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Bound Entanglement for Bipartite and Tripartite Quantum Systems

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Abstract We construct bound entangled states in bipartite systems and tripartite systems. A class of bound entangled states in $3k \otimes 3k$ quantum systems is constructed. Moreover, we construct a class of entangled states in $4 \otimes 4 \otimes 4$ quantum systems and classify those states with respect to their distillability. The class of states are bound entangled for arbitrary bipartite split.

Keywords Bound entangled · Positive partial transpose

1 Introduction

Serving as an important resource for quantum theory, quantum entanglement has been widely applied in the rapidly expanding filed of quantum information processing. However, the fundamental problem to check whether a given state is entangled or not has not been fully resolved. In recent years, there were considerable efforts to analyze the entanglement of quantum states, and great progress was achieved. The first significant progress is known as the PPT criterion [1], the separable states remain positive if subjected to partial transpose. The reduction criterion [2] is equivalent to the PPT criterion for $2 \otimes N$ composite systems, but it is not sufficient for separability in general. The range criterion [3] which says that the separable state can span its range is also a necessary condition for separability.

A special kind of states that entangled but not distillable are called bound entangled states, it plays a very important role in quantum information [4–7]. The first example of bound entangled states was proposed by Horodecki [3]. Based on the unextendible product basis, a class of bound entangled states was constructed by Bennett et al. [8]. A class of bound entangled states in $4 \otimes 4$ quantum systems was constructed by Fei et al. [9]. An N-qubit bound entangled state which violates the Bell inequality if and only if $N \geq 6$ was constructed by Dür [10]. Although more and more efforts have been done to analyze

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entanglement of quantum states [11–14], and more and more bound entangled states were presented [15–18], a full comprehensive understanding of bound entangled states is still a challenge.

Following the work of Fei et al., we present the construction of bound entangled states in $3k \otimes 3k$ systems and $4 \otimes 4 \otimes 4$ systems. The structure of this paper is as follows: In Section 2, we present the construction of bound entangled states in $3k \otimes 3k$ systems, and examples of the bound entangled states in $3 \otimes 3$ and $6 \otimes 6$ systems are given. In Section 3, we give a detailed description about the construction of bound entangled states in $4 \otimes 4 \otimes 4$ systems. Finally, conclusion and discussion are given in Section 4.

2 Construction of Bound Entangled States in $3k \otimes 3k$ Systems

Suppose $|\psi\rangle$ is a bipartite pure state acting on $\mathcal{H} \otimes \mathcal{H}$, where $\mathcal{H}$ is a complex Hilbert space with $\dim \mathcal{H} = 3k, \quad k \in Z^+$. $\{e_i\}_{i=1}^{3k}$ denotes the orthonormal basis of $\mathcal{H}$, and

$$|\psi\rangle = \sum_{i,j=1}^{3k} a_{ij} e_i \otimes e_j, \quad a_{ij} \in \mathcal{C}, \tag{1}$$

with $\sum\limits_{i,j=1}^{3k} a_{ij} a_{ij}^* = 1$.

Consider antisymmetric $3k \times 3k$ matrix $\widetilde{A}$ having entries given by a_{ij} in (1) such that

$$\widetilde{A} = \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{pmatrix}, \quad \text{where } A = \begin{pmatrix} 0 & b & a \\ -b & 0 & c \\ -a & -c & 0 \end{pmatrix}, \quad a, b, c \in \mathcal{C}.$$

Since $\widetilde{A}$ is an antisymmetric matrix, then we have the following standard form when performing the similarity transformations on $\widetilde{A}$:

$$\widetilde{A}_1 = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_1 \end{pmatrix}, \quad \text{where } A_1 = \begin{pmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

or

$$\widetilde{A}_2 = \begin{pmatrix} A_2 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & A_2 \end{pmatrix}, \quad \text{where } A_2 = \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ -\lambda & 0 & 0 \end{pmatrix}.$$

Let $\lambda = b$ in A_1 and in A_2, with $|b|^2 = \frac{1}{2k}$, t denotes the transposition. Then we have pure states

$$\begin{aligned} |\phi_1\rangle &= (0, b, 0, 0, \cdots, 0, -b, 0, 0, 0, \cdots, 0, 0, b, 0, 0, \cdots, 0, -b, 0, 0, 0, \cdots, 0)^t, \\ |\phi_2\rangle &= (0, 0, b, 0, \cdots, 0, 0, \cdots, 0, -b, 0, 0, 0, \cdots, 0, 0, 0, b, 0 \cdots, 0, 0, \cdots, 0, \\ &\quad -b, 0, 0)^t. \end{aligned}$$

Construct states

$$\sigma=(1-\varepsilon)L_{3k}+\varepsilon\sigma_0,\quad 0<\varepsilon\leq\frac{2}{7k-2}. \tag{2}$$

where we define $\sigma_0=\frac{1}{2}\sigma_1+\frac{1}{2}\sigma_2,\ \sigma_1=|\phi_1><\phi_1|,\ \sigma_2=|\phi_2><\phi_2|$, L_{3k} is a $9k^2\times 9k^2$ matrix having the following elements to be its only nonzero entries:

$$(L_{3k})_{(m-1)\times 3k+m(m-1)\times 3k+m}=\frac{1}{7k^2-4k},\ \ m=1,2,\cdots,3k.$$

$$(L_{3k})_{(3m-l)\times 3k+3n-l'(3m-l)\times 3k+3n-l'}=\frac{1}{7k^2-4k},\ \ m,n=1,2,\cdots,k,\ \ m\neq n.$$

l,l' satisfies $l'=\begin{cases}0,2 & l=1,\\ 1,2 & l=2,\\ 0,1,2 & l=3.\end{cases}$

Next we will represent the matrix σ by partitioned matrix, in the following, P_{mn} denotes the elementary matrix obtained by interchanging the m-th row and the n-th row of the identity matrix $I_{q\times q}$. We set

$$C_1=\frac{1}{4k}I_{3\times 3}+P_{12}C_2,\ C_3=P_{23}C_2,\ C_4=-P_{12}C_2,\ \ \text{where}\ \ C_2=\begin{pmatrix}0 & 0 & 0\\ -\frac{1}{4k} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Let

$$C_{11}=\begin{pmatrix}
C_1 & 0 & \cdots & 0 & C_2 & 0 & \cdots & 0 & C_3 & 0 & \cdots & 0\\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
C_2^\dagger & 0 & \cdots & 0 & C_4 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
C_3^\dagger & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & C_4 & 0 & \cdots & 0\\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0
\end{pmatrix},$$

$$C_{ij}=(I_{3\times 3}\otimes Q_{1i})C_{11}(I_{3\times 3}\otimes Q_{1j}),\ \ i,j=1,\cdots,k,\ \ i\leq j.$$

where $Q_{1l}=P_{3(3l)}P_{2(3l-1)}P_{1(3l-2)},\ \ l=1,\cdots,k$. Thus we can obtain the partitioned matrix σ_0 as follows,

$$\sigma_0=\begin{pmatrix}C_{11} & C_{12} & \cdots & C_{1k}\\ C_{12}^\dagger & C_{22} & \cdots & C_{2k}\\ \vdots & \vdots & \ddots & \vdots\\ C_{k1}^\dagger & C_{k2}^\dagger & \cdots & C_{kk}\end{pmatrix}.$$

Then we set

$$D_1=\begin{pmatrix}\frac{1}{7k^2-4k} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad D_2=\frac{1}{7k^2-4k}I_{3\times 3},\ D_3=P_{12}D_1P_{12},\ \ D_4=P_{13}D_1P_{13},$$
$$D_5=D_1+P_{12}D_1P_{12},\ D_6=D_1+P_{13}D_1P_{13}.$$

Let

$$D_{11}=\begin{pmatrix} D_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_2 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & D_3 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & D_5 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & D_5 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & D_4 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & D_6 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & D_6 \end{pmatrix},$$

$$D_{ii}=(I_{3\times3}\otimes Q_{1i})D_{11}(I_{3\times3}\otimes Q_{1i}),\quad i=1,\cdots,k.$$

Thus the partitioned matrix L_{3k} can be of the following form,

$$L_{3k}=\begin{pmatrix} D_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_{kk} \end{pmatrix}.$$

Obviously, the matrix σ can be written as

$$\sigma=\varepsilon\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1k} \\ C_{12}^{\dagger} & C_{22} & \cdots & C_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ C_{k1}^{\dagger} & C_{k2}^{\dagger} & \cdots & C_{kk} \end{pmatrix}+(1-\varepsilon)\begin{pmatrix} D_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_{kk} \end{pmatrix},\quad 0<\varepsilon\leq\frac{2}{7k-2}.$$

The matrix σ^{T_2} is just the partial transposition acting on the second system of σ. It is easy to see σ^{T_2} is a nonzero Hermitian row diagonally dominant matrix when $0<\varepsilon\leq\frac{2}{7k-2}$, thus σ^{T_2} is positive semidefinite [19]. Hence σ is PPT when $0<\varepsilon\leq\frac{2}{7k-2}$. Next we will show that σ is entangled by using the range criterion.

Assume that the basis is ordered as $e_1\otimes e_1,\cdots,e_1\otimes e_{3k},\cdots,e_{3k}\otimes e_1,\cdots,e_{3k}\otimes e_{3k}$, then any vector belonging to the range of σ (Ran σ)can be represented as

$$\mu=(\mu_1,\cdots,\mu_j,\cdots,\mu_k)^t, \tag{3}$$

where

$$\begin{aligned} \mu_1 =\ & (A_1, B, C, D_1, A_2, A_3, D_2, A_4, A_5, \cdots, D_{k-1}, A_{2k-2}, A_{2k-1}, -B, A_{2k}, 0, A_{2k+1}, D_k, \\ & 0, A_{2k+2}, D_{k+1}, 0, \cdots, A_{3k-1}, D_{2k-2}, 0, -C, 0, A_{3k}, A_{3k+1}, 0, D_{2k-1}, A_{3k+2}, 0, \\ & D_{2k}, \cdots,\ A_{4k-1}, 0, D_{3k-3})^t, \\ \mu_j =\ & (D_{\beta-3k+4}, A_{\alpha-j+2}, A_{\alpha-j+3}, D_{\beta-3k+5}, A_{\alpha-j+4}, A_{\alpha-j+5}, \cdots, D_{\beta-3k+j+2}, A_{\alpha+j-2}, \\ & A_{\alpha+j-1}, A_{\alpha+j}, B, C, D_{\beta-3k+j+3}, A_{\alpha+j+1}, A_{\alpha+j+2}, D_{\beta-3k+j+4}, A_{\alpha+j+3}, A_{\alpha+j+4}, \cdots, \\ & D_{\beta-2k+2}, A_{\alpha-j+2k-1}, A_{\alpha-j+2k}, A_{\alpha-j+2k+1}, D_{\beta-2k+3}, 0, A_{\alpha-j+2k+2}, D_{\beta-2k+4}, 0, \cdots, \\ & A_{\alpha+2k-1}, D_{\beta-2k+j+1}, 0, -B, A_{\alpha+2k}, 0, A_{\alpha+2k+1}, D_{\beta-2k+j+2}, 0, A_{\alpha+2k+2}, D_{\beta-2k+j+3}, 0, \cdots, \\ & A_{\alpha-j+3k}, D_{\beta-k+1}, 0, A_{\alpha-j+3k+1}, 0, D_{\beta-k+2}, A_{\alpha-j+3k+2}, 0, D_{\beta-k+3}, \cdots, A_{\alpha+3k-1}, 0, \\ & D_{\beta-k+j}, -C, 0, A_{\alpha+3k}, A_{\alpha+3k+1}, 0, D_{\beta-k+j+1}, A_{\alpha+3k+2}, 0, D_{\beta-k+j+2}, \cdots, A_{\alpha-j+4k}, 0, D_{\beta})^t, \\ & \alpha=4(j-1)k,\ \beta=3(k-1)j,\quad A_1,\cdots,A_{4k^2-k}, D_1,\cdots,D_{3(k-1)k}, B, C\in\mathcal{C}. \end{aligned}$$

On the other hand, if σ is separable, then any vector belonging to Ran σ also can be of the following form,

$$\mu_{sep} = (a_1, \cdots, a_{3k})^t \otimes (b_1, \cdots, b_{3k})^t \tag{4}$$

Comparing (3) with (4), we have the following equalities:

$$a_{3m-1}b_{3n} = 0, \tag{5}$$

$$a_{3m}b_{3n-1} = 0, \tag{6}$$

$$a_{3m-2}b_{3m-1} = -a_{3m-1}b_{3m-2} = B, \tag{7}$$

$$a_{3m-2}b_{3m} = -a_{3m}b_{3m-2} = C, \tag{8}$$

where $m, n = 1, \cdots, k$.

We consider the following cases.

i) The case of $a_{3m-2} \neq 0,\ a_1 = \cdots = a_{3m-3} = a_{3m-1} = \ldots a_{3k} = 0,\ m = 1, 2, \cdots, k$. From (7),(8), we have

$$\nu_{3m-2,3m-2} = (0, \cdots, 0, b_{3m-2}, 0, \cdots, 0)^t . \nu_{3m-2,3n-2} = (0, \cdots, 0, b_{3n-2}, 0, \cdots, 0)^t,$$
$$\nu_{3m-2,3n-1} = (0, \cdots, 0, b_{3n-1}, 0, \cdots, 0)^t,$$

$$\nu_{3m-2,3n} = (0, \cdots, 0, b_{3n}, 0, \cdots, 0,)^t,$$

$$n = 1, 2, \cdots, (m-1), (m+1), \cdots, k.$$

ii) The case of $a_{3m-1} \neq 0,\ a_1 = \cdots = a_{3m-2} = a_{3m} = \ldots a_{3k} = 0,\ m = 1, 2, \cdots, k$. From (5),(7), we have

$$\nu_{3m-1,m} = (0, \cdots, 0, b_{3m-1}, 0, \cdots, 0)^t.$$

$$\nu_{3m-1,n} = (0, \cdots, 0, b_{3n-2}, 0, \cdots, 0)^t, \nu_{3m-1,n+k} = (0, \cdots, 0, b_{3n-1}, 0, \cdots, 0)^t$$

$$n = 1, 2, \cdots, (m-1), (m+1), \cdots, k.$$

iii) The case of $a_{3m} \neq 0,\ a_1 = \cdots = a_{3m-1} = a_{3m+1} = \ldots a_{3k} = 0,\ m = 1, 2, \cdots, k$. From (6), (8), we have

$$\nu_{3m,m} = (0, \cdots, 0, b_{3m}, 0, \cdots, 0)^t.$$

$$\nu_{3m,n} = (0, \cdots, 0, b_{3n-2}, 0, \cdots, 0)^t, \nu_{3m,n+k} = (0, \cdots, 0, b_{3n}, 0, \cdots, 0)^t$$
$$n = 1, 2, \cdots, (m-1), (m+1), \cdots, k.$$

iv) The case of $a_2 = a_5 = \cdots = a_{3k-1} = 0, a_1a_3a_4 \cdots a_{3k-3}a_{3k-2}a_{3k} \neq 0$. From (6),(8), we have $\phi_k \otimes \psi_k = (a_1, 0, a_3, \cdots, a_{3k-2}, 0, a_{3k})^t \otimes (b_1, 0, b_3, \cdots, b_{3k-2}, 0, b_{3k})^t$,

v) The case of $a_3 = a_6 = \cdots = a_{3k} = 0, a_1a_2a_4a_5 \cdots a_{3k-2}a_{3k-1} \neq 0$. From (5),(7), we have $\phi'_k \otimes \psi'_k = (a_1, a_2, 0, \cdots, a_{3k-2}, a_{3k-1}, 0)^t \otimes (b_1, b_2, 0, \cdots, b_{3k-2}, b_{3k-1}, 0)^t$.

Let $\mu_i = (0, \cdots, 0, a_i, 0, \cdots, 0)^t$. Thus we have linearly independent vectors $\mu_{3m-2} \otimes \nu_{3m-2,j}, j = 1, 2, \cdots, 3m-2, 3m+1, \cdots, 3k$. $\mu_{3m-1} \otimes \nu_{3m-1,j}, \mu_{3m} \otimes \nu_{3m,j}, j = 1, \cdots, k+m-1, k+m+1, \cdots, 2k$. $\phi_k \otimes \psi_k, \phi'_k \otimes \psi'_k$ spanning Ran σ.

Since all of the vectors $\mu_{3m-2} \otimes \nu^\star_{3m-2,j},\ j = 1, \cdots, 3m-2, 3m+1, \cdots, 3k.\ \mu_{3m-1} \otimes \nu^\star_{3m-1,j},\ \mu_{3m} \otimes \nu^\star_{3m,j},\ j = 1, \cdots, k+m-1, k+m+1, ..., 2k.\ \phi_k \otimes \psi^\star_k,\ \phi'_k \otimes (\psi'_k)^\star$ are linearly independent with the vector $\mu_0 = (1, 0, \cdots, 0)^t \otimes (0, 1, 0, \cdots, 0)^t$, which belongs just to Ran σ^{T_2}. Then we can get that σ is entangled by the range criterion.

Therefore, for any $0 < \varepsilon \leq \frac{2}{7k-2}$, σ is a bound entangled state.

Example 1 For the case of $k = 1$, we consider the matrix ρ_0 as follows,

$$\rho_0 = \begin{pmatrix} \frac{1-\varepsilon}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\varepsilon}{4} & 0 & -\frac{\varepsilon}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\varepsilon}{4} & 0 & 0 & 0 & -\frac{\varepsilon}{4} & 0 & 0 \\ 0 & -\frac{\varepsilon}{4} & 0 & \frac{\varepsilon}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-\varepsilon}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\varepsilon}{4} & 0 & 0 & 0 & \frac{\varepsilon}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-\varepsilon}{3} \end{pmatrix}.$$

Since $\rho_0^{T_2}$ is positive semidefinite when $0 < \varepsilon \leq \frac{2}{5}$[19], and it is easy to show ρ_0 is entangled by the range criterion, then ρ_0 is bound entangled.

Example 2 For the case of $k = 2$, we consider the matrix ρ as follows,

$$\rho = \varepsilon \begin{pmatrix} C_1 & 0 & C_2 & 0 & C_3 & 0 & 0 & C_1 & 0 & C_2 & 0 & C_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_2^\dagger & 0 & C_4 & 0 & 0 & 0 & 0 & C_2^\dagger & 0 & C_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_3^\dagger & 0 & 0 & 0 & C_4 & 0 & 0 & C_3^\dagger & 0 & 0 & 0 & C_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_1 & 0 & C_2 & 0 & C_3 & 0 & 0 & C_1 & 0 & C_2 & 0 & C_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_2^\dagger & 0 & C_4 & 0 & 0 & 0 & 0 & C_2^\dagger & 0 & C_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_3^\dagger & 0 & 0 & 0 & C_4 & 0 & 0 & C_3^\dagger & 0 & 0 & 0 & C_4 \end{pmatrix} + (1-\varepsilon) \begin{pmatrix} D_{11} & 0 \\ 0 & D_{22} \end{pmatrix},$$

where

$$D_{11} = \begin{pmatrix} D_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & D_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & D_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & D_6 \end{pmatrix}, \quad D_{22} = \begin{pmatrix} D_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & D_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & D_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & D_4 \end{pmatrix}.$$

According to [19], we can get that ρ^{T_2} is positive semidefinite when $0 < \varepsilon \le 1/6$. Next we will show ρ is entangled by the range criterion. It is easy to obtain the linearly independent vectors $\mu_1 \otimes \nu_{1j}, j = 1,4,5,6$. $\mu_2 \otimes \nu_{2j}, \mu_3 \otimes \nu_{3j}, j = 1,2,4$. $\mu_4 \otimes \nu_{4j}, j = 1,\cdots,4$. $\mu_5 \otimes \nu_{5j}, \mu_6 \otimes \nu_{6j}, j = 1,2,3$. $\phi_2 \otimes \psi_2,\ \phi_2' \otimes \psi_2'$ spanning Ran ρ, where $\mu_i = (0,\cdots,0,a_i,0,\cdots,0)$.

$$\begin{aligned}
&\nu_{11} = (b_1,0,0,0,0,0)^t, \quad \nu_{14} = (0,0,0,b_4,0,0)^t, \quad \nu_{15} = (0,0,0,0,b_5,0)^t,\\
&\nu_{16} = (0,0,0,0,0,b_6)^t, \quad \nu_{21} = (0,b_2,0,0,0,0)^t,\\
&\nu_{22} = \nu_{14}, \quad \nu_{24} = \nu_{15}, \quad \nu_{31} = (0,0,b_3,0,0,0)^t, \quad \nu_{32} = \nu_{14},\\
&\nu_{34} = \nu_{16}, \quad \nu_{41} = \nu_{11}, \quad \nu_{42} = \nu_{21}, \quad \nu_{43} = \nu_{31}, \quad \nu_{44} = \nu_{14}, \quad \nu_{51} = \nu_{11},\\
&\nu_{52} = \nu_{15}, \quad \nu_{53} = \nu_{21}, \quad \nu_{61} = \nu_{11}, \quad \nu_{62} = \nu_{16}, \quad \nu_{63} = \nu_{31}.
\end{aligned}$$

Since all of the vectors $\mu_1 \otimes \nu_{1j}^\star,\ j = 1,4,5,6.\ \mu_2 \otimes \nu_{2j}^\star,\ \mu_3 \otimes \nu_{3j}^\star,\ j = 1,2,4.\ \mu_4 \otimes \nu_{4j}^\star,\ j = 1,\cdots,4.\ \mu_5 \otimes \nu_{5j}^\star,\ \mu_6 \otimes \nu_{6j}^\star,\ j = 1,2,3.\ \phi_2 \otimes \psi_2^\star,\ \phi_2' \otimes (\psi_2')^\star$ are linearly independent with the vector $\mu_0 = (1,0,0,0,0,0)^t \otimes (0,1,0,0,0,0)^t$, which belongs just to Ranρ^{T_2}. Then we can get that ρ is entangled.

Therefore, for any $0 < \varepsilon \le 1/6,\ \rho$ is bound entangled.

3 Construction of Bound Entangled States in 4 ⊗ 4 ⊗ 4 Systems

Suppose $|\psi\rangle$ is a tripartite pure state acting on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, where $\mathcal{H}$ is a complex Hilbert space with $\dim \mathcal{H} = 4$. $\{e_i\}_{i=1}^4$ denotes the orthonormal basis of $\mathcal{H}$, and

$$|\psi\rangle = \sum_{h,i,j=1}^{4} a_{hij} e_h \otimes e_i \otimes e_j, \quad a_{hij} \in \mathcal{C}, \tag{9}$$

with $\sum\limits_{h,i,j=1}^{4} a_{hij} a_{hij}^* = 1$.

Consider antisymmetric matrix B having entries given by a_{hij} in (9) such that

$$B = \begin{pmatrix}
0 & b & a & -c & 0 & 0 & 0 & 0\\
-b & 0 & c & d & 0 & 0 & 0 & 0\\
-a & -c & 0 & -e & 0 & 0 & 0 & 0\\
c & -d & e & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & b & a & -c\\
0 & 0 & 0 & 0 & -b & 0 & c & d\\
0 & 0 & 0 & 0 & -a & -c & 0 & -e\\
0 & 0 & 0 & 0 & c & -d & e & 0
\end{pmatrix}, \tag{10}$$

where $a, b, c, d, e \in \mathcal{C}$.

In a similar way, we have the following equivalent standard form:

$$B_1 = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_2 & 0 \end{pmatrix}$$

or

$$B_2 = \begin{pmatrix} 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & 0 & 0 & -\lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 \end{pmatrix}.$$

Let $\lambda_1 = b, \lambda_2 = -c$ in B_1, with $|b|^2 + |c|^2 = \frac{1}{4}$, we have the pure state

$$|\varphi_{+b}\rangle = (0, b, 0, \cdots, 0, -b, 0, \cdots, 0, -c, 0, \cdots, 0, c, 0, \cdots, 0, b, 0, \cdots, 0, -b, 0, \cdots, 0, -c, 0, \cdots, 0, c, 0)^t.$$

Let $\lambda_1 = -b, \lambda_2 = -c$ in B_1, we have the pure state

$$|\varphi_{-b}\rangle = (0, -b, 0, \cdots, 0, b, 0, \cdots, 0, -c, 0, \cdots, 0, c, 0, \cdots, 0, -b, 0, \cdots, 0, b, 0, \cdots, 0, -c, 0, \cdots, 0, c, 0)^t.$$

Let $\lambda_1 = a, \lambda_2 = -d$ in B_2, with $|a|^2 + |d|^2 = \frac{1}{4}$, we have the pure state

$$|\varphi_{+a}\rangle = (0, 0, a, 0, \cdots, 0, -d, 0, 0, 0, 0, -a, 0, \cdots, 0, d, 0, \cdots, 0, a, 0, \cdots, 0, -d, 0, 0, 0, 0, -a, 0, \cdots, 0, d, 0, 0)^t.$$

Let $\lambda_1 = -a, \lambda_2 = -d$ in B_2, we have the pure state

$$|\varphi_{-a}\rangle = (0, 0, -a, 0, \cdots, 0, -d, 0, 0, 0, 0, a, 0, \cdots, 0, d, 0, \cdots, 0, -a, 0, \cdots, 0, -d, 0, 0, 0, 0, -a, 0, \cdots, 0, d, 0, 0)^t.$$

Construct states

$$\varrho = (1 - \varepsilon)L_8 + \varepsilon\varrho_0, \quad 0 < \varepsilon \leq \frac{1}{17}. \tag{11}$$

where ϱ_0 is defined as $\varrho_0 = \frac{1}{2}\varrho_a + \frac{1}{2}\varrho_b$, $\varrho_a = \frac{1}{2}|\varphi_{+a}\rangle\langle\varphi_{+a}| + \frac{1}{2}|\varphi_{-a}\rangle\langle\varphi_{-a}|$, $\varrho_b = \frac{1}{2}|\varphi_{+b}\rangle\langle\varphi_{+b}| + \frac{1}{2}|\varphi_{-b}\rangle\langle\varphi_{-b}|$. L_8 is a 64×64 matrix having the following elements to be its only nonzero entries:

$$\begin{aligned}
(L_8)_{1,1} = (L_8)_{5,5} &= (L_8)_{6,6} = (L_8)_{7,7} = \frac{1}{32};\\
(L_8)_{10,10} = (L_8)_{13,13} &= (L_8)_{14,14} = (L_8)_{16,16} = \frac{1}{32};\\
(L_8)_{19,19} = (L_8)_{21,21} &= (L_8)_{23,23} = (L_8)_{24,24} = \frac{1}{32};\\
(L_8)_{28,28} = (L_8)_{30,30} &= (L_8)_{31,31} = (L_8)_{32,32} = \frac{1}{32};\\
(L_8)_{33,33} = (L_8)_{34,34} &= (L_8)_{35,35} = (L_8)_{37,37} = \frac{1}{32};\\
(L_8)_{41,41} = (L_8)_{42,42} &= (L_8)_{44,44} = (L_8)_{46,46} = \frac{1}{32};\\
(L_8)_{49,49} = (L_8)_{51,51} &= (L_8)_{52,52} = (L_8)_{55,55} = \frac{1}{32};\\
(L_8)_{58,58} = (L_8)_{59,59} &= (L_8)_{60,60} = (L_8)_{64,64} = \frac{1}{32}.
\end{aligned}$$

Since there are three different bipartite splits of the systems: A-(BC), B-(AC), (AB)-C, then below we will discuss the separability of ϱ in the three cases respectively.

I) The case of A-(BC). To consider the matrix ϱ by partitioned matrix, we set

$$E_1 = -(E_2P_{12} + E_3P_{13}), \quad E_2 = \begin{pmatrix} 0 & 0\,0\,0 \\ -\frac{|b|^2}{2} & 0\,0\,0 \\ 0 & 0\,0\,0 \\ 0 & 0\,0\,0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0\,0\,0 \\ 0 & 0\,0\,0 \\ -\frac{|a|^2}{2} & 0\,0\,0 \\ 0 & 0\,0\,0 \end{pmatrix},$$

$$E_4 = \begin{pmatrix} 0 & 0 & 0\,0 \\ 0 & 0 & 0\,0 \\ 0 & 0 & 0\,0 \\ 0 & -\frac{|d|^2}{2} & 0\,0 \end{pmatrix}, \quad E_5 = -(P_{13}E_3 + E_6P_{34}), \quad E_6 = \begin{pmatrix} 0\,0 & 0 & 0 \\ 0\,0 & 0 & 0 \\ 0\,0 & 0 & 0 \\ 0\,0 & -\frac{|c|^2}{2} & 0 \end{pmatrix}.$$

Let

$$E_{ij} = (I_{4\times 4} \otimes Q'_{1i})E_{11}(I_{4\times 4} \otimes Q'_{1j}), \quad i, j = 1, 2.\ i \le j.$$

where

$$E_{11} = \begin{pmatrix}
E_1 & 0 & E_2 & 0 & E_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
E_2^\dagger & 0 & E_1 & 0 & 0 & 0 & E_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
E_3^\dagger & 0 & 0 & 0 & E_5 & 0 & E_6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & E_4^\dagger & 0 & E_6^\dagger & 0 & E_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$Q'_{1l} = P_{4(4l)}P_{3(4l-1)}P_{2(4l-2)}P_{1(4l-3)}, \quad l = 1, 2.$$

thus the partitioned matrix ϱ_0 can be of the following form

$$\varrho_0 = \begin{pmatrix} E_{11} & E_{12} \\ E_{12}^{\dagger} & E_{22} \end{pmatrix}.$$

Then we set

$$F_2 = \frac{1}{32} I_{4\times 4} - P_{14} F_1 P_{14}, \quad F_3 = P_{12} F_1 P_{12}, \quad F_4 = \frac{1}{32} I_{4\times 4} - P_{13} F_1 P_{13},$$

$$F_5 = P_{13} F_1 P_{13}, \quad F_6 = \frac{1}{32} I_{4\times 4} - P_{12} F_1 P_{12}, \quad F_7 = P_{14} F_1 P_{14}, \quad F_8 = \frac{1}{32} I_{4\times 4} - F_1,$$

where $F_1 = \begin{pmatrix} \frac{1}{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Let

$$F_{ii} = (I_{4\times 4} \otimes Q'_{1i}) F_{11} (I_{4\times 4} \otimes Q'_{1i}), \ i = 1, 2.$$

where

$$F_{11} = \begin{pmatrix} F_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & F_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & F_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & F_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & F_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & F_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & F_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_8 \end{pmatrix},$$

thus the partitioned matrix L_8 is as follows,

$$L_8 = \begin{pmatrix} F_{11} & 0 \\ 0 & F_{22} \end{pmatrix}.$$

It is easy to see the matrix ϱ can be written as

$$\varrho = \varepsilon \begin{pmatrix} E_{11} & E_{12} \\ E_{12}^{\dagger} & E_{22} \end{pmatrix} + (1-\varepsilon) \begin{pmatrix} F_{11} & 0 \\ 0 & F_{22} \end{pmatrix}, \quad 0 < \varepsilon \le \frac{1}{17}.$$

Consider the matrix ϱ^{T_A} which denotes the partial transposition acting on A system of ϱ, we have ϱ^{T_A} is positive semidefinite when $0 < \varepsilon \le \frac{1}{9}$, then it is also positive semidefinite when $0 < \varepsilon \le \frac{1}{17}$. Thus ϱ is PPT when $0 < \varepsilon \le \frac{1}{17}$.

Next we will show that ϱ is entangled.

Assume that the basis is ordered as $e_1 \otimes e_1 \otimes e_1, \ e_1 \otimes e_1 \otimes e_2, \ e_1 \otimes e_1 \otimes e_3, \ e_1 \otimes e_2 \otimes e_1, \ e_1 \otimes e_2 \otimes e_2, \ e_1 \otimes e_2 \otimes e_3, \ \cdots, \ e_3 \otimes e_3 \otimes e_3$, then any vector which belongs to Ran ϱ can be presented as

$$\begin{aligned} \mu = \ &(A_1, B, C, 0, A_2, A_3, A_4, 0, -B, A_5, 0, D, A_6, A_7, 0, A_8, -C, 0, A_9, E, \\ &A_{10}, 0, A_{11}, A_{12}, 0, -D, -E, A_{13}, 0, A_{14}, A_{15}, A_{16}, A_{17}, A_{18}, A_{19}, 0, \\ &A_{20}, B, C, 0, A_{21}, A_{22}, 0, A_{23}, -B, A_{24}, 0, D, A_{25}, 0, A_{26}, A_{27}, -C, 0, \\ &A_{28}, E, 0, A_{29}, A_{30}, A_{31}, 0, -D, -E, A_{32})^t, \end{aligned} \tag{12}$$

where $A_1, A_2, \cdots, A_{32}, B, C, D, E \in \mathcal{C}$. On the other hand, if μ is separable, it also can be of the following form

$$\mu_{sep} = (r_1, r_2, r_3, r_4)^t \otimes (s_1, \cdots, s_{16})^t, \tag{13}$$

where $r_1, r_2, r_3, r_4, s_1, \cdots, s_{16} \in \mathcal{C}$.

Comparing (12) with (13), we have the following equalities:

$$r_1 s_2 = -r_1 s_9 = r_3 s_6 = -r_3 s_{13} = B, \tag{14}$$

$$r_1 s_3 = -r_2 s_1 = r_3 s_7 = -r_4 s_5 = C, \tag{15}$$

$$r_1 s_{12} = -r_2 s_{10} = r_3 s_{16} = -r_4 s_{14} = D, \tag{16}$$

$$r_2 s_4 = -r_2 s_{11} = r_4 s_8 = -r_4 s_{15} = E, \tag{17}$$

$$r_1 s_4 = r_1 s_8 = r_1 s_{11} = r_1 s_{15} = 0, \tag{18}$$

$$r_2 s_2 = r_2 s_6 = r_2 s_9 = r_2 s_{13} = 0, \tag{19}$$

$$r_3 s_4 = r_3 s_8 = r_3 s_{11} = r_3 s_{15} = 0, \tag{20}$$

$$r_4 s_2 = r_4 s_6 = r_4 s_9 = r_4 s_{13} = 0. \tag{21}$$

In a similar way, we consider the following cases:

1) $r_1 \neq 0, r_2 = r_3 = r_4 = 0$. From (14-16),(18), we obtain the vector

$$(r_1, 0, 0, 0)^t \otimes (s_1, 0, 0, 0, s_5, s_6, s_7, 0, 0, s_{10}, 0, 0, s_{13}, s_{14}, 0, s_{16})^t, \tag{22}$$

2) $r_2 \neq 0, r_1 = r_3 = r_4 = 0$. From (15-17),(19), we obtain the vector

$$(0, r_2, 0, 0)^t \otimes (0, 0, s_3, 0, s_5, 0, s_7, s_8, 0, 0, 0, s_{12}, 0, s_{14}, s_{15}, s_{16})^t, \tag{23}$$

3) $r_3 \neq 0, r_1 = r_2 = r_4 = 0$. From (14-16),(20), we obtain the vector

$$(0, 0, r_3, 0)^t \otimes (s_1, s_2, s_3, 0, s_5, 0, 0, 0, s_9, s_{10}, 0, s_{12}, 0, s_{14}, 0, 0)^t, \tag{24}$$

4) $r_4 \neq 0, r_1 = r_2 = r_3 = 0$. From (15-17),(21), we have the vector

$$(0, 0, 0, r_4)^t \otimes (s_1, 0, s_3, s_4, 0, 0, s_7, 0, 0, s_{10}, s_{11}, s_{12}, 0, 0, 0, s_{16})^t, \tag{25}$$

5) $r_1 r_2 \neq 0, r_3 = r_4 = 0$. From (15),(16),(18),(19), we obtain the vector

$$(r_1, r_2, 0, 0)^t \otimes (0, 0, 0, 0, s_5, 0, s_7, 0, 0, 0, 0, 0, 0, s_{14}, 0, s_{16})^t, \tag{26}$$

6) $r_1 r_3 \neq 0, r_2 = r_4 = 0$. From(15),(16),(18),(20), we obtain the vector

$$(r_1, 0, r_3, 0)^t \otimes (s_1, s_2, 0, 0, s_5, s_6, 0, 0, s_9, s_{10}, 0, 0, s_{13}, s_{14}, 0, 0)^t, \tag{27}$$

7) $r_1 r_4 \neq 0, r_2 = r_3 = 0$. From (15),(16),(18),(21), we obtain the vector

$$(r_1, 0, 0, r_4)^t \otimes (s_1, 0, 0, 0, 0, 0, s_7, 0, 0, s_{10}, 0, 0, 0, 0, 0, s_{16})^t, \tag{28}$$

8) $r_2 r_3 \neq 0, r_1 = r_4 = 0$. From (15),(16),(19),(20), we obtain the vector

$$(0, r_2, r_3, 0)^t \otimes (0, 0, s_3, 0, s_5, 0, 0, 0, 0, 0, 0, s_{12}, 0, s_{14}, 0, 0)^t, \tag{29}$$

9) $r_2 r_4 \neq 0, r_1 = r_3 = 0$. From (15),(16),(19), we obtain the vector

$$(0, r_2, 0, r_4)^t \otimes (0, 0, s_3, s_4, 0, 0, s_7, s_8, 0, 0, s_{11}, s_{12}, 0, 0, s_{15}, s_{16})^t, \tag{30}$$

10) $r_3 r_4 \neq 0, r_1 = r_2 = 0$. From (15),(16),(20),(21), we obtain the vector

$$(0, 0, r_3, r_4)^t \otimes (s_1, 0, s_3, 0, 0, 0, 0, 0, 0, s_{10}, 0, s_{12}, 0, 0, 0, 0)^t, \tag{31}$$

11) $r_1 r_2 r_3 \neq 0, r_4 = 0$. From (15),(16),(18),(19), we obtain the vector

$$(r_1, r_2, r_3, 0)^t \otimes (0, 0, 0, 0, s_5, 0, 0, 0, 0, 0, 0, 0, 0, s_{14}, 0, 0)^t, \tag{32}$$

12) $r_1 r_2 r_4 \neq 0, r_3 = 0$. From (15),(16),(18),(19), we obtain the vector

$$(r_1, r_2, 0, r_4)^t \otimes (0, 0, 0, 0, 0, 0, s_7, 0, 0, 0, 0, 0, 0, 0, 0, s_{16})^t, \tag{33}$$

13) $r_1r_3r_4 \neq 0, r_2 = 0$. From (15),(16),(18),(19), we obtain the vector

$$(r_1, 0, r_3, r_4)^t \otimes (s_1, 0, 0, 0, 0, 0, 0, 0, 0, s_{10}, 0, 0, 0, 0, 0, 0)^t, \quad (34)$$

14) $r_2r_3r_4 \neq 0, r_1 = 0$. From (15),(16),(19),(20), we obtain the vector

$$(0, r_2, r_3, r_4)^t \otimes (0, 0, s_3, 0, 0, 0, 0, 0, 0, 0, 0, s_{12}, 0, 0, 0, 0)^t, \quad (35)$$

15) $r_1r_2r_3r_4 \neq 0$. From (15),(16),(18),(19), we obtain the vector

$$(r_1, r_2, r_3, r_4)^t \otimes (s_1, 0, s_3, 0, 0, s_6, 0, s_8, s_9, 0, s_{11}, 0, s_{13}, 0, s_{15}, 0)^t. \quad (36)$$

Obviously, the vectors (22-36) are linearly independent and span Ran ϱ. Performing the partial complex conjugations with respect to A system of the above vectors, we can get that the resulting vectors can't span the range of ϱ^{T_A}, since the vector $(1, 0, 0, 0)^t \otimes (0, 0, 1, 0, \cdots, 0)^t$ belonging to the range of ϱ^{T_A} is also linearly independent with the resulting vectors. Hence, ϱ is entangled by the range criterion.

Therefore, for any $0 < \varepsilon \leq \frac{1}{17}$, ϱ is bound entangled.

II) The case of B-(AC). Similar to I), we can obtain that ϱ is PPT when $0 < \varepsilon \leq \frac{1}{17}$, and show that ϱ is entangled by the range criterion. Therefore, for any $0 < \varepsilon \leq \frac{1}{17}$, ϱ is bound entangled.

III) The case of (AB)-C. Consider the matrix ϱ^{T_C} that achieved by performing the partial transposition on C system of ϱ , it's easy to obtain that ϱ^{T_C} is positive semidefinite when $0 < \varepsilon \leq \frac{1}{17}$. Thus ϱ is PPT when $0 < \varepsilon \leq \frac{1}{17}$. In a similar way, we can also obtain the following linearly independent vectors that span Ran ϱ:

$$\begin{aligned}
&(c_1, c_2, 0, c_4, 0, c_6, 0, 0, c_9, c_{10}, c_{11}, 0, c_{13}, 0, 0, 0)^t \otimes (d_1, 0, 0, 0)^t,\\
&(0, c_2, c_3, c_4, 0, 0, 0, c_8, c_9, 0, c_{11}, c_{12}, 0, 0, s_{15}, 0)^t \otimes (0, d_2, 0, 0)^t,\\
&(0, c_2, 0, 0, c_5, c_6, 0, c_8, c_9, 0, 0, 0, c_{13}, c_{14}, c_{15}, 0)^t \otimes (0, 0, d_3, 0)^t,\\
&(0, 0, 0, c_4, 0, c_6, c_7, c_8, 0, 0, c_{11}, 0, c_{13}, 0, c_{15}, c_{16})^t \otimes (0, 0, 0, d_4)^t,\\
&(c_1, c_2, c_3, c_4, 0, 0, 0, 0, c_9, c_{10}, c_{11}, c_{12}, 0, 0, 0, 0)^t \otimes (d_1, d_2, 0, 0)^t,\\
&(c_1, c_2, 0, 0, c_5, c_6, 0, 0, c_9, c_{10}, 0, 0, c_{13}, c_{14}, 0, 0)^t \otimes (d_1, 0, d_3, 0)^t,\\
&(0, 0, 0, c_4, 0, c_6, 0, 0, 0, 0, c_{11}, 0, c_{13}, 0, 0, 0)^t \otimes (d_1, 0, 0, d_4)^t,\\
&(0, c_2, 0, 0, 0, 0, 0, c_8, c_9, 0, 0, 0, 0, 0, c_{15}, 0)^t \otimes (0, d_2, d_3, 0)^t,\\
&(0, 0, c_3, c_4, 0, 0, c_7, c_8, 0, 0, c_{11}, c_{12}, 0, 0, c_{15}, c_{16})^t \otimes (0, d_2, 0, d_4)^t,\\
&(0, 0, 0, 0, c_5, c_6, c_7, c_8, 0, 0, 0, 0, c_{13}, c_{14}, c_{15}, c_{16})^t \otimes (0, 0, d_3, d_4)^t,\\
&(c_1, c_2, 0, 0, 0, 0, 0, 0, c_9, c_{10}, 0, 0, 0, 0, 0, 0)^t \otimes (d_1, d_2, d_3, 0)^t,\\
&(0, 0, c_3, c_4, 0, 0, 0, 0, 0, 0, c_{11}, c_{12}, 0, 0, 0, 0)^t \otimes (d_1, d_2, 0, d_4)^t,\\
&(0, 0, 0, 0, c_5, c_6, 0, 0, 0, 0, 0, 0, c_{13}, c_{14}, 0, 0)^t \otimes (d_1, 0, d_3, d_4)^t,\\
&(0, 0, 0, 0, 0, 0, c_7, c_8, 0, 0, 0, 0, 0, 0, c_{15}, c_{16})^t \otimes (0, d_2, d_3, d_4)^t. \quad (37)
\end{aligned}$$

where $c_1, \cdots, c_{16}, d_1, \cdots, d_4 \in \mathcal{C}$. Since the vector $(1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0,)^t \otimes (0, 0, 1, 0)^t$ belonging to Ran ϱ^{T_C} is linearly independent with the partial complex conjugations of these vectors, then ϱ is entangled.

Therefore, for any $0 < \varepsilon \leq \frac{1}{17}$, ϱ is bound entangled.

4 Conclusion and Discussion

We have constructed a class of bound entangled states in $3k \otimes 3k$ quantum systems and given two examples of the bound entangled states in $3 \otimes 3$ and $6 \otimes 6$ quantum systems. We have also constructed a class of bound entangled states in $4 \otimes 4 \otimes 4$ quantum systems, and such tripartite bound entangled states may be constructed in $k \otimes k \otimes k$ quantum systems if k^3 can be a square number, since in this case we can define a $\sqrt{k^3} \times \sqrt{k^3}$ matrix $\widetilde{B}$ similar to the matrix B at the beginning of Section 3. We hope our results will be helpful for the future research on the construction of bound entangled states in multipartite quantum systems.

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