MV} \vee \mathcal{OM}$. For instance, a non-trivial horizontal sum of non-Boolean MV-algebras is a simple algebra in $\mathcal{E}$, but it is neither an MV-algebra nor an orthomodular lattice.

The concept that plays a central role in our proof is compatibility (see [7], Sect. 1.10). In general, two elements x, y in an effect algebra are said to be *compatible*, in symbols $x \leftrightarrow y$, if there exist x_1, y_1, z such that $x = x_1 + z$, $y = y_1 + z$ and $x_1 + z + y_1$ is defined. It is obvious that $0 \leftrightarrow x \leftrightarrow 1$ and $x \leftrightarrow x'$ for any x. Also, if $x \leq y$ or $y \leq x$, then $x \leftrightarrow y$. Another important fact that we will use repeatedly is that $x \leftrightarrow y$ iff $x \leftrightarrow y'$. In lattice effect algebras we have

$$x \leftrightarrow y \quad \text{iff} \quad (x \vee y) - y = x - (x \wedge y) \quad \text{iff} \quad (x \vee y) - y \leq x \quad \text{iff} \quad x - (x \wedge y) \leq y',$$

and hence, in the language of the variety $\mathcal{E}$,

$$x \leftrightarrow y \quad \text{iff} \quad x \oslash y = x \ominus y \quad \text{iff} \quad x \oslash y \leq x \quad \text{iff} \quad x \ominus y \leq y'.$$

Since $x \leftrightarrow y$ iff $x \leftrightarrow y'$, it is also true that

$$x \leftrightarrow y \quad \text{iff} \quad x \oplus y = y \oplus x \quad \text{iff} \quad x \leq x \oplus y.$$

Consequently, since MV-effect algebras are lattice effect algebras where $x \leftrightarrow y$ for all x, y, the variety $\mathcal{MV}$ may be axiomatized, relative to $\mathcal{E}$, by any of the identities $x \oslash y = x \ominus y$, $x \oslash y \leq x$, $x \ominus y \leq y'$, $x \oplus y = y \oplus x$ or $x \leq x \oplus y$.

A *block* of an effect algebra is a maximal set of mutually compatible elements. For lattice effect algebras Riečanová [17] proved that if x, y are compatible with a given z, then so are $x \vee y$, $x \wedge y$, $x + y$ (when defined) and $x - y$ (when defined). It follows that every block B of any algebra $\boldsymbol{A} \in \mathcal{E}$ is a subuniverse of $\boldsymbol{A}$, and obviously, the subalgebra $\boldsymbol{B}$ is an MV-algebra. The intersection of the blocks of $\boldsymbol{A}$ is called the *compatibility centre* and we denote it by $K(\boldsymbol{A})$. Clearly, $K(\boldsymbol{A}) = \{a \in A \mid a \leftrightarrow x \text{ for all } x \in A\}$ and $K(\boldsymbol{A})$ is a subuniverse of $\boldsymbol{A}$.

Though the addition $\oplus$ is neither commutative nor associative,[1] we may unambiguously write $n \cdot x = x \oplus \cdots \oplus x$ (with n occurrences of x) for any positive integer $n \in \mathbb{N}$, because every element x belongs to a block which is in fact an MV-algebra. We will mostly write just nx instead of $n \cdot x$.

By [3], the variety $\mathcal{E}$ is congruence regular, i.e., any congruence θ of an algebra $\boldsymbol{A} \in \mathcal{E}$ is determined by each of its classes $[a]_\theta$, and in particular, by $[0]_\theta$. We say that $I \subseteq A$ is an *ideal*[2] of $\boldsymbol{A}$ if $I = [0]_\theta$ for some congruence θ of $\boldsymbol{A}$. Pulmannová and Vinceková [16] proved that $\emptyset \neq I \subseteq A$ is an ideal if and only if

(i) $x \oplus y \in I$ for all $x, y \in I$;
(ii) $x \oslash y \in I$ for all $x \in I$ and $y \in A$.

An alternative characterization of ideals can be found in [5]. Of course, if $\boldsymbol{A} \in \mathcal{MV}$, then the condition (ii) amounts to saying that I is downwards closed.

For any ideal I of an algebra $\boldsymbol{A} \in \mathcal{E}$, the only congruence $\theta(I)$ with the property that $[0]_{\theta(I)} = I$ is given by

$$\langle x, y \rangle \in \theta(I) \quad \text{iff} \quad x \oslash y, y \oslash x \in I \quad \text{iff} \quad x \ominus y, y \ominus x \in I.$$

When ordered by set inclusion, the ideals of $\boldsymbol{A} \in \mathcal{E}$ form a distributive lattice that is isomorphic to the congruence lattice of $\boldsymbol{A}$ under the mutually inverse assignments $I \mapsto \theta(I)$ and $\theta \mapsto [0]_\theta$.

In general, we do not have a reasonable description of the ideal $\mathrm{Ig}(X)$ generated by a given subset $X \subseteq A$, but if X is an ideal of the compatibility centre or $X = \{a\}$ where $a \in K(\boldsymbol{A})$, then $\mathrm{Ig}(X)$ can be described easily:

Lemma 1 (cf. [12], Prop. 1) *Let $\boldsymbol{A} \in \mathcal{E}$. If J is an ideal of the MV-algebra $\boldsymbol{K}(\boldsymbol{A})$, then* $\mathrm{Ig}(J) = \{x \in A \mid x \leq a$ *for some* $a \in J\}$. *In particular, for any $a \in K(\boldsymbol{A})$,* $\mathrm{Ig}(a) = \{x \in A \mid x \leq na$ *for some* $n \in \mathbb{N}\}$.

[1] In fact, $\oplus$ is commutative iff it is associative, which happens exactly in MV-algebras.

[2] In the literature on (lattice) effect algebras, the name "ideal" is often used for subsets which are closed with respect to $+$ and downwards closed, while our ideals defined above correspond to the so-called "Riesz ideals".

Proof The first part is but a translation of [12], Prop. 1, into the language of the variety $\mathcal{E}$. Since $\boldsymbol{K}(\boldsymbol{A})$ is an MV-algebra, it is obvious that for any $a \in K(\boldsymbol{A})$, the ideal of $\boldsymbol{K}(\boldsymbol{A})$ generated by a is $\{x \in K(\boldsymbol{A}) \mid x \leq na$ for some $n \in \mathbb{N}\}$, and consequently, the ideal of $\boldsymbol{A}$ generated by a is $\{x \in A \mid x \leq na$ for some $n \in \mathbb{N}\}$. □

We should notice that in any algebra $\boldsymbol{A} \in \mathcal{E}$ we have

$$x \leq y \quad \Rightarrow \quad x \oplus z \leq y \oplus z, \quad x \oslash z \leq y \oslash z \quad \text{and} \quad z \ominus y \leq z \ominus x, \tag{9}$$

which follows directly from (1) and (3). In particular, we have $y \leq x \oplus y$, $x \oslash y \leq y'$ and $x \ominus y \leq x$. We will also need the identity

$$(x \wedge y) \oplus z = (x \oplus z) \wedge (y \oplus z), \tag{10}$$

which is over $\mathcal{E}$ equivalent to the identities $(x \vee y) \oslash z = (x \oslash z) \vee (y \oslash z)$ and $x \ominus (y \wedge z) = (x \ominus y) \vee (x \ominus z)$. It is not hard to show (cf. [7], Prop. 1.8.6) that any lattice effect algebra satisfies the equality $(x \wedge y) + z = (x + z) \wedge (y + z)$ provided $x + z$, $y + z$ are defined, whence it follows that the identity (10) holds in every algebra in $\mathcal{E}$. In fact, (9) and (10) hold in all "basic algebras" (i.e., algebras satisfying (4)–(7), see [15]).

Lemma 2 *Let $\boldsymbol{A} \in \mathcal{E}$. If $a_1, \ldots, a_n, b \in A$ are such that $a_i \leftrightarrow b$ and $a_i \wedge b = 0$ for each $i = 1, \ldots, n$, then $s(a_1, \ldots, a_n) \leftrightarrow b$ and $s(a_1, \ldots, a_n) \wedge b = 0$ for every additive term*[3] *$s(x_1, \ldots, x_n)$.*

Proof We know that if $a_i \leftrightarrow b$ for every $i = 1, \ldots, n$, then also any "sum" of the a_i's is compatible with b. The rest is an easy induction: Suppose that $s = p \oplus q$ where p, q are additive terms for which the statement holds true. Let $c = p(a_1, \ldots, a_n)$ and $d = q(a_1, \ldots, a_n)$. Then $b \wedge c = b \wedge d = 0$ and, since $b \leftrightarrow d$ and using (10), we have $b \wedge s(a_1, \ldots, a_n) = b \wedge (c \oplus d) = b \wedge (b \oplus d) \wedge (c \oplus d) = b \wedge ((b \wedge c) \oplus d) = b \wedge d = 0$. □

Lemma 3 *Let $\boldsymbol{A} \in \mathcal{E}$ and $a \in A$. Suppose that $(x \oslash (x \oplus y)) \wedge a = 0$ for all $x, y \in A$. Then $a \in K(\boldsymbol{A})$ and the polar $a^\perp = \{x \in A \mid x \wedge a = 0\}$ is an ideal of $\boldsymbol{A}$ such that* $\mathrm{Ig}(a) \cap a^\perp = \{0\}$.

Proof We have $x \oplus y = y \oplus x$ whenever $x \leq y'$. Hence, by (8), $x' \oplus y \leq a \oplus (x' \oplus y) = (a' \oslash (x' \oplus y))'$ implies

$$\begin{aligned}((x \oslash y) \wedge a)' &= (x' \oplus y) \vee a' = [a' \oslash (x' \oplus y)] \oplus (x' \oplus y) \\ &= (x' \oplus y) \oplus [a' \oslash (x' \oplus y)] = x' \oplus ([a' \oslash (x' \oplus y)] \oplus y),\end{aligned}$$

whence $((x \oslash y) \wedge a) \ominus x = x' \oslash ((x \oslash y) \wedge a)' = x' \oslash (x' \oplus z)$ where $z = (a' \oslash (x' \oplus y)) \oplus y$. But then $((x \oslash y) \wedge a) \ominus x \leq a$ yields $((x \oslash y) \wedge a) \ominus x = (x' \oslash (x' \oplus z)) \wedge a = 0$ by the assumption about a. Thus $(x \oslash y) \wedge a \leq x$ for all $x, y \in A$.

Then $x \oslash a' = (x \oslash a') \wedge a \leq x$ for any $x \in A$, because we have $x \oslash a' \leq 1 \oslash a' = a$. This shows that $a' \in K(\boldsymbol{A})$, whence also $a \in K(\boldsymbol{A})$. Now, (i) if $x, y \in a^\perp$, then $x \oplus y \in a^\perp$ by Lemma 2; and (ii) if $x \in a^\perp$, then $(x \oslash y) \wedge a \leq x \wedge a = 0$, so $x \oslash y \in a^\perp$. Hence $a^\perp$ is an ideal of $\boldsymbol{A}$.

Finally, if $x \in \mathrm{Ig}(a) \cap a^\perp$, then $x \leq na$ for some $n \in \mathbb{N}$ by Lemma 1, and $x \wedge a = 0$, which entails $x \wedge na = 0$ by Lemma 2. Thus $x = x \wedge na = 0$. □

[3] By an additive term we mean a term built up from the variables using the addition $\oplus$ only.

Now, let $\mathcal{K}$ be a finitely based subvariety of $\mathcal{E}$ incomparable with $\mathcal{MV}$, and suppose that $\mathcal{K}$ is axiomatized, relative to $\mathcal{E}$, by an identity which is of the form

$$t(x_1, \ldots, x_k) = 0.$$

Since $\mathcal{K}$ is finitely based, it can always be axiomatized by a single identity of such a form. Indeed, any identity $p = q$ can be replaced with $p \diamond q = 0$ and $q \diamond p = 0$ where $\diamond$ is $\ominus$ or $\oslash$, and any finite family of identities $t_1 = 0, \ldots, t_n = 0$ can be replaced with the identity $t_1 \vee \cdots \vee t_n = 0$.

Theorem 4 *If $\mathcal{K}$ is as above, then the join of $\mathcal{MV}$ and $\mathcal{K}$ is axiomatized, relative to $\mathcal{E}$, by the identity*

$$(x \oslash (x \oplus y)) \wedge t(z_1, \ldots, z_k) = 0. \tag{11}$$

Proof Let $\mathcal{K}'$ be the subvariety of $\mathcal{E}$ axiomatized by (11). Since $x \oslash (x \oplus y) = 0$ holds in $\mathcal{MV}$ and $t(z_1, \ldots, z_k) = 0$ in $\mathcal{K}$, it is obvious that (11) holds in $\mathcal{MV}$ as well as in $\mathcal{K}$. Hence $\mathcal{MV} \vee \mathcal{K} \subseteq \mathcal{K}'$.

Conversely, let $\boldsymbol{A} \in \mathrm{Si}(\mathcal{K}')$ and suppose that $\boldsymbol{A} \notin \mathcal{K}$, i.e., there exist $a_1, \ldots, a_k \in A$ such that $t(a_1, \ldots, a_k) \neq 0$. Let $b = t(a_1, \ldots, a_k)$. Then $(x \oslash (x \oplus y)) \wedge b = 0$ for any $x, y \in A$, and so, by Lemma 3, $b^{\perp}$ is an ideal such that $\mathrm{Ig}(b) \cap b^{\perp} = \{0\}$. Since $\boldsymbol{A}$ is a subdirectly irreducible algebra and $\mathrm{Ig}(b) \neq \{0\}$, we have $b^{\perp} = \{0\}$. But $x \oslash (x \oplus y) \in b^{\perp}$ for all $x, y \in A$, hence $\boldsymbol{A}$ satisfies the identity $x \oslash (x \oplus y) = 0$, i.e. $x \leq x \oplus y$. In other words, $\boldsymbol{A} \in \mathcal{MV}$. Therefore, $\mathrm{Si}(\mathcal{K}') \subseteq \mathrm{Si}(\mathcal{MV}) \cup \mathrm{Si}(\mathcal{K}) = \mathrm{Si}(\mathcal{MV} \vee \mathcal{K})$, whence $\mathcal{K}' \subseteq \mathcal{MV} \vee \mathcal{K}$. □

Let us return to orthomodular lattices. The variety $\mathcal{OM}$ is axiomatized, relative to $\mathcal{E}$, by the identity $x \wedge x' = 0$, i.e., the term $t(x_1, \ldots, x_k)$ from the above theorem is $t(x) = x \wedge x'$, and hence by Theorem 4 we obtain:

Corollary 5 *The join of $\mathcal{MV}$ and $\mathcal{OM}$ is axiomatized, relative to $\mathcal{E}$, by the identity*

$$(x \oslash (x \oplus y)) \wedge z \wedge z' = 0.$$

Let $\langle D; +, 0, 1\rangle$ be the so-called distributive diamond, i.e., the lattice effect algebra with universe $D = \{0, a, b, 1\}$ such that $a+a = 1 = b+b$, while $a+b$ and $b+a$ are not defined. This effect algebra has a prominent role for distributive lattice effect algebras because by [10], every finite distributive lattice effect algebra is isomorphic to a direct product of chains and diamonds. Let $\boldsymbol{D}$ be the corresponding algebra in $\mathcal{E}$. Note that $a \oplus b = b$ and $b \oplus a = a$. Relative to the variety $\mathcal{DE}$ of *distributive* members of $\mathcal{E}$, the variety $\mathrm{V}(\boldsymbol{D})$ generated by $\boldsymbol{D}$ was axiomatized by the identity $(x \ominus y) \ominus 2z = (x \ominus 2z) \ominus (y \ominus 2z)$ in [11], and by $2x = 3x$ in [4]. Using Theorem 4 we can simplify the axiomatization of the join of $\mathcal{MV}$ and $\mathrm{V}(\boldsymbol{D})$ which was given in [1]. We may take the term $t(x) = 3x \oslash 2x$.

Corollary 6 *The join of $\mathcal{MV}$ and $\mathrm{V}(\boldsymbol{D})$ is axiomatized, relative to $\mathcal{DE}$, by the identity*

$$(x \oslash (x \oplus y)) \wedge (3z \oslash 2z) = 0.$$

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