Algebraic Structures, Physics and Geometry from a Unified Field Theoretical Framework

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Abstract Starting from a Unified Field Theory (UFT) proposed previously by the author, the possible fermionic representations arising from the same spacetime are considered from the algebraic and geometrical viewpoint. We specifically demonstrate in this UFT general context that the underlying basis of the single geometrical structure $P(G, M)$ (the principal fiber bundle over the real spacetime manifold M with structural group G) reflecting the symmetries of the different fields carry naturally a biquaternionic structure instead of a complex one. This fact allows us to analyze algebraically and to interpret physically in a straightforward way the Majorana and Dirac representations and the relation of such structures with the spacetime signature and non-hermitian (CP) dynamic operators. Also, from the underlying structure of the tangent space, the existence of hidden (super) symmetries and the possibility of supersymmetric extensions of these UFT models are given showing that Rothstein's theorem is incomplete for that description. The importance of the Clifford algebras in the description of all symmetries, mainly the interaction of gravity with the other fields, is briefly discussed.

Keywords Unified field theories · Affine geometries · Dirac · Majorana · Poisson manifolds · Supermanifolds

1 Fermionic Symmetry and Matter Fields

Reviewing some concepts from earlier references [2], in [1] it was discussed that according to Wigner, from the quantum viewpoint a matter field can be defined by a spinor field $\Psi^k(x^\lambda)$ where $k = 1, 2; \lambda = 1, 2, 3, 4$ and in the case of Lorentzian metric, $x^4 = ix^0$. These

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fields can be taken as elements of some internal space located at x^λ of the 4 dimensional spacetime manifold. The elementary field ("particle" was used by Weyl) is defined by the following transformation property

$$\Psi'^k(x^\lambda) = U_j^k(x, 2)\Psi^j(x^\lambda) \tag{1}$$

where the U_j^k is the 2×2 matrix representation of the unitary group $U(2, \mathbb{C})$ and is a continous function of x^λ. If the argumentation given by Weyl runs in the correct way, strictly speaking and accordling to the analysis that follows, a biquaternionic structure is the most adequate to derive the Dirac equation. From the algebraic viewpoint the only generalized quaternion algebra over $\mathbb{C}$ is the ring of 2×2 matrices over $\mathbb{C}$ and moreover, the Clifford algebra of a two-dimensional space with a nondegenerate quadratic form is central, simple and it is a generalized quaternion algebra.

From what is written above, it is necessary to fully analyze the underlying structure of the theory (and in particular the model) presented in [3–5, 7, 10] not only from the physical and geometrical viewpoint but as well as first principles. The target is clear: to find the fundamental essence of unification as the natural world presents us.

The organization of the article is as follows: Sections 2 and 3 are devoted to describe the spacetime manifold: Dirac structure and the relation with Clifford algebras as the natural language of the description [11]. In Section 4 the emerging character of the biquaternionic structure and the connection with the Dirac equation is explicitly presented and analyzed. In Section 5 the Majorana representation is introduced and discussed from the point of view of a bi-quaternionic structure. In Sections 6, 7 and 8 physical aspects are discussed considering the relationship between the structure of the tangent space, the signature of spacetime and the algebra $\mathbb{H}$. Section 9 deals to the study and description of the spacetime manifold from the point of view of supersymmetry and the Poisson structures: the Rothstein theorem is discussed in these context. Finally in Section 10 conclusions and outlook are listed.

2 The Real Dirac Structure of the Spacetime Manifold

The principal fiber bundle (PFB) $P(G, M)$ with the structural group G determines the (Dirac) geometry of the spacetime. We suppose now G with the general form

$$G = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \quad G^+ G = I_4 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix} \tag{2}$$

A, B 2×2 matrices and containing a manifestly symplectic structure. Consequently, there exists a fundamental tensor $J_\mu{}^\lambda J_\lambda{}^\nu = \delta_\mu^\nu$ invariant under G with structure

$$J = \begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix} \tag{3}$$

of such manner that

$$G = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = AI_4 + BJ \tag{4}$$

Where however, there exists a Lorentzian metric $g_{\lambda\mu}$ [33], i.e., the metric of a curved space-time manifold with signature $(+ - - -)$, that is also invariant under G due its general form (2). Finally, a third fundamental tensor $\sigma_{\lambda\mu}$ is also invariant under G where the following relations between the fundamental tensors are

$$J_\lambda{}^\nu = \sigma_{\lambda\mu} g^{\lambda\nu}, \qquad g_{\mu\nu} = \sigma_{\lambda\mu} J_\nu^\lambda, \qquad \sigma_{\lambda\mu} = J_\lambda^\nu g_{\mu\nu} \tag{5}$$

where

$$g^{\lambda\nu} = \frac{\partial g}{\partial g_{\lambda\nu}} \quad (g \equiv \det(g_{\mu\nu})) \tag{6}$$

Then, the necessary fundamental structure is given by

$$G = L(4) \cap Sp(4) \cap K(4) \tag{7}$$

which leaves concurrently invariant the three fundamental forms

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{8}$$

$$\sigma = \sigma_{\lambda\mu}dx^\lambda \wedge dx^\mu \tag{9}$$

$$\phi = J_\nu^\lambda w^\nu v_\lambda \tag{10}$$

where w^ν are components of a vector $w^\nu \in V^*$: the dual vector space. In expression (5) $L(4)$ is the Lorentz group in 4D, $Sp(4)$ is the Symplectic group in 4D real vector space and $K(4)$ denotes the almost complex group that leaves ϕ invariant.

For instance, G leaves the geometric (Clifford) product invariant

$$\gamma_\mu\gamma_\nu = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) + \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) \tag{11}$$

$$= \gamma_\mu \cdot \gamma_\nu + \gamma_\mu \wedge \gamma_\nu = g_{\mu\nu} + \sigma_{\mu\nu} \tag{12}$$

where the γ_μ are now regarded as a set of orthonormal basis vectors, of such a manner that any vector can be represented as $\mathbf{v} = v^\lambda\gamma_\lambda$ and the invariant (totally antisymmetric) tensor as

$$\varepsilon_{\alpha\beta\gamma\delta} \equiv \gamma_\alpha \wedge \gamma_\beta \wedge \gamma_\gamma \wedge \gamma_\delta \tag{13}$$

In resume, the fundamental structure of the spacetime is then represented by P(G, M), where G is given by (5), which leaves the fundamental forms invariant (5), implying that

$$\nabla_\lambda g_{\mu\nu} = 0 \tag{14}$$

$$\nabla_\nu \sigma_{\lambda\mu} = 0 \tag{15}$$

$$\nabla_\lambda J_\nu^\lambda = 0 \tag{16}$$

where ∇_λ denotes the covariant derivative of the G connection. It is interesting to note that it is only necessary to consider two of the above three equations: the third follows automatically. Then, we will consider (14), (15) because in some sense they represent the boson and fermion symmetry respectively. Notice that this structure is naturally a heterotic one carrying a $\mathbb{H}(n)$ representation of its own.

Remark 1 As will be clear later, there exists a kind of supermanifold underlying structure in this UFT and also in other unified theories.

3 Interlude: Clifford Algebras as Natural Language

It has turned out that the Clifford algebras provide very promising tools for description and generalization of geometry and physics [13–17], also [32]. As it was pointed out before [17] there exist two kinds of the Clifford algebras, orthogonal and symplectic [18]. In the orthogonal Clifford algebras, the symmetric product of two basis vectors $\mathbf{v} = v^\lambda\gamma_\lambda$ is the inner product and it gives the orthogonal metric, while the antisymmetric product gives a basis bivector. In the symplectic Clifford algebras [11], the antisymmetric product of two basis vectors q_a is the inner product and it gives the symplectic metric, whilst the symmetric

product gives a basis bivector. Both kinds of the Clifford algebras are included into the expressions involving the three G invariant forms. Consequently, there exist in the model a boson $\leftrightarrow$fermion symmetry and spacetime$\leftrightarrow$phase space. An interesting point that we use but will not discuss in detail here, is that the generators of an orthogonal Clifford algebra can be transformed into a basis (the Witt basis) in which they behave as fermionic creation and annihilation operators. The generators of a symplectic Clifford algebra behave as bosonic creation and annihilation operators as it is well know [17]. Consequently, both kinds of operators can be united into a single structure so that they form a basis of a 'superspace'.

Remark 2 This important fact allows to incorporate from the very fundamental structure of the manifold M a consistent quantum theory with a clear geometrical meaning.

4 Dirac Equation and $\mathbb{H}$ Structure

As we have considered previously [3–7, 10], the G-structure must describe the spinorial field through the appearance of the Dirac equation in the tangent space. The physical choice for the structure of G can be given by

$$G^+ G = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \begin{pmatrix} a_0\sigma_0 & \sigma \cdot a \\ -\sigma \cdot a & a_0\sigma_0 \end{pmatrix} \begin{pmatrix} a_0\sigma_0 & -\sigma \cdot a \\ \sigma \cdot a & a_0\sigma_0 \end{pmatrix} \tag{17}$$

$$= \begin{pmatrix} (a_0\sigma_0)^2 + (\sigma \cdot a)^2 & 0 \\ 0 & (a_0\sigma_0)^2 + (\sigma \cdot a)^2 \end{pmatrix} = \mathbb{I}_4 \tag{18}$$

where a_b are physical quantities to be determined). Then,

$$(a_0\sigma_0)^2 + (\sigma \cdot a)^2 = 1 \Rightarrow a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1 \tag{19}$$

and consequently the physical meaning of the coefficients a are immediatly determined:

$$a_0 = \frac{\widehat{p_0}}{m}, a_1 = i\frac{\widehat{p_1}}{m}, a_2 = i\frac{\widehat{p_2}}{m}, a_3 = i\frac{\widehat{p_3}}{m} \tag{20}$$

leading the relativistic relation

$$\widehat{p_0^2} - \widehat{p_1^2} - \widehat{p_2^2} - \widehat{p_3^2} = m^2 \tag{21}$$

where the introduction of the momentum operators $\widehat{p}_\mu$ and the mass parameter m was performed. For instance, from the explicit structure of G and the meaning of a_b we obtain

$$G\mathbf{v} = \mathbf{u} \tag{22}$$

$$G^t\mathbf{u} = \mathbf{v} \tag{23}$$

with $\mathbf{u} = \begin{pmatrix} u^0 \\ u^1 \\ u^2 \\ u^3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix}$. Explicitly in the abstract form, we have ($h = 0, 1$)

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \begin{pmatrix} u^h \\ u^{h+2} \end{pmatrix} = \begin{pmatrix} v^h \\ v^{h+2} \end{pmatrix} \tag{24}$$

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} v^h \\ v^{h+2} \end{pmatrix} = \begin{pmatrix} u^h \\ u^{h+2} \end{pmatrix} \tag{25}$$

Then, having $4D$ real vector space with G as its automorphism such that $G \subset L(4)$ determines the real structure of the Dirac equation in the form

$$(\gamma_0 p_0 - i\gamma \cdot \mathbf{p})\mathbf{u} = m\mathbf{v} \tag{26}$$

$$(\gamma_0 p_0 + i\gamma \cdot \mathbf{p})\mathbf{v} = m\mathbf{u} \tag{27}$$

with

$$\gamma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix} \tag{28}$$

where σ are the Pauli matrices and $\mathbf{p} = (\widehat{p}_1, \widehat{p}_2, \widehat{p}_3)$

4.1 Biquaternionic Structure

Considering the above, we see the possibility that, writing $\mathbf{u}$ and $\mathbf{v}$ in the following form

$$\eta^h = u^h + iu^{h+2} \tag{29}$$

$$\xi^h = v^h + iv^{h+2} \tag{30}$$

the Dirac equation becomes

$$Q\eta = \xi \text{ and } \overline{Q}\xi = \eta \tag{31}$$

where Q and $\overline{Q}$ are the following elements of the field of the biquaternions

$$Q = a_0\sigma_0 - i\sigma \cdot a = A - iB \tag{32}$$

$$\overline{Q} = a_0\sigma_0 + i\sigma \cdot a = A + iB \tag{33}$$

where the upper bar is quaternionic conjugation

The Clifford algebra in real Minkowski space is $\mathbb{H}_2$ but its complexification is $\mathbb{H}_2 \otimes \mathbb{C} = \mathbb{C}_4$, which is the Dirac algebra. One may use the differential form basis and the vee ($\vee$) product in order to derive results for the Dirac gamma matrices which are useful in quantum field theory. It is interesting to see that the complexification of the quaternionic structure is necessary to incorporate in any theory of massive particles with spin 1/2 when we have $(\mathbb{C}, 4, (1, -1 - 1 - 1))$ [12–16].

5 Majorana Representation for Symmetric Equation

Despite having a real representation of the Dirac equation from the G structure, we see that it is possible to perform a unitary transformation to G for which the Dirac equation becomes with real coefficients and symmetric for both: fermions and antifermions. Consequently, it will be important to know how this transformation affects the underlying structure of the spacetime from the quaternionic viewpoint. The explicit unitary transformation is

$$U = U^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \sigma_2 \\ \sigma_2 & -1 \end{pmatrix} \tag{34}$$

and it was given by Ettore Majorana in 1937 [9]. The transformation changes the four dimensional structure of G, namely $a_0 I_4 + \gamma \cdot a$ (γ in the standard form [8]) to $a_0 I_4 + \gamma' \cdot a$ with

$$\gamma_3' \rightarrow -i\sigma_1 \otimes \sigma_0 \tag{35}$$

$$\gamma_2' \rightarrow \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \tag{36}$$

$$\gamma_1' \rightarrow i\sigma_3 \otimes \sigma_0 \tag{37}$$

and in order to be complete $\beta' \rightarrow \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}$. Explicitly

$$G' \rightarrow \begin{pmatrix} a_0\sigma_0 + i(\sigma_3 a_1 + \sigma_1 a_3) & -\sigma_2 a_2 \\ \sigma_2 a_2 & a_0\sigma_0 + i(\sigma_3 a_1 + \sigma_1 a_3) \end{pmatrix} \tag{38}$$

$$G^{T'} \rightarrow \begin{pmatrix} a_0\sigma_0 - i(\sigma_3 a_1 + \sigma_1 a_3) & \sigma_2 a_2 \\ -\sigma_2 a_2 & a_0\sigma_0 - i(\sigma_3 a_1 + \sigma_1 a_3) \end{pmatrix} \tag{39}$$

Notice that G' and $G^{T'}$ $\left(G'G^{T'} = G^{T'}G' = \mathbb{I}_4 \right)$ are related by complex conjugation, as expected due to the performed Majorana transformation, being the relativistic relation of previous sections without changes.

6 Non-Compact Fundamental $\mathbb{H}$-Structure, G and the 2+2 Spacetime

In Ref. [31] we have presented a Majorana-Weyl representation that is given by the 2 by 2 following operators

$$\sigma_\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{40}$$

where the required condition over such matrices $\sigma_\alpha \wedge \sigma_\beta = \sigma_\gamma$, $\sigma_\beta \wedge \sigma_\gamma = \sigma_\alpha$ and $\sigma_\gamma \wedge \sigma_\alpha = -\sigma_\beta$, evidently holds (Lie group, with α, β, γ :fixed indices) given the underlying non-compact $SL(2R)$ symmetry.

As we have seen previously, the G-structure must describe the spinorial field through the appearance of the Dirac equation in the tangent space. The physical choice for the structure of G can be given by

$$G^+G = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \begin{pmatrix} a_0\sigma_0 & \sigma \cdot a \\ -\sigma \cdot a & a_0\sigma_0 \end{pmatrix} \begin{pmatrix} a_0\sigma_0 & -\sigma \cdot a \\ \sigma \cdot a & a_0\sigma_0 \end{pmatrix} \tag{41}$$

$$= \begin{pmatrix} (a_0\sigma_0)^2 + (\sigma \cdot a)^2 & 0 \\ 0 & (a_0\sigma_0)^2 + (\sigma \cdot a)^2 \end{pmatrix} = \mathbb{I}_4 \tag{42}$$

where we remind that a_b are physical quantities. Then, only from the G-structure and not from any extra assumption, we have as before

$$(a_0\sigma_0)^2 + (\sigma \cdot a)^2 = 1 \Rightarrow a_0^2 + a_1^2 - a_2^2 + a_3^2 = 1 \tag{43}$$

notice the change of sign of $a_2{}^2$ due to the non compact substructure introduced by $\sigma_\beta^2 = (-i\sigma_2)^2 = -1$; consequently the physical role of the coefficients a cannot be easily identified as before. We have here two possibilities:

i)   if the definition is the same for the a_b, we have

$$a_0 = \frac{\widehat{p_0}}{m}, a_1 = i\frac{\widehat{p_1}}{m}, a_2 = i\frac{\widehat{p_2}}{m}, a_3 = i\frac{\widehat{p_3}}{m} \tag{44}$$

leading to the relativistic relation

$$\widehat{p_0^2} - \widehat{p_1^2} + \widehat{p_2^2} - \widehat{p_3^2} = m^2 \tag{45}$$

where the introduction of the momentum operators $\widehat{p}_\mu$ and the mass parameter m was performed. In such a case, evidently the signature of the spacetime is $(+ - + -)$

The structure of the Dirac equation has now the form

$$(\gamma_0 p_0 + i\gamma_2 \widehat{p}_2 - i\gamma \cdot \mathbf{p})\,\mathbf{u} = m\mathbf{v} \tag{46}$$

$$(\gamma_0 p_0 + i\gamma_2 \widehat{p}_2 + i\gamma \cdot \mathbf{p})\,\mathbf{v} = m\mathbf{u} \tag{47}$$

with

$$\gamma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix} \tag{48}$$

where σ are the representation given now by matrices (40) and $\mathbf{p} = (\widehat{p}_1, \widehat{p}_3)$

ii) if the definition for the a_b is

$$a_0 = \frac{\widehat{p}_0}{m}, \, a_1 = i\frac{\widehat{p}_1}{m}, \, a_2 = \frac{\widehat{p}_2}{m}, \, a_3 = i\frac{\widehat{p}_3}{m} \tag{49}$$

leading to the relativistic relation

$$\widehat{p}_0^2 - \widehat{p}_1^2 - \widehat{p}_2^2 - \widehat{p}_3^2 = m^2 \tag{50}$$

where the introduction of the momentum operators $\widehat{p}_\mu$ and the mass parameter m was performed. In such a case, evidently the signature of the spacetime is conserved as $(+ - --)$ with an evident emergent non hermiticity of the respective dynamical operators.

The structure of the Dirac equation has now the form

$$(\gamma_0 p_0 - \gamma_2 \widehat{p}_2 - i\gamma \cdot \mathbf{p})\,\mathbf{u} = m\mathbf{v} \tag{51}$$

$$(\gamma_0 p_0 + \gamma_2 \widehat{p}_2 + i\gamma \cdot \mathbf{p})\,\mathbf{v} = m\mathbf{u}$$

with

$$\gamma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix} \tag{52}$$

where σ are the representation given now by matrices (40) and $\mathbf{p} = (\widehat{p}_1, \widehat{p}_3)$.

Remark 3 From the point of view of Unification there exists a kind of "duality" between non-hermitian structures and spacetime signatures (this fact can be crucial to understand what happens in high dimensional theories where exist an interplay between "duality, spacetime signature and spinors phase transitions" as described in [30])

7 Relation Between Spacetime Signatures and Related Dynamics

From the argumentation given before, if certainly there exists a precise relation between the spacetime signatures, physically we have two related dynamics. As it is well known, the Palatini variational principle determines the connection required for the space-time symmetry as well as the field equations. As we have shown in [3–5], if by construction any geometrical Lagrangian or action yields the G-invariant conditions (namely, the intersection of the 4-dimensional Lorentz group L_4, the symplectic $Sp(4)$ and the almost complex group $K(4)$), as an immediate consequence the gravitational, Dirac and Maxwell equations arise from a such geometrical Lagrangian L_g as a causally connected closed system. From the tangent space viewpoint, the self-consistency is given by [3–7]

$$f_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\sigma^{\rho\sigma} = *\sigma_{\mu\nu} \tag{53}$$

where $\sigma_{\nu\lambda}$ is related to the torsion by $\frac{1}{6}(\partial_\mu \sigma_{\nu\lambda} + \partial_\nu \sigma_{\lambda\mu} + \partial_\lambda \sigma_{\mu\nu}) = T^\rho_{\nu\mu}\sigma_{\rho\lambda}$ and $f_{\mu\nu}$ can plays naturally the role of electromagnetic field. As the simplest illustration, due to the fact

that we are in the tangent space, the second order version of the Dirac eq. takes the familiar form:

$$\left\{ \left(\widehat{P}_\mu - e\widehat{A}_\mu \right)^2 - m^2 - \frac{1}{2}\sigma^{\mu\nu} f_{\mu\nu} \right\} u^\lambda = 0 \tag{54}$$

$$\left\{ \left(\widehat{P}_\mu - e\widehat{A}_\mu \right)^2 - m^2 + e\Sigma \cdot H - ie\alpha \cdot E \right\} u^\lambda = 0 \tag{55}$$

where we have introduced

$$\sigma_{\mu\nu} = (\alpha, i\Sigma), \ f^{\mu\nu} = (-E, H) \tag{56}$$

(corresponding to Galilean-type coordinates) and the fact that the momentum $\widehat{p} = \widehat{P}_\mu - e\widehat{A}_\mu$ is generalized due to the gauge freedom and the existence of a vector torsion h_α (see also Appendix) that in the case of ref. [3–5, 7, 10] is the dual of a totally antisymmetric torsion field $h_\alpha = \varepsilon_\alpha^{\nu\rho\sigma} T_{\nu\rho\sigma}$. The torsion field appears as a consequence of the existence in the very structure of the tangent space, of the third fundamental tensor $\sigma_{\lambda\mu}$. From the above "euristic" perspective we make the following remarks:

i)   The equation is symmetric: for u^λ and the same obviously for v^λ (remember that $\Psi = \mathbf{u} + i\mathbf{v}$).
ii) Because the geometrical propierties of the tangent space (G-structure) are translated to the fields and viceversa, physically the contraction $\sigma^{\mu\nu} f_{\mu\nu}$ represents the interplay between spin and electromagnetic field,
iii) In the case of 2+2 signature the "electromagnetic field" has 4 electric components and 2 magnetic ones, and in the case with 3+1 signature the quantity $E^2 + H^2$ (e.g. "energy") can be negative due to the non-hermitian character of the generalized momentum operators.

Here we can make some interlude with respect to the above results, particularly item iii). Interestlingly with the point of view of symmetry structure induced by G, we find a convergence of some isolate (from recent references) results. Some of these consequences (enumerated below) of that paper involving a $(2 + 2)$ "by hand" signatures, can be explained due to the existence of the $SL(2R)$ symmetry of a "hidden" (bi)quaternionic structure:

1) Bars from the viewpoint of 2t-physics [21] considered as a minimal model the structure of (2+2)-physics
2) Since time ago, it was suspected, looking at some structures in string theory, two dimensional black holes [22] and conformal field theory [23], that the (2+2)-signature is deeply linked to the SL(2,R)-group.
3) the (2+2)-signature is conjectured as an important physical concept in a number of physical scenarios, including the background for N = 2 strings [24, 25] (see also Refs [26]), Yang-Mills theory in Atiyah-Singer background [28] (see also Refs. [29] for the mathematical importance of the (2+2)-signatures), Majorana-Weyl spinor in supergravity [27]

In the next Section we will bring the conceptual and mathematical consistency to the above issues.

8 G-Structure, Spacetime and Fields at $T_p(M)$

It is well known that to every Lie algebra a local Lie group corresponds only being the G-structure a global affair (important issue without answer till today). Starting from the six dimensional group $SL(2\mathbb{C})$ it contains

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \; \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; \sigma_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{57}$$

$$\rho_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \rho_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \rho_3 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{58}$$

The bispinor can be constructed on the tangent space $T_p(M)$ by complexification

$$\Psi'^B = U_A^B(P)\Psi^A(P) \qquad A, B = 1, 2 \tag{59}$$

where, due to the Ambrose-Singer theorem [18], the key link of the theory is given by

$$U_A^B(P) = \delta_A^B + \mathcal{R}_{A\mu\nu}^B dx^\mu \wedge dx^\nu \tag{60}$$
$$= \delta_A^B + \omega^k(\mathcal{T}_k)_A^B$$

then

$$\mathcal{R}_{A\mu\nu}^B dx^\mu \wedge dx^\nu \equiv \omega^k(\mathcal{T}_k)_A^B \tag{61}$$

immediately we can make the folllowing observations:

  i)   there exists a true and direct correspondence Manifold group structure, tangent space, curvature and physical fields.
 ii)   the reason of the interplay described in i) is due to the unified character of the theory: all the "matter and energy" content come from the same spacetime manifold.
iii)   the underlying (super) symmetry is quite evident from the link given above: the curvature involves fermionic and bosonic structues (e.g. mixed indices), then is not difficult to see that other fields with different amount of spin can appear. Even more, due to the geometrical and group theoretical meaning of the above expression, the possible transformations have local (diffeomorphyc) character that make the role of the supersymmetry and the role of the supergravity and superspace concept to be taken under consideration.

9 Incompleteness of Rothstein's Theorems: Physics Geometrization vs. Supermanifold Construction

9.1 Poisson Structure, Quantization and Supersymmetry

Symplectic geometry grew out of the theoretical study of classical and quantum mechanics. At first it was thought that it differs considerably from Riemannian geometry, which developed from the study of curves and surfaces in three dimensional Euclidean space, and went on to provide the language in which General Relativity is studied. This fact was understandable given that symplectic geometry started from the study of phase spaces for mechanical systems but, with the subsequent seminal works of Cartan that introduce the symplectic structure into the geometry of the spacetime calculus, that thinking changed radically.

The existence of a symplectic structure on a manifold is a very significant constraint and many simple and natural constructions in symplectic geometry lead to manifolds

which cannot possess a symplectic structure (or to spaces which cannot possess a manifold structure). However these spaces often inherit a bracket of functions from the Poisson bracket on the original symplectic manifold. It is a (semi-)classical limit of quantum theory and also is the theory dual to Lie algebra theory and, more generally, to Lie algebroid theory.

Poisson structures are the first stage in quantization, in the specific sense that a Poisson bracket is the first term in the power series of a deformation quantization. Poisson groups are also important in studies of complete integrability.

From the point of view of the Poisson structure associated to the differential forms induced by the unitary transformation from the G-valued tangent space implies automatically, the existence of an *even non-degenerate (super)metric*. The remaining question of the previous section was if the induced structure from the tangent space (via Ambrose-Singer theorem) was intrinsically related to a supermanifold structure (e.g.hidden supersymmetry, etc.). Some of these results were pointed out in the context of supergeometrical analysis by Rothstein and by others authors [17, 19, 20], corroborating this fact in some sense. Consequently we have actually several models coming mainly from string theoretical frameworks that are potentially ruled out. Let us see this issue with more detail: from the structure of the tangent space $T_p(M)$ we have seen

$$
\begin{aligned}
U_A^B(P) &= \delta_A^B + \mathcal{R}_{A\mu\nu}^B dx^\mu \wedge dx^\nu \\
&= \delta_A^B + \omega^k (\mathcal{T}_k)_A^B
\end{aligned}
\tag{62}
$$

where the Poisson structure is evident (as the dual of the Lie algebra of the group manifold) in our case leading to the identification

$$
\mathcal{R}_{A\mu\nu}^B dx^\mu \wedge dx^\nu \equiv \omega^k (\mathcal{T}_k)_A^B
\tag{63}
$$

We have in the general case, a (matrix) automorphic structure. The general translation to the spacetime from the above structure in the tangent space takes the form

$$
\begin{aligned}
\widetilde{\omega} = \frac{1}{2} &\left[\omega_{ij} + \frac{1}{2} \left(\omega_{kl} \left(\Gamma_{ai}^k \Gamma_{bj}^l - \Gamma_{bj}^k \Gamma_{ai}^l \right) + g_{bd} R_{ija}^d \right) d\psi^a d\psi^b \right] dx^i \wedge dx^j \\
&+ \omega_{ij} A_{bm}^j dx^m dx^i d\psi^b + \\
&+ \frac{1}{2} \left[g_{ab} + \frac{1}{2} \left(g_{cd} \left(\Gamma_{ib}^c \Gamma_{ja}^d - \Gamma_{ja}^c \Gamma_{ib}^d \right) + \omega_{lj} R_{abi}^l \right) dx^i \wedge dx^j \right] d\psi^a d\psi^b \\
&+ g_{ab} A_{id}^b d\psi^d d\psi^a dx^i
\end{aligned}
\tag{64}
$$

Because covariant derivatives are defined in the usual (group theoretical) way

$$
D\psi^a = d\psi^a - \Gamma_{ib}^i d\psi^b dx^i
\tag{65}
$$
$$
Dx^i = dx^i - \Gamma_{aj}^i dx^j d\psi^a
\tag{66}
$$

we can rewrite $\widetilde{\omega}$ in a compact form as

$$
\begin{aligned}
\widetilde{\omega} = \frac{1}{2} &\left[\left(\omega_{ij} Dx^i \wedge Dx^j + \frac{1}{2} g_{bd} R_{ija}^d d\psi^a d\psi^b dx^i \wedge dx^j \right) \right. \\
&\left. + \left(g_{ab} D\theta^a D\theta^b + \frac{1}{2} \omega_{lj} R_{abi}^l dx^i \wedge dx^j d\theta^a d\theta^b \right) \right]
\end{aligned}
\tag{67}
$$

At the tangent space, where that unitary transformation makes the link, the first derivatives of the metric are zero, remaining only the curvatures, we arrive to

$$\widetilde{\omega} = \frac{1}{2}\left[\left(\eta_{ij} + \frac{1}{2}\epsilon_{bd}R^d_{ija}d\psi^a d\psi^b\right)dx^i \wedge dx^j + \left(\epsilon_{ab} + \frac{1}{2}\eta_{lj}R^l_{abi}dx^i \wedge dx^j\right)d\psi^a d\psi^b\right] \tag{68}$$

Here the Poisson structure can be checked

$$\eta_{ij} + \frac{1}{2}\epsilon_{bd}R^d_{ija}d\psi^a d\psi^b = \left(\delta^k_j + \frac{1}{2}\epsilon_{bd}\eta^{kl}R^d_{lja}d\psi^a d\psi^b\right)\eta_{ki} \tag{69}$$

$$\epsilon_{ab} + \frac{1}{2}\eta_{lj}R^l_{abi}dx^i \wedge dx^j = \left(\delta^c_b + \frac{1}{2}\eta_{lj}\epsilon^{cd}R^l_{dbi}dx^i \wedge dx^j\right)\epsilon_{ac} \tag{70}$$

In expressions (64–70) the curvatures, the differential forms and the other geometrical operators depend also on the field where they are defined: $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. In the quaternionic $\mathbb{H}$-case (that can correspond to the SU(2)-structure of the UFT of Borchsenius for example) the metric is quaternion valuated with the propierty $\omega^\dagger_{[ij]} = -\omega_{[ji]}$ and the covariant derivative can be straightforwardly defined as expressions (65, 66) but with the connection and coordinates also quaternion valuated. The fundamental point in a such a case going towards a fully reliable gravitational theory is to fix the connection in order to have a true link with the physical situation. The matrix representation of structures (69, 70) are automorphic ones: e.g. they belong to the identity and to the symplectic block generating the corresponding trascendent (parameter depending) functions. Now, we will analize the above fundamental structure under the light of the supersymplectic structure given by Rothstein (notation as in Ref. [19, 20])

$$\widetilde{\omega} = \frac{1}{2}\left(\omega_{ij} + \frac{1}{2}g_{bd}R^d_{ija}\theta^a\theta^b\right)dx^i dx^j + g_{ab}D\theta^a D\theta^b \tag{71}$$

where the usual set of Grassmann supercoordinates were introduced: $x^1, ...x^j; \theta^1.....\theta^d$; the superspace metrics were defined as: $\omega_{ij} = \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$, $g_{ab} = \left(\frac{\partial}{\partial\theta^a}, \frac{\partial}{\partial\theta^b}\right)$ and

$$\nabla_{\frac{\partial}{\partial x^i}}(\theta^a) = A^i_{ib}\theta^b \tag{72}$$

Due to the last expression, we can put $\widetilde{\omega}$ in a compact form with the introduction of a suitable covariant derivative: $D\theta^a = d\theta^a - A^i_{ib}\theta^b dx^i$. With all the definitions at hands, the Poisson structure of $\widetilde{\omega}$ in the case of Rothstein's is easily verified

$$\omega_{ij} + \frac{1}{2}g_{bd}R^d_{ija}\theta^a\theta^b = \left(\delta^k_i + \underbrace{\frac{1}{2}g_{bd}\omega^{lk}R^d_{ila}\theta^a\theta^b}_{\equiv B}\right)\omega_{kj} \tag{73}$$

The important remark of Rothstein [19, 20] is that the matrix representation of the structure B has *nilpotent* entries, schematically

$$\widetilde{\omega}^{-1} = \left[\omega^{-1}\left(I - B + B^2 - B^3....\right)\right]^{ij}\nabla_i \wedge \nabla_j + g^{ab}\frac{\partial}{\partial\theta^a} \wedge \frac{\partial}{\partial\theta^b} \tag{74}$$

where, as is obvious $B^n = 0$ for $n > 1$ and $n \in \mathbb{N}$

Remarks: From the above analysis, we can compare the Rothstein case with the general one arriving to the following points:

i) In the Rothstein case only a part of the full induced metric from the tangent space is preserved ("one way" extension [11–16, 19, 20])

ii)   The geometrical structures (particularly, the fermionic ones) are extended "by hand" motivated, in general, to give by differentiation of the corresponding closed forms, the standard supersymmetric spaces (e.g. Kahler, CP^n, etc.) [19, 20]. In fact it is easily seen from the structure of the covariant derivatives: in the Rothstein case there are Grassmann coordinates instead of the coordinate differential 1-forms contracted with the connection.

iii) In the Rothstein case the matrix representation (73) coming from the Poisson structure is nilpotent (characteristic of Grassmann manifolds) in sharp contrast with the general representation (68–70) coming from the tangent space of the UFT that is automorphic.

Remark 4 was noted in [13, 14] that the following facts arise:*i)* A Grassmann algebra, as used in supersymmetry, is equivalent, in some sense, to the spin representation of a Clifford algebra. *ii)* The questions about the nature and origin of the vector space on which this orthogonal group acts are completely open. *iii)* If it is a tangent space or the space of a local internal symmetry, the vectors will be functions of space-time, and the Clifford algebra will be local. *iv)* In other cases we will have a global Clifford algebra. Consequently, the geometric structure of the UFT presented here falls precisely in such a case.

9.2 UFT and Supermanifold Structure

The UFT structure induced from the tangent space by means of the Ambrose-Singer [18] theorem (62, 63) verifies straigforwardly the Darboux-Kostant theorem: e.g. it has a super-manifold structure. Darboux-Kostant's theorem [17] is the supersymmetric generalization of Darboux's theorem and statement that:

Given a $(2n|q)$-dimensional supersymplectic supermanifold $(M, \mathcal{A}_M, \omega)$, it states that for any open neighbourhood U of some point m in M there exists a set $(q_1, ..., q_n, p1, ..., pn; \xi_1, ..., \xi_q)$ of local coordinates on $V\mathcal{E}(U)$ so that ω on U can be written in the following form,

$$\omega|_U \equiv \widetilde{\omega} = \sum_{i=1}^{n} dpi \wedge dq^i + \sum_{a=1}^{q} \frac{\epsilon}{2}(\xi^a)^2, \qquad (\epsilon = \pm 1) \tag{75}$$

Proof   by simple inspection we can easily see that the expression (68) has the structure (75). That means that we have locally a supersymplectic vector superspace induced (globally) by a supersymplectic supermanifold.   □

10 Concluding Discussion and Perspectives

Here we discuss some of the results obtained in this work and describe their possible generalizations. We also briefly state other results as follows

From the point of view of the geometry and unification:

- i) The cornerstone of a consistent UFT must be a G-structure (for the tangent bundle $T(M)$) which reflects the symmetries of the different fields considered.
- ii) The difference between the QFT here and the standard QFT in curved spacetime is that whilst the latter does not alter the spacetime structure (whose structure group

remains Lorentzian), the former alters the spacetime structure radically since the structure group for the (reduced) tangent bundle is now the correspondent to the induced QFT (the same curvature of the tangent space)

- iii) The radical difference between spacetime signature and non-hermitian dynamic operators is induced by the same G-structure.
- iv) Torsion, through its dual four-dimensional vector, plays a key role both in the signature of spacetime and the CP invariant character of the field dynamics.
- v) From points iii) and iv) is clear that fermionic phase transitions in the early universe as the paradigm of energy and dark matter could have a satisfactory explanation seriously considering a theory as presented here endowed with a G structure.

From the point of view of the boson-fermion symmetries

- iv) the Darboux-Kostant theorem is fulfilled in our case showing that M fits the characteristic of a general supermanifold in addition to all those the considerations given in [13, 14, 17, 19, 20].
- v) The Rothstein theorem is incomplete to decribe the spacetime manifold being it with a more general structure from the algebraic and geometrical viewpoint.

Outlook: there are several topics that must be analyzed in future works:

- vi) There exists a deep relation of our research with early works where quaternionic and even octonionic structures (as the Moffat-Boer theory) were considered in the context of gravity: will be good to make a deep study of this issue considering the boson-fermion symmetry and the link with the quantum-gravity trouble.
- vii) the possibility, following an old Dirac's conjecture, to find a discrete quaternionic structure inside the Poincare group: this fact will be give us the possibility of spacetime discretization without break Lorentz symmetries.
- viii) The introduction of group theoretical methods of compactification as in [31]
- ix) the relation with nonlinearly realized symmetries and quantization.

Appendix: Generalized Hodge-de Rham Decomposition, the Vector Torsion *h* and the Fermion Interaction

As pointed out in references [3–5, 7, 10] the torsion vector $h = h_\alpha dx^\alpha$ (the 4-dimensional dual of the torsion field $T_{\beta\gamma\delta}$) plays multiple roles and can be constrained in several different physical situations. Mathematically, it is defined by the Hodge-de Rham decomposition given by the **4-dimensional Helmholtz theorem** which states:

If $h = h_\alpha dx^\alpha \notin F'(M)$ is a 1-form on M, then there exist a zero-form Ω, a 2-form $\alpha = A_{[\mu\nu]}dx^\mu \wedge dx^\nu$ and a harmonic 1-form $q = q_\alpha dx^\alpha$ on M that

$$h = d\Omega + \delta\alpha + q \rightarrow h_\alpha = \nabla_\alpha\Omega + \varepsilon_\alpha^{\beta\gamma\delta}\nabla_\beta A_{\gamma\delta} + q_\alpha . \tag{76}$$

Notice that even if it is not harmonic, and assuming that $q_\alpha = (P_\alpha - eA_\alpha)$ is a vector, an axial vector can be added so that the above expression takes the form

$$h_\alpha = \nabla_\alpha \Omega + \varepsilon_\alpha^{\beta\gamma\delta} \nabla_\beta A_{\gamma\delta} + \varepsilon_\alpha^{\beta\gamma\delta} M_{\beta\gamma\delta} + (P_\alpha - eA_\alpha) \tag{77}$$

$$= \nabla_\alpha \Omega + \varepsilon_\alpha^{\beta\gamma\delta} \nabla_\beta A_{\gamma\delta} + \gamma^5 b_\alpha + (P_\alpha - eA_\alpha), \tag{78}$$

where $M_{\beta\gamma\delta}$ is a completely antisymmetric tensor. In such a way, $\varepsilon_\alpha^{\beta\gamma\delta} M_{\beta\gamma\delta} \equiv \gamma^5 b_\alpha$ is an axial vector.

One can immediately see that, due to the theorem given above, one of the roles of h_α is precisely to be a generalized energy-momentum vector, avoiding the addition "by hand" of a matter Lagrangian in the action. As it is well known, the addition of the matter Lagrangian leads, in general, to non-minimally coupled terms into the equations of motion of the physical fields. Consequently, avoiding the addition of energy-momentum tensor, the fields and their interactions are effectively restricted thanks to the same geometrical structure in the space-time itself.

References

1. Weyl, H.: Space-Time-Matter. Dover (1952)
2. Xin, Y.: General Relativity on Spinor-Tensor Manifold. In: Bergman, P.G., de.Sabbata, V., Treder, H.J. (eds.) Quantum Gravity - Int. School on Cosmology & Gravitation, XIV Course, pp. 382–411. World Scientific (1996)
3. Cirilo-Lombardo, D.J.: Int. J. Theor. Phys. **49**, 1288 (2010)
4. Cirilo-Lombardo, D.J.: Int. J. Theor. Phys. **50**, 1699 (2011)
5. Cirilo-Lombardo, D.J.: Int. J. Theor. Phys. **50**, 3621 (2011)
6. Cirilo-Lombardo, D.J.: J. Math. Phys. **48**, 032301 (2007). Class. Quant. Grav. **22**, 4987 (2005)
7. Cirilo-Lombardo, D.J.: Astropart. Phys. **50-52**, 51 (2013)
8. Beresteskii, V.B., Lifshitz, E.M., Pitaevskii, L.P.: Quantum electrodynamics. Pergamon Press, New York (1982)
9. Majorana, E.: Teoria Simmetrica Dell' Elettrone E Del Positrone. Il Nuovo Cimento (1924-1942) **14**(4), 171–184 (1937)
10. Cirilo-Lombardo, D.J.: Phys. Part. Nucl. **44**(5), 848–865 (2013)
11. Pavsic, M.: AACA **22**, 449–481 (2012)
12. Albert, A.A.: Structure of Algebras, Amer. Math. Soc., Providence, R.I. (1961)
13. Winnberg, J.O.: J. Math. Phys. **18**, 625 (1977)
14. Pavsic, M.: J. Phys. Conf. Ser. **33**, 422–427 (2006)
15. Salingaros, N.A., Wene, G.P.: Acta Applicandae Mathematicae **4**, 271–292 (1985)
16. Pavsic, M.: AACA **20**, 781–801 (2010). Phys. Lett. **B692**, 212–217 (2010)
17. Kostant, B.: Lecture Notes in Mathematics. In: Bleuler, K., Reetz, A. (eds.) Proceedings of the Conf. on Diff. Geom. Meth. in Math. Phys., Bonn 1975, vol. 570, p. 177. Springer-Verlag, Berlin (1977)
18. Ambrose, W., Singer, I.M.: Trans. Am. Math. Soc. **75**(3), 428–443 (1953)
19. Rothstein, M.: Lecture Notes in Physics. In: Bartocci, C., Bruzzo, U., Cianci, R. (eds.) Proceedings of the Conf. on Diff. Geom. Meth. in Math. Phys., Rapallo 1990, vol. 375, pp. 331. Springer-Verlag, Berlin (1991)
20. Bartocci, C., Bruzzo, U., Hernandez Ruiperez, D.: The geometry of supermanifolds, p. 1991. Kluwer, Dordrecht
21. Bars, I., Chen, S.H.: Phys. Rev. D **79**, 085021 (2009)
22. Witten, E.: Phys. Rev. D **44**, 314 (1991)
23. Ooguri, H., Vafa, C.: Nucl. Phys. **B367**, 83 (1991). Nucl. Phys. **B361**, 469 (1991)
24. Sezgin, E.: Is there a stringy description of selfdual supergravity in (2+2)-dimensions? Published in Trieste 1995, High energy physics and cosmology 360–369; arXiv:hep-th/9602099
25. Khviengia, Z., Lu, H., Pope, C.N., Sezgin, E., Wang, X.J., Xu, K.W.: Nucl. Phys. **B444**, 468 (1995). arXiv:hep-th/9504121
26. Ketov, S.V.: Class. Quantum Grav. **10**, 1689 (1993). arXiv:hep-th/9302091
27. Ketov, S.V., Nishino, H., Gates, S.J. Jr.: Phys. Lett. **B 307**, 323 (1993). arXiv:hep-th/9203081

28. De Andrade, M.A., Del Cima, O.M., Colatto, L.P.: Phys. Lett. **B 370**, 59 (1996). arXiv:hep-th/9506146
29. Atiyah, M.F., Ward, R.S.: Commun. Math. Phys. **55**, 117 (1977)
30. Hull, C.M.: JHEP **9811**, 017 (1998). arXiv:hep-th/9807127
31. Cirilo-Lombardo, D.J.: Eur. Phys. J. **C72**, 2079 (2012)
32. Pavsic, M.: The Landscape of theoretical physics: A Global view. From point particles to the brane world and beyond, in search of a unifying principle, p. 386. Kluwer (2001). (Fundamental theories of physics. 119). arXiv:gr-qc/0610061
33. Bishop, R.L., Goldberg, S.I.: Tensor analysis on manifolds. Dover (1980). see in particular Chapter 5 pp. 208 about the meaning of Lorentz metric