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FRW Viscous Cosmological Models of Generalized Chapyglin Gases

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Abstract Exact solutions of some viscous cosmological Friedmann–Robertson–Walker (FRW) models of generalized Chapyglin gases in four–dimensional space–time are obtained analytically. In addition to generalized Chapyglin gases, analytical solutions are obtained for isothermal perfect gases where the relationship between the pressure and the density does not depend on the scale factor. It is shown that the two viscosity laws employed in the study reported here provide similar dynamical equations and that, depending on the equation of state, flat universes may expand as either power laws or exponential functions of the cosmic time. It is also shown that finite time singularities may occur in some of the models considered here. The stability of the solutions obtained is also analyzed.

Keywords Bulk viscosity · Cosmology · Chapyglin gas · Four–dimensional space–time

1 Introduction

The cosmological observations made in the last half a century [1, 2] indicate that, in order to account for the present acceleration phase of the universe within the realms of Einstein's theory of relativity, one has to introduce a non-gravitational type of matter with a huge negative pressure called dark energy [3]. The most common candidate for dark energy is to introduce the cosmological constant Λ and assume that this matter behaves as an isothermal perfect fluid with $p = -\rho$ as equation of state, where p and ρ are the pressure and density, respectively. However, this approach may be disregarded entirely due to the inconsistencies observed in the value of Λ in Ia–type supernovae experiments and the value of Λ interpreted as vacuum energy.

Recently, attempts have been made to employ barotropic approximations and a generalization of Chapyglin's equation of state, i.e., $p = -\frac{A}{\rho^\alpha}$, where A is a positive constant and

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$0 < \alpha \leq 1$; the Chapyglin gas corresponds to $\alpha = 1$ [4]. It must be noted that the generalized Chapyglin gas (GCG) corresponds to a perfect one upon choosing $A = 1$ and $\alpha = -1$ and to a barotropic fluid for $\alpha < 0$, and has been previously used by, for example Saadat and Pourhassan [5] who studied viscous FRM cosmologies with constant viscosity for two specified scale factors, i.e., $a(t) = a(0)t^n$ and $a(t) = a(0)t^n e^{\beta t}$, where t is time, β is a constant and $n > 0$.

Some authors [6] have used equations of state that are a combination of those of a perfect gas and that proposed by Chapyglin, e.g.,

$$p = B\rho - \frac{A}{\rho^\alpha},$$

and studied the critical points of the resulting dynamical system. The above equation reduces to that of a GCG for $B = 0$ and to that of an ideal gas for $A = 0$, and may be referred to as an equation of state for a modified Chapyglin gas (MCG). Other authors, e.g., Sadeghi et al. [7], have introduced modified Chaplygin gases given by the following equation of state

$$p = B\rho - \frac{1}{\rho^\alpha}\left[\frac{A}{1+\omega} - 1 + \left(\rho^{1+\alpha} - \frac{A}{1+\omega} + 1\right)^{-\omega}\right],$$

where B and ω are constants, which reduces to the equation of state for a generalized Chapyglin gas for $B = 0$ and $\omega = 0$, to that of an isothermal ideal gas, i.e., $p = B\rho$, for $A = 0$ and $\omega = 0$, and to that of Mazumder et al. [6] for $\omega = 0$. Other MCG equations of state include [7] $p = -\frac{1}{\rho^\alpha}\left[A + \left(\rho^{1+\alpha} - A\right)^{-\omega}\right]$ which reduces to the GCG equation of state for $\omega = \infty$ and to $p = -\rho$ for $\omega = -1$.

More recently, some authors [5, 8–10] have introduced equations of state for extended Chapyglin gases (ECG), e.g.,

$$p = \sum_{n=0} B_n \rho^n - \frac{A}{\rho^\alpha},$$

which reduce to the GCG equation of state for $B_n = 0$ and to barotropic equations for $A = 0$. In particular, for $B_n = 0$ with $n = 0, 2, 3, \cdots$ and $A = 0$, the last equation becomes $p = B_1\rho$.

The effects of bulk viscosity in cosmological models have been included by some authors; for example, Saadat [11] considered the Friedmann-Robertson–Walker model in four–dimenional space–time, a perfect gas law equation of state (cf. his (8)) and obtained some solutions for flat universes in the absence of viscosity and for n–dimensional flat universes with constant viscosity. Saadat and Farahani [12] considered a Chapyglin gas with $\alpha = 1$ and a constant viscosity coefficient and assumed that the scale factor is proportional to a power of the cosmic time, thereby, not satisfying the two equations that govern the FRW model. On the other hand, Amani and Pourhassan [13] employed a viscous generalized Chapyglin gas with arbitrary α and constant viscosity coefficient, but their (10) is not the integral of their (9). Katore et al. [14] considered a FRW bulk viscosity cosmology in multi–dimensional space–time where the equation of state is that of a perfect gas, the viscosity coefficient is a monomial of the density, and the adiabatic coefficient that appears in the perfect gas law depends in a nonlinear fashion on the scale factor.

More recently, Saadat and Pourhassan [15] considered a generalized Chapyglin gas with a viscosity coefficient that is a monomial of the density and obtained a solution that relates the cosmic time to the density through a hypergeometric function. The same authors [16] considered an ECG equation of state and an ansatz that relates the density to the cosmic time with a constant viscosity. Saadat and Pourhassan [5] studied FRM cosmologies with a GCG equation of state and a constant viscosity for two specified scale factors. A constant viscosity and an MCG equation of state were used by Saadat and Pourhassan [17] and Pourhassan [18] in their studies of FRW cosmologies, while Naji et al. [19] employed an MCG equation of state in their studies of the effects of shear and bulk viscosities on cosmological parameters.

In this paper, we study the effects of the viscosity and the equation of state on four–dimensional FRW cosmological models of flat universes. The equation of state considered here corresponds to a generalized Chapyglin gas and includes that of an ideal gas, whereas the viscosity coefficient is assumed to vary with the Hubble's parameter. Two viscosity laws are considered and analytical solutions are provided; attempts to obtain analytical (closed–form) solutions with MCG and ECG equations of state and the two viscosity laws considered in this paper are in progress and may be reported in the future.

The paper has been arranged as follows. In Section 2, a brief summary of the field equations is presented. Section 2 is followed by two sections where two bulk viscosity models are analyzed as functions of the exponents of the density that appear in the generalized Chapyglin's gas law and the viscosity coefficient. Although the formulation presented in Sections 3 and 4 is carried out in terms of Hubble's parameter, it can be performed with respect to the density without any difficulty. Furthermore, only analytical solutions for the scale factor as a function of time are presented in Sections 3 and 4; the dependence of the density, viscosity and pressure on time may be obtained easily from Friedmann's equation, once the scale factor is known as a function of time. The stability of the solutions obtained in Sections 3 and 4 is analyzed in Section 5. A final section summarizes the most important findings of the paper.

2 Field Equations and Viscous Gas Models

The Friedmann–Robertson–Walker (FRW) universe in four–dimensional space–time is governed by the following metric [20–22]

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right), \tag{1}$$

where $a(t)$ is the scale factor, t is the cosmic time, $d\Omega^2 = d\theta^2 + \sin^2\theta d\psi^2$, k is the curvature of space, i.e., $k = 0$, 1 and -1 correspond to flat, spherical and hyperbolic spaces, respectively, and r , θ $(0 \leq \theta < \pi)$ and ψ $(0 \leq \psi < 2\pi)$ denote spherical coordinates, i.e., the radial coordinate and the azimuthal and polar angles, respectively .

The energy–momentum equation tensor of a viscous fluid is given by the following relation [23–26]

$$T^{\mu\nu} = (\rho + \bar{p})u_\mu u_\nu - \bar{p}g_{\mu\nu}, \tag{2}$$

under the assumptions that $c = 1$, $\Lambda = 0$ and $8\pi G = 1$, where u^μ is the velocity vector with the normalization condition $u^\mu u_\nu = -1$, and the total pressure is given by

$$\bar{p} = p - 3\mu H, \tag{3}$$

where μ is the viscosity, $H = \frac{\dot{a}}{a}$ is the Hubble expansion parameter, and the dot denotes differentiation with respect to t.

From the above three equations, it is easy to deduce the following field equations

$$H^2 = \frac{\rho}{3} - \frac{k}{a^2}, \tag{4}$$

and

$$\dot{\rho} + 3(\rho + p)H - 9\mu H^2 = 0, \tag{5}$$

where we have used the condition that $\partial_\nu T^{\mu\nu} = 0$ and the dot denotes differentiation with respect to t.

Equations (4) and (5) with the equation of state

$$p = (\omega - 1)\rho, \tag{6}$$

where ω ($0 \leq \omega \leq 2$) is the adiabatic parameter and may depend on the cosmic time, has been used with constant ω in Ref. [11] and with ω depending on the scale factor in Ref. [14].

In this paper, we consider a generalized Chapyglin gas, i.e., the equation of state is

$$p = -\frac{A}{\rho^\alpha}, \tag{7}$$

where A is a positive constant and $0 < \alpha \leq 1$. We also consider (7) for other values of α which do not correspond to a generalized Chapyglin gas; for example, for $\alpha = -1$ and $A = 1 - \omega$, (7) reduces to (6) which is the equation of state for an isothermal ideal gas, and negative values of α result in barotropic equations of state where the pressure depends on a (positive) power of the density.

The derivatives with respect to the cosmic time which appear in H and in (5) may be changed to derivatives with respect to a as, for example, $\dot{P} = P'\dot{a} = P'Ha$, where the prime denotes differentiation with respect to $a(t)$. With such a change of variables, it is easy to deduce form (4) and (5) that

$$a^3 HH' = k + \frac{a^2}{2\rho^\alpha}\left(3\mu\rho^\alpha H - \rho^{\alpha+1} + A\right), \tag{8}$$

where

$$\rho = \frac{3}{a^2}\left(a^2 H^2 + k\right). \tag{9}$$

Equations (8) and (9) are nonlinearly coupled and their solutions may have to be obtained numerically. However, analytical solutions to these equations may be obtained as indicated next.

3 Viscous Cosmology in Flat Universes: $\mu = \bar{\mu} H$

For $k = 0$, i.e., a flat universe, and $\mu = \bar{\mu}\sqrt{\Theta}$, where $\Theta = H^2$ and $\bar{\mu}$ is a constant, (8) and (9) can be written as

$$\Theta' = \frac{3}{a\Theta^\alpha}\left((\bar{\mu} - 1)\Theta^{\alpha+1} + \bar{A}\right), \tag{10}$$

the solution of which is

$$\Theta = H^2 = \left(\frac{1}{\bar{\mu} - 1}\left(Ba^{3(\alpha+1)(\bar{\mu}-1)} - \bar{A}\right)\right)^{\frac{1}{\alpha+1}}, \tag{11}$$

where $\bar{A} = \frac{A}{3^{\alpha+1}}$, B is an integration constant that must be determined from the initial conditions, $\bar{\mu} \neq 1$ and $\alpha \neq -1$. The cases $\bar{\mu} = 1$ and $\alpha = -1$ will be considered below.

Upon making use of the definition of H, (11) yields

$$C + 3t(\alpha + 1)(\bar{\mu} - 1)^{\frac{2\alpha+1}{2(\alpha+1)}} \equiv I(\Phi) = \int \frac{d\Phi}{\Phi(B\Phi - \bar{A})^{\frac{1}{2(\alpha+1)}}}, \tag{12}$$

where C is an integration constant that must be determined from the initial conditions, $\bar{\mu} \neq 1, \alpha \neq -1$ and $\Phi = a^{3(\alpha+1)(\bar{\mu}-1)}$.

Case 1 $\alpha = \frac{1}{2n} - 1 \in \aleph$. The integral in (12) can be performed analytically for $\alpha = \frac{1}{2n} - 1 \in \aleph$ where $\aleph$ denotes the set of natural numbers, and, for example, for $n = 1$, yields [27]

$$I(\Phi) = \frac{1}{\bar{A}} \ln \frac{B\Phi - \bar{A}}{\Phi}, \tag{13}$$

and, therefore, using (12)

$$a(t) = \left(\frac{\bar{A}}{B - e^{\bar{A}\left(C + 3t(\alpha+1)(\bar{\mu}-1)^{\frac{2\alpha+1}{2(\alpha+1)}}\right)}} \right)^{\frac{1}{3(\alpha+1)(\bar{\mu}-1)}}, \tag{14}$$

where $\bar{\mu} \neq 1$, $a(0) = \left[\bar{A}/(B - e^{\bar{A}C})\right]^{\frac{1}{3(\alpha+1)(\bar{\mu}-1)}}$ and the positivity of $a(0)$ requires that $\bar{A}/(B - e^{\bar{A}C}) \geq 0$. For a generalized Chapyglin gas, $\bar{A} > 0$, and the positivity of $a(0)$ requires that $B > e^{\bar{A}C}$.

The argument of the exponential term in (14) is positive for large t if $(1 + \alpha)(\bar{\mu} - 1)^{\frac{2\alpha+1}{2(\alpha+1)}} > 0$, which requires that $\bar{\mu} > 1$. For these conditions, there may exist a critical cosmic time $t_c = -\frac{\bar{A}C}{R} \ln B$ where $R \equiv 3\bar{A}(\alpha + 1)(\bar{\mu} - 1)^{\frac{2\alpha+1}{2(\alpha+1)}}$ at which a becomes unbounded, provided that $\frac{\bar{A}C}{R} \ln R$ is negative; otherwise, the solution remains bounded and $a(t)$ tends to zero as t tends to infinity. On the other hand, for $\bar{\mu} < 1$, the argument of the exponential term in (14) is negative and, therefore, $a(t)$ tends to the asymptotic value $a_\infty = \left(\frac{\bar{A}}{B}\right)^{\frac{1}{3(\alpha+1)(\bar{\mu}-1)}}$ as $t \to \infty$.

Equation (14) and the discussion of the previous paragraph clearly indicate that the scale factor $a(t)$ and, therefore, the Hubble expansion parameter, $H(t) = \frac{\dot{a}}{a}$, the deceleration parameter, $q(t) = -\frac{\ddot{a}a}{\dot{a}^2}$, and the jerk parameter, $j(t) = \frac{\dddot{a}}{aH^3}$, where $\dddot{a}$ denotes the third–order derivative of $a(t)$ with respect to t, depend on the equation of state parameters A and α, the viscosity coefficient $\bar{\mu}$ and the initial conditions through the integration constants B and C.

Since, for flat universe, $\rho = 3H^2$ (cf. (4)), both the density and the pressure (cf. (3)) depend on these variables and may be calculated from (14). Furthermore, as indicated above, the initial conditions play a paramount role in determining the cosmology dynamics and the existence of finite time singularities.

Case 2 $\bar{\mu} = 1$. For $\bar{\mu} = 1$, (12) yields

$$H(a) = \left((\alpha + 1)(3\bar{A} \ln a + B)\right)^{\frac{1}{\alpha+1}}, \tag{15}$$

where B is an integration constant, and (15) has the following solution

$$\phi = \left(\frac{2\alpha + 1}{2(\alpha + 1)} \left(3(\alpha + 1)^{\frac{1}{2(\alpha+1)}} \bar{A}t + C\right) \right)^{\frac{2(\alpha+1)}{2\alpha+1}}, \tag{16}$$

for $\alpha \neq -1$ and $\alpha \neq -\frac{1}{2}$, where $\phi = 3\bar{A} \ln a + B$, and, therefore,

$$a(t) = a(0) \exp\left(\frac{1}{3\bar{A}} \left((P(Qt + C))^{\frac{1}{P}} - (PC)^{\frac{1}{P}} \right) \right), \tag{17}$$

where $P = \frac{2\alpha+1}{2(\alpha+1)}$ and $Q = 3\bar{A}(\alpha+1)^{\frac{1}{2(\alpha+1)}}$. Therefore, $a(t)$ increases as a power of the cosmic time if $\bar{A}(2\alpha+1)(1+\alpha)^{-\frac{2\alpha+1}{2(\alpha+1)}}$ is positive. This condition is clearly satisfied for $0 < \alpha \leq 1$ and $A > 0$, i.e., for the generalized Chapyglin gas considered in this paper. Furthermore, for $R \equiv \frac{(PQ)^{\frac{1}{P}}}{3\bar{A}} > 0$, (17) indicates that $a(t) \sim a(0)\exp(Rt^{\frac{1}{P}})$ provided that $P > 0$. This exponential behavior is quite different from that used in the quasi–exponential phase of the universe, i.e., $a(t) = a(0)t^n e^{\beta t}$, where β is a positive constant [11, 30].

As stated in the Introduction, for $\alpha < 0$, the generalized Chapyglin gas equation of state becomes $p = -A\rho^{|\alpha|}$ which corresponds to a barotropic fluid; the case $\alpha = -1$, results in the ideal gas law $p = -A\rho$ which has been previously considered by Saadat [11] and Mohanty and Pradman [28]. Using Saadat's (8), one obtains $A = 1 - \omega$ where ω denotes the adiabatic parameter and $0 \leq \omega \leq 2$, so that $-1 \leq A \leq 1$. ω depends on the scale factor [14, 29], although it was assumed to be constant in Ref. [11].

As indicated above, the formulation presented here can also deal with barotropic fluids and, in particular, with the ideal equation of state. For this reason, in the following subsections and section, we consider negative values of α.

Case 3 $\bar{\mu} = 1$ and $\alpha = -1$. For $\bar{\mu} = 1$ and $\alpha = -1$, the solution of (12) together with the definition of $H(t)$ yield

$$a(t) = \left(-\frac{3\bar{A}}{2}(t\sqrt{B} + C)\right)^{-\frac{2}{3\bar{A}}}, \tag{18}$$

which, for positive $\bar{A}$, indicates that $a(t)$ becomes unbounded at $t = -\frac{C}{\sqrt{B}}$ provided that $\frac{C}{\sqrt{B}}$ is negative. If this condition is not met and $\bar{A}$ is positive, (18) indicates that $a(t)$ tends to zero as t tends to infinity. On the other hand, if $\bar{A} < 0$, $a(t) \sim D^r t^r$ where $D = \frac{3}{2}|\bar{A}|\sqrt{B}$ and $r = \frac{2}{3|\bar{A}|}$; this value of r is in full accord with the power of t that appears in (14) of Ref. [11] if $|\bar{A}| = \omega$, although the factor that multiplies the power of time in the corresponding expressions is different.

Case 4 $\bar{\mu} = 1$ and $\alpha = -\frac{1}{2}$. For $\bar{\mu} = 1$ and $\alpha = -\frac{1}{2}$, the solution of (12) together with the definition of $H(t)$ yield

$$3\bar{A}\ln(a(t)) + B = Ce^{\frac{3\bar{A}}{2}t}, \tag{19}$$

which, for positive $\bar{A}$, indicates that a increases exponentially with the cosmic time as

$$a(t) = a(0)\exp\left(\frac{C}{3\bar{A}}\left(e^{\frac{3\bar{A}}{2}t} - 1\right)\right). \tag{20}$$

Equation (20) indicates that $a(t)$ is a double–exponential function of the cosmic time and that $a(t) \sim a(0)\exp\left(\frac{C}{3\bar{A}}e^{\frac{3\bar{A}}{2}t}\right)$ for $t >> 1$ if C and $\bar{A}$ are positive. On the other hand, for $\bar{A} > 0$ and $C < 0$, the right–hand side of (20) tends to zero as t tends to infinity.

The double–exponential dependence on the cosmic time observed in (20) is in marked contrast with the monomial, exponential and mixed monomial–exponential behaviors [30], i.e., $a(t) = a(0)t^n$, $a(t) = a(0)e^{\beta t}$ and $a(t) = a(0)t^n e^{\beta t}$, respectively, which have been used in some models of cosmology, e.g., [5].

4 Viscous Cosmology in Flat Universes: $\mu = \bar{\mu} H^{-(2\alpha+1)}$

If $k = 0$ and $\mu = \bar{\mu}\Theta^{-(\alpha+\frac{1}{2})}$, where $\Theta = H^2$ and $\bar{\mu}$ is a constant, (4) and (5) can be written as

$$\Theta' = \frac{3}{a\Theta^{\alpha}}\left(\hat{A} - \Theta^{\alpha+1}\right), \tag{21}$$

where $\hat{A} = \bar{A} + \bar{\mu}$.

Equation (21) can be integrated to obtain

$$C - 3(\alpha+1)t \equiv I(\Phi) = \int \frac{d\Phi}{\Phi(\hat{A} - B\Phi)^{\frac{1}{2(\alpha+1)}}}, \tag{22}$$

for $\alpha \neq -1$.

The integral that appears in the right–hand side of (22) is closely related to that of (12) and, therefore, one may obtain analytical solutions for the same cases as those considered in the previous section. These cases are not reported here; however, we consider some cases that correspond to perfect gases.

Case 5 $\alpha = -1$. For $\alpha = -1$, the solution to (21) is

$$a^{\frac{3}{2}(1-\hat{A})} = \frac{3}{2}(1-\hat{A})(C + t\sqrt{B}), \tag{23}$$

where B and C are integration constants.

Equation (23) indicates that $a(t)$ increases or decreases with the cosmic time if $\hat{A} < 1$ or $\hat{A} > 1$, respectively. For $t >> 1$ and $\hat{A} < 1$, (23) indicates that $a(t) \sim D^r t^r$ where $D = \frac{3}{2}(1-\hat{A})\sqrt{B}$ and $r = \frac{2}{3(1-\hat{A})}$ which coincides with a well–known power–law acceleration in cosmology [11, 30]. It is interesting to note that this asymptotic behavior is identical to that found in (14) of Ref. [11] if $B = 1$.

Case 6 $\alpha = -1$ and $\hat{A} = 1$. For $\alpha = -1$ and $\hat{A} = 1$, (21) implies that $\Theta' = 0$ and, therefore, $H = B$, so that $a(t) = Ce^{\sqrt{B}t}$ in this case and, therefore, the scale factor increases exponentially with time. This scalar factor is a special case of $a(t) = a(0)t^n e^{\beta t}$ [30] and corresponds to $n = 0$ and $\beta = \sqrt{B}$, and the latter depends on the initial conditions as indicated in Section 2.

Although not shown here, the density may be calculated as $\rho = 3H^2 = 3\Theta$ from the scale factor $a(t)$ determined in previous paragraphs. It must also be pointed out that the viscosity coefficients employed in this and the previous section are of the form $\mu = f(\Theta) = f(H^2) = f\left(\frac{\rho}{3}\right)$ and, therefore, correspond to $\mu = \frac{1}{3}\bar{\mu}\rho$ and $\mu = \bar{\mu}\left(\frac{\rho}{3}\right)^{-(2\alpha+1)}$, respectively. This means that the results of Section 3 correspond to dark matter whose dynamic viscosity, i.e., $\nu = \frac{\mu}{\rho}$, is constant, whereas those of Section 4 correspond to a fluid which thickens, i.e., it becomes more viscous, with the density if $2(\alpha+1) < 1$, a fluid with constant viscosity if $\alpha = -\frac{1}{2}$ and a thinning fluid for $2(\alpha+1) > 1$. This indicates that it is interesting to consider the case $\alpha = -\frac{1}{2}$.

Case 7 $\alpha = -\frac{1}{2}$. For $\alpha = -\frac{1}{2}$, (21) yields

$$\sqrt{\Theta} = H = \hat{A} - Ba^{-\frac{3}{2}}, \tag{24}$$

where B is an integration constant.

Upon introducing $\Phi = a^{-\frac{3}{2}}$, (24) can be integrated to obtain

$$\Phi = a^{-\frac{3}{2}} = Ce^{\frac{3}{2}Bt} + \frac{\hat{A}}{B}, \tag{25}$$

which indicates that, for $B > 0$, $a(t) \sim C^{-\frac{2}{3}} e^{-Bt}$ for $t >> 1$, whereas, for $B < 0$, $a(t)$ tends to $(\frac{B}{\hat{A}})^{\frac{2}{3}}$ as $t \to \infty$, thus showing that the cosmology behavior depends on the initial conditions through the integration constants B and C and the viscosity coefficient $\bar{\mu}$ through $\hat{A}$.

5 Stability

The stability of the solutions obtained in the two previous sections may be analyzed by means of a variety of methods, i.e., linearization, perturbation theory, etc. It may be also analyzed by considering that the speed of sound, c_s, defined as $c_s^2 = \frac{d\bar{p}}{d\rho} \geq 0$ [31, 32] which is the approach followed here.

From (3) and (9), one obtains $c^2 = \frac{\alpha A}{\rho^{\alpha+1}} - \frac{d(\mu H)}{dH^2}$ which is equal to $\frac{\alpha A}{\rho^{\alpha+1}} - \bar{\mu}$ and $\frac{\alpha}{H^{2(\alpha+1)}}\left(\bar{\mu} + \frac{A}{3^{\alpha+1}}\right)$ for the viscosity laws employed in Sections 3 and 4, respectively.

In the next two sections, we analyze the stability of the viscous models presented in Sections 3 and 4.

5.1 Viscous Cosmology in Flat Universes: $\mu = \bar{\mu}H$

By using (11), it is an easy matter to to show that the condition of stability requires that

$$\frac{\bar{\mu}-1}{3^{\frac{\alpha+1}{2}}}\sqrt{\frac{\alpha A}{\bar{\mu}}} + \bar{A} \geq Ba^R, \tag{26}$$

where $R \equiv 3(\alpha+1)(\bar{\mu}-1)$. We next consider the cases analyzed in Section 3.

Case 1 $\alpha = \frac{1}{2n} - 1 \in \aleph$. For this case, (14) shows that $a(t)$ is a monotonic function of t, and, therefore, the stability condition requires that

$$\frac{\bar{\mu}-1}{3^{\frac{\alpha+1}{2}}}\sqrt{\frac{\alpha A}{\bar{\mu}}} + \bar{A} \geq \max\left(\frac{B\bar{A}}{B - e^{\bar{A}C}}, \bar{A}\right), \tag{27}$$

if $\bar{A}Q < 0$, or

$$\frac{\bar{\mu}-1}{3^{\frac{\alpha+1}{2}}}\sqrt{\frac{\alpha A}{\bar{\mu}}} + \bar{A} \geq \max\left(\frac{B\bar{A}}{B - e^{\bar{A}C}}, 0\right), \tag{28}$$

if $\bar{A}Q > 0$.

Equations (27) and (28) clearly indicate that the stability depends on the parameters that characterize the equation of state of generalized Chapyglin gases, i.e., A and α, the viscosity coefficient $\bar{\mu}$ and the initial conditions through the constants of integration B and C.

Case 2 $\bar{\mu} = 1$. In this case, the stability condition requires that

$$\frac{1}{3^{\frac{\alpha+1}{2}}}\sqrt{\frac{\alpha A}{\bar{\mu}}} \geq (\alpha+1)(B + 3\bar{A}\ln(a(0)) + (P(Qt+C))^{\frac{1}{P}} - (PQ)^{\frac{1}{P}}), \tag{29}$$

which will be clearly violated if $(\alpha+1)(PQ)^{\frac{1}{P}} > 0$; the time, at which this violation occurs, can be easily determined by replacing the $\geq$ sign by the $=$ in (29). Such a violation is clearly a function of α, μ, A, and the constants of integration B and C, in accord with the analytical solution reported in Section 3 for this case. Note that $a(0)$ depends on the same parameters.

On the other hand, if $(\alpha+1)(PQ)^{\frac{1}{P}} < 0$, the solution given by (17) is stable if

$$\frac{1}{3^{\frac{\alpha+1}{2}}}\sqrt{\frac{\alpha A}{\bar{\mu}}} = \frac{1}{3^{\frac{\alpha+1}{2}}}\sqrt{\alpha A} \geq (\alpha+1)(B + 3\bar{A}\ln(a(0))), \tag{30}$$

which, once again, indicates that the stability depends on α, A, and the constants of integration B and C.

Case 3 $\bar{\mu} = 1$ and $\alpha = -1$. In this case, one may obtain, after some algebraic manipulations, that the condition of stability is

$$\left(\frac{\alpha A}{\bar{\mu}}\right)^{\alpha+1} \geq \frac{3B}{\left(-\frac{3\bar{A}}{2}(\sqrt{B}t + C)\right)^2}, \tag{31}$$

which implies that $\left(\sqrt{\frac{-A}{2}}\right)^{\alpha+1} = 1 \geq \frac{4B}{3\bar{A}^2C^2}$ and, therefore, depends on B, C, and A.

Case 4 $\bar{\mu} = 1$ and $\alpha = -\frac{1}{2}$. In this case, it is easily seen that the stability condition demands that

$$\left(\frac{\alpha A}{\bar{\mu}}\right)^{\alpha+1} \geq \frac{3C^2}{4}e^{3\bar{A}t}, \tag{32}$$

which will be violated if $\bar{A} > 0$; the time at which this occurs may be determined by replacing the $\geq$ sign by the $=$ sign in the above equation. On the other hand, for $\bar{A} < 0$, the stability condition requires that $\sqrt{-\frac{A}{2}} \geq \frac{3C^2}{4}$.

5.2 Stability of the Viscous Cosmology in Flat Universes: $\mu = \bar{\mu}H^{-(2\alpha+1)}$

For the viscous law of Section 4, $c_s^2 \geq 0$ if $\alpha\left(\bar{\mu} + \frac{A}{3^{\alpha+1}}\right) \geq 0$ which is clearly satisfied by the generalized Chapyglin gas considered in this paper because $0 < \alpha \leq 1$, $A > 0$ and $\bar{\mu} > 0$. The condition $c_s^2 \geq 0$ is also satisfied by barotropic fluids, i.e., those for which $\alpha < 0$ provided that $\bar{\mu} + \frac{A}{3^{\alpha+1}} \leq 0$ which implies that $A \leq -3^{\alpha+1}\bar{\mu}$; otherwise, the solution is unstable.

6 Conclusions

The FRW model in four–dimensional flat space-time universes has been studied with equations of state where the pressure is only dependent of the density and the bulk viscosity is a function of the Hubble's parameter or density, and analytical solutions have been obtained.

For a viscosity that depends either linearly on or is inversely proportional to a power of the density, it has been found that the scale factor may be an exponential or a double exponential function of the cosmic time for a generalized Chapyglin gas. For the equation of state of a perfect gas law, the scale factor may be a power law or a double exponential function of the cosmic time for a viscosity that depends linearly on the density, and a power law, an exponential or a double exponential function of the cosmic time when the viscosity is inversely proportional to a power of the density. The scale factor and, therefore, the Hubble, deceleration and jerk parameters and the pressure and density, have been found to depend on the two parameters that characterize the equation of state of a generalized Chapyglin gas, the parameter for the viscosity law and the initial conditions.

It has also been shown that, when the viscosity is inversely proportional to a power of the density and the equation of state is that of a generalized Chapyglin gas, the resulting cosmologies are stable, whereas, when the viscosity is a linear function of the density, instabilities and finite time singularities may occur. Such instabilities and/or singularities have been found to depend on the initial conditions, the parameters that characterize the equation of state and the parameter of the viscosity law.

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