Dynamic Order Algebras as an Axiomatization of Modal and Tense Logics

Ivan Chajda · Jan Paseka

Abstract The aim of the paper is to introduce and describe tense operators in every propositional logic which is axiomatized by means of an algebra whose underlying structure is a bounded poset or even a lattice. We introduce the operators G, H, P and F without regard what propositional connectives the logic includes. For this we use the axiomatization of universal quantifiers as a starting point and we modify these axioms for our reasons. At first, we show that the operators can be recognized as modal operators and we study the pairs (P, G) as the so-called dynamic order pairs. Further, we get constructions of these operators in the corresponding algebra provided a time frame is given. Moreover, we solve the problem of finding a time frame in the case when the tense operators are given. In particular, any tense algebra is representable in its Dedekind-MacNeille completion. Our approach is fully general, we do not relay on the logic under consideration and hence it is applicable in all the up to now known cases.

Keywords Propositional logic · Modal logic · Bounded poset · Tense logic · Tense operators · Dynamic order algebra

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1 Introduction

It is known that propositional logics, both classic or non-classic, does not incorporate the dimension of time. To obtain a tense logic we enrich the given propositional logic by new unary operators which are usually denoted by G, H, F and P, see e.g. [3, 9, 10] and [5].

It is worth noticing that the operators G and H can be considered as a certain kind of modal operators which were already studied for intuitionistic calculus by Wijesekera [16], in the de Morgan framework by Cattaneo, Ciucci and Dubois [4] and in a general setting by Ewald [12]. For the logic of quantum mechanics (see e.g. [11] for details of the so-called quantum structures), the underlying algebraic structure is e.g. an orthomodular lattice or the so-called effect algebra (see [11, 13]) and the corresponding tense logic was treated in [6, 7, 15], in a bit more general setting also in [1].

The aforementioned operators G, H, F and P are usually called *tense operators*. They are in certain sense quantifiers which quantify over the time dimension of the logic under consideration. The semantical interpretation of these tense operators G and H is as follows. Consider a pair $(T, \leq)$ where T is a non-void set and $\leq$ is a partial order on T. Let $x \in T$ and $f(x)$ be a formula of a given logical calculus. By the domain of truth-values we will consider a bounded poset, where the value 1 means "true" and the value 0 means "false". The values sharply between 0 and 1 express to which extent the proposition under consideration is true.

We say that $G(f(t))$ *is valid with a value w* if for any $s \geq t$ the formula $f(s)$ is valid with a value at least w. Analogously, $H(f(t))$ *is valid with a value w* if for any $s \leq t$ the formula $f(s)$ is valid with a value at least w. Thus the unary operators G and H constitute an algebraic counterpart of the tense operations "it is always going to be the case that" and "it has always been the case that", respectively. These tense operators were firstly introduced as operators on Boolean algebras (see [3] for a overview).

Analogously, the operators F and P can be considered in a certain sense as existential quantifiers "it will at some time be the case that" and "it has at some time been the case that". If the logical calculus under consideration has a connective negation (denoted by $'$) and if it satisfies the double negation law (i.e. $x'' = x$ for all its propositions x) then we can define P and F by means of G and H as follows:

$$P(x) = H\left(x'\right)' \quad \text{and} \quad F(x) = G\left(x'\right)'.$$

However, this is not possible e.g. in intuitionistic logic (see e.g. [2, 12]) where $x''' = x'$ but x'' need not be equal to x.

Since the tense operators G and H can be considered as universal quantifiers in the aforementioned case, we can try to axiomatize them by means of some of the known axiomatization for universal quantifiers which is as follows:

(U1) $\forall(1) = 1$,
(U2) $\forall(x) \leq x$,
(U3) $\forall(x \wedge y) = \forall(x) \wedge \forall(y)$,
(U4) $\forall(\forall(x)) = \forall(x)$,
(U5) $\forall\left((\forall(x))'\right) = (\forall(x))'$.

Let us mention that this system of axioms implicitly assumes that the logical calculus in question is ordered (and hence the relation $\leq$ in (U2) is a partial ordering) with the greatest element 1 and it can be considered as a meet-semilattice (where $\wedge$ in (U3) is the infimum). However, when studying the logic of quantum mechanics via effect algebras, the underlying algebra is an ordered set with the greatest element 1 but it need not have a semi-lattice or lattice structure, see [11, 13]. Hence, when tense operators were introduced for the

corresponding effect algebras in [7] and [15], a bit more modified axioms were used. More-over, the axiom (U5) yields that $\forall(0) = 0$ thus we inserted it directly in the first axiom. Due to the fact that the underlying algebra need not be a semilattice, the axiom (U3) can be re-placed by a weaker form that ensures monotonicity (which is an easy conclusion of (U3)). i.e.,

$$x \leq y \quad \text{implies} \quad \forall(x) \leq \forall(y).$$

It is a remarkable fact that the above mentioned system of axioms (U1)-(U5) does not con-sider other logical connectives except negation and the connective $\wedge$ (which need not be a conjunction, see [1, 9, 10] or [11] for MV-algebras and their generalizations).

This motivates us to neglect logical connectives of the logic in question. Hence, we will consider an arbitrary propositional logic and we will form a quotient set of classes of equiva-lent propositions. Then we will assume that these classes (usually considered as propositions) form a bounded ordered set $(A; \leq, 0, 1)$ where 1 denotes the tautology and 0 denotes the contradiction. This quadruple will be called an *algebra of the propositional logic*. In what follows, we will work only with this algebra of the propositional logic $\mathcal{A} = (A; \leq, 0, 1)$.

The paper is organized as follows. After introducing several necessary algebraic con-cepts, we introduce tense operators in an arbitrary logic without regard what logical connectives are considered. In Section 3 we get a simple construction of tense operators which uses lattice theoretical properties of the underlying ordered set. In the case when the underlying ordered set is a not a complete lattice, we show how to apply the lattice comple-tion for this construction. In Sections 4 and 5 we solve the problem of a representation of dynamic order algebras and tense algebras. This means that we get a procedure how to con-struct a corresponding time frame to be in accordance with the construction from Section 3. In particular, in Section 6 we show that any dynamic order or tense algebra is representable in its Dedekind-MacNeille completion.

Let us recall several well-known but useful concepts and results from algebra.

If R is a binary relation on a set A, i.e. $R \subseteq A \times A$, by its converse is meant the relation $R^{op} = \{(x, y) \in A \times A \mid (y, x) \in R\}$.

Let $(P_1; \leq)$, $(P_2; \leq)$ be two ordered sets. By an *antitone mapping* $f : P_1 \to P_2$ is meant a mapping satisfying $x \leq y$ implies $f(y) \leq f(x)$. Analogously, $g : P_1 \to P_2$ is *order-preserving* (or *monotone*) if $x \leq y$ implies $g(x) \leq g(y)$. A pair (f, g) of order-preserving mappings $f : P_1 \to P_2$ and $g : P_2 \to P_1$ is a *Galois connection* or an *adjunction* between P_1 and P_2 provided that

$$f(x) \leq y \text{ if and only if } x \leq g(y).$$

In an adjunction (f, g) the mapping f is called the *left adjoint* (*lower adjoint*) and the mapping g is called the *right adjoint* (*upper adjoint*). The mapping f is then uniquely determined by the mapping g and vice versa. Note that the pair (f, g) of order-preserving mappings $f : P_1 \to P_2$ and $g : P_2 \to P_1$ is an adjunction if and only if

$$f(x) = \min\{y \in P_2 \mid x \leq g(y)\} \text{ for all } x \in P_1$$

if and only if

$$g(y) = \max\{x \in P_1 \mid f(x) \leq y\} \text{ for all } y \in P_2$$

if and only if

$$x \leq g(f(x)) \quad \text{and} \quad f(g(y)) \leq y \text{ for all } x \in P_1, y \in P_2.$$

A pair (f, g) of mappings $f : P_1 \to P_2$ and $g : P_2 \to P_1$ is an *antitone Galois connection* between P_1 and P_2 provided that (f, g) is a Galois connection between $(P_1; \leq)$ and $(P_2; \leq^{op})$.

A *morphism* $f : P_1 \to P_2$ *of bounded posets* is an order, top element and bottom element preserving map. A morphism $f : P_1 \to P_2$ of bounded posets is *order reflecting* if $(f(a) \leq f(b)$ if and only if $a \leq b)$ for all $a, b \in P_1$.

Observation 1 [7] *Let P_1, P_2 be bounded posets, T a set and $h_t : P_1 \to P_2, t \in T$, morphisms of bounded posets. The following conditions are equivalent:*

(i) $((\forall t \in T),\ h_t(a) \leq h_t(b)) \Longrightarrow a \leq b$ *for any elements $a, b \in P_1$;*
(ii)   *The map $h : P_1 \to P_2^T$ defined by $h(a) = (h_t(a))_{t \in T}$ for all $a \in P_1$ is order reflecting.*

We then say that $\{h_t : P_1 \to P_2 \mid t \in T\}$ is a *full set of order preserving maps with respect to P_2*. Note that we may in this case identify P_1 with a subposet of P_2^T since h is an order reflecting morphism of bounded posets.

2 The Definition

Consider now an algebra $\mathcal{A} = (A; \leq, 0, 1)$ introduced in the previous section. We can define the dynamic order pair (G, P) on $\mathcal{A}$ as follows:

Definition 1 A couple (G, P) of partial mappings $P, G : A \to A$ is called a *dynamic order pair* on $\mathcal{A}$ if the following holds:

(P1) $G(0) = 0, G(1) = 1, P(0) = 0$ and $P(1) = 1$.
(P2) $x \leq y$ implies $G(x) \leq G(y)$ whenever $G(x), G(y)$ exist and $P(x) \leq P(y)$ whenever $P(x), P(y)$ exist.
(P3) $x \leq GP(x)$ if $P(x)$ exists and $GP(x)$ exists, $PG(y) \leq y$ if $G(y)$ exists and $PG(y)$ exists.

The operator P is called a *weak dynamic order operator* and the operator G is called a *strong dynamic order operator* (shortly *dynamic order operators*). The triple $\mathcal{B} = (\mathcal{A}; G, P)$ is called a *partial dynamic order algebra*. If both G and P are total mappings we speak about a *dynamic order algebra*. We say that a partial dynamic order algebra $(\mathcal{A}; G, P)$ is *complete* if $\mathcal{A}$ is a complete lattice.

If $(\mathcal{A}_1; G_1, P_1)$ and $(\mathcal{A}_2; G_2, P_2)$ are partial dynamic order algebras, then a *morphism of partial dynamic order algebras $f : (\mathcal{A}_1; G_1, P_1) \to (\mathcal{A}_2; G_2, P_2)$* is a morphism of bounded posets such that $f(G_1(x)) = G_2(f(x))$, for any $x \in A_1$ such that $G_1(x)$ is defined and $f(P_1(y)) = P_2(f(y))$, for any $y \in A_1$ such that $P_1(y)$ is defined.

We say that a partial map $G : A \to A$ is *contractive (transitive)* if $G(x) \leq x$ ($G(x) \leq GG(x)$) for all $x \in A$ such that $G(x)$ is defined (for all $x \in A$ such that $G(x)$ and $GG(x)$ are defined). A partial map G that is both contractive and transitive is called a *conucleus*.

A dynamic order pair (G, P) is called *modal* if G is a conucleus and we speak about a *partial modal algebra*. If both G and P are total mappings we speak about a *modal algebra*.

Remark 1 Let us note that the partial mappings defined only for 0 and 1 by $G(0) = P(0) = 0$ and $G(1) = P(1) = 1$ are dynamic order operators on every bounded poset. Hence, every bounded poset can be organized into a partial dynamic order algebra.

Theorem 1 *Let $\mathcal{A}$ be a bounded poset, $G, P : A \to A$ mappings. Then the following conditions are equivalent:*

1. $(\mathcal{A}; G, P)$ *is a dynamic order algebra.*
2. *G is a mapping of A into itself satisfying*

(D1)   *G has a left adjoint P.*
(D2)   *$G(0) = 0$ and $P(1) = 1$.*

Proof $\Longrightarrow$: Let $(A; G, P)$ be a dynamic order algebra. Assume that $x, y \in A$. Let us check (D1). Then by (P2) P and G are order-preserving and again by (P2) and (P3) $P(x) \leq y$ implies $x \leq GP(x) \leq G(y)$. By (P2) and (P3), $x \leq G(y)$ implies $P(x) \leq PG(y) \leq y$ and (D1) is valid. (D2) follows immediately from (P1).

$\Longleftarrow$: Let (D1) be satisfied. Then both G and P are order-preserving mappings so we get (P2). Since G is a right adjoint it preserves arbitrary existing infima. Particularly, $1 = \bigwedge \emptyset$ and hence $G(1) = 1$. Similarly, since P is a left adjoint that preserves arbitrary suprema and $0 = \bigvee \emptyset$, we obtain $P(0) = 0$. So we get (P1) by the preceding and (D2). Since (P, G) is an adjunction we obtain (P3). $\qquad\square$

Remark 2 The preceding Theorem 2 allows us to work with dynamic order algebras equipped with only one dynamic order operator G satisfying conditions (D1)-(D2).

3 The Construction

In what follows we want to provide a meaningful procedure giving dynamic order operators on every bounded poset which will be in accordance with an intuitive idea of time dependency.

By a *frame* (see e.g. [10]) is meant a couple (T, R) where T is a non-void set and R is a binary relation on T. Furthermore, we will assume that for all $x \in T$ there are $y, z \in T$ such that zRx and xRy. The set T is considered to be a *time scale*, the relation R expresses a relationship "to be before" and "to be after". Having a bounded poset $\mathcal{A} = (A; \leq, 0, 1)$ and a non-void set T, we can produce the direct power $\mathcal{A}^T = (A^T; \leq, o, j)$ where the relation $\leq$ is defined and evaluated on $p, q \in A^T$ componentwise, i.e. $p \leq q$ if $p(t) \leq q(t)$ for each $t \in T$. Moreover, o, j are such elements of A^T that $o(t) = 0$ and $j(t) = 1$ for all $t \in T$.

Theorem 2 *Let $\mathcal{A} = (A; \leq, 0, 1)$ be a bounded poset and let (T, R) be a frame. Then for every complete lattice M where A is a subposet of M we can define partial mappings G, P of A^T into itself as follows: for all $p \in A^T$, $G(p)$ is defined if and only if, for all $s \in T$, $\bigwedge_M \{p(t) \mid sRt\} \in A$ in which case*

$$G(p)(s) = \bigwedge_M \{p(t) \mid sRt\} = \bigwedge_A \{p(t) \mid sRt\}$$

and for all $p \in A^T$, $P(p)$ is defined if and only if, for all $s \in T$, $\bigvee_M \{p(t) \mid tRs\} \in A$ in which case

$$P(p)(s) = \bigvee_M \{p(t) \mid tRs\} = \bigvee_A \{p(t) \mid tRs\}$$

and partial mappings H, F of A^T into itself as follows: for all $p \in A^T$, $H(p)$ is defined if and only if, for all $s \in T$, $\bigwedge_M \{p(t) \mid tRs\} \in A$ in which case

$$H(p)(s) = \bigwedge_M \{p(t) \mid tRs\} = \bigwedge_A \{p(t) \mid tRs\}.$$

and for all $p \in A^T$, *$F(p)$ is defined if and only if, for all $s \in T$, $\bigvee_M \{p(t) \mid sRt\} \in A$ in which case*

$$F(p)(s) = \bigvee_M \{p(t) \mid sRt\} = \bigvee_A \{p(t) \mid sRt\}.$$

If this is the case, then G, P, H, F are dynamic order operators on A^T such that, for all $p \in A^T$,

$$G(p) \leq F(p) \quad and \quad H(p) \leq P(p)$$

whenever the respective sides of the relation $\leq$ are defined, i.e., both $\mathcal{D} = \left(A^T; G, P\right)$ and $\mathcal{E} = \left(A^T; H, F\right)$ are partial dynamic order algebras. Moreover, the following holds:

(a) *If R is reflexive then G and H are contractive.*
(b) *If R is transitive then G and H are transitive.*
(c) *If R is both reflexive and transitive then both $\mathcal{D} = (A^T; G, P)$ and $\mathcal{E} = \left(A^T; H, F\right)$ are partial modal algebras.*

Proof Before going to the proof note that operators H and F were defined in the exactly same manner as operators G and P, the only distinction is the fact that for operators G and P we used the time frame (T, R) and for operators H and F we used the time frame (T, R^{op}). This observation will be used very often in this and other proofs.

Trivially we can verify $G(o) = o$, $G(j) = j$ and $P(o) = o$, $P(j) = j$ due to the fact that $o(t) = 0$ and $j(t) = 1$ for each $t \in T$; thus (P1) holds. Let us check (P2). Assume that $G(p), G(q)$ exist and that $p \leq q$. Then, for all $t \in T$, $p(t) \leq q(t)$. This yields that $G(p)(s) = \bigwedge_M \{p(t) \mid sRt\} \leq \bigwedge_M \{q(t) \mid sRt\} = G(q)(s)$ for all $s \in T$. Therefore $G(p) \leq G(q)$. Similarly, $P(p) \leq P(q)$ whenever $P(p), P(q)$ exist and $p \leq q$.

It remains to prove (P3). Assume that $P(p)$ and $GP(p)$ exist. We have, for all $p \in A^T$ and all $t \in T$,

$$P(p)(u) = \bigvee_A \{p(t) \mid tRu\}$$

and, for all $p \in A^T$ and all $s \in T$,

$$GP(p)(s) = \bigwedge_A \left\{ \bigvee_A \{p(t) \mid t \in T, tRu\} \mid u \in T, sRu \right\}.$$

There are two possibilities. First, the set $\{u \in T \mid sRu\}$ is empty. In this case we get that $\bigwedge_B \left\{ \bigvee_B \{p(t) \mid t \in T, tRu\} \mid u \in T, sRu \right\} = 1$. Second, the set $\{u \in T \mid sRu\}$ is non-empty. Since every member of the infimum is greater or equal to $p(s)$, we conclude $GP(p)(s) \geq p(s)$ for each $s \in T$, i.e., in both cases we obtain $p \leq GP(p)$. Analogously it can be shown $PG(p) \leq p$.

Now, assume that $p \in A^T$ and both $G(p)$ and $F(p)$ are defined. Let us verify that $G(p) \leq F(p)$. Let $s \in T$. Then, by the assumption that (T, R) is a frame, there is an element $t \in T$ such that sRt. It follows that

$$G(p)(s) \leq p(t) \leq F(p)(s).$$

An analogous result follows for the operator H.

Let us check (a). Assume that $p \in A^T$, $s \in T$ and $G(p)$ is defined. Since R is reflexive then from sRs we obtain that $G(p)(s) = \bigwedge \{p(t) \mid t \in T, sRt\} \leq p(s)$. Let us proceed similarly for (b). Assume that $p \in A^T$ and both $G(p)$ and $GG(p)$ are defined. We have, for all $s \in T$,

$$\begin{aligned} G(p)(s) &= \bigwedge \{p(u) \mid u \in T, sRu\} \leq \bigwedge \{p(u) \mid t, u \in T, sRt, tRu\} \\ &= \bigwedge \{\bigwedge \{p(u) \mid u \in T, tRu\} \mid t \in T, sRt\} \\ &= \bigwedge \{G(p)(t) \mid t \in T, sRt\} = GG(p)(s) \end{aligned}$$

since $\{u \in T \mid \exists t \in T, sRt, tRu\} \subseteq \{u \in T \mid sRu\}$ by transitivity. An analogous result follows for the operator H. The validity of (c) follows immediately from (a) and (b). □

Remark 3 If the relation R on a non-void set T is a quasiorder, i.e. our frame is $(T, \leq)$ with $\leq$ reflexive and transitive, then $s \leq t$ expresses the fact that t "follows" s and s "is before" t. More precisely, $P(p)$ says that *p was true in past with at least the same degree as p is in present* and $G(p)$ says that *p will be true in future with at most the same degree as it is now*. Analogously, $F(p)$ can be interpreted as "a case in future of an element p" and $H(p)$ is "a history" of p.

Corollary 1 *Let* $\mathcal{M} = (M; \leq, 0, 1)$ *be a complete lattice and let* (T, R) *be a frame (a quasiorder). Define mappings* $\widehat{G}, \widehat{P}, \widehat{H}, \widehat{F}$ *of* M^T *into itself as follows: For all* $p \in M^T$,

$$\widehat{G}(p)(x) = \bigwedge_M \{p(y) \mid xRy\}.$$
$$\widehat{P}(p)(x) = \bigvee_M \{p(y) \mid yRx\}.$$
$$\widehat{H}(p)(x) = \bigwedge_M \{p(y) \mid yRx\}.$$

and

$$\widehat{F}(p)(x) = \bigvee_M \{p(y) \mid xRy\}.$$

Then $\widehat{G}, \widehat{P}, \widehat{H}, \widehat{F}$ *are dynamic order operators on* $\mathcal{M}^T$ *such that*

$$\widehat{G} \leq \widehat{F} \quad \text{and} \quad \widehat{H} \leq \widehat{P}(p)$$

and both $(\mathcal{M}^T; \widehat{G}, \widehat{P})$ *and* $(\mathcal{M}^T; \widehat{H}, \widehat{F})$ *are dynamic order algebras (modal algebras).*

4 Representation of Dynamic Order Algebras

In Theorem 3, we presented a construction of natural dynamic order operators when a bounded poset and a frame are given. However, we can ask, for a given dynamic order algebra $(\mathcal{A}; G, P)$, whether there exist a frame (T, R) and a bounded poset $\mathcal{M} = (M; \leq, 0, 1)$ such that the dynamic order operators G, P can be derived by this construction where $\mathcal{A}$ is embedded into the power algebra $\mathcal{M}^T$. Hence, we ask, if every element p of A is in the form $(p(t))_{t \in T}$ in M^T, $G(p)(s) = \bigwedge_M \{p(t) \mid sRt\}$ and $P(p)(s) = \bigvee_M \{p(t) \mid tRs\}$. If such a representation exists then one can recognize the time variability of elements of A expressed as time dependent functions $p : T \to M$.

From Corollary 1 we immediately see that $(\mathcal{M}^T; \widehat{G}, \widehat{P})$ is automatically a dynamic order algebra.

Proposition 1 *Let* $\mathcal{A} = (A; \leq, 0, 1)$ *be a bounded poset equipped with a full set* S *of morphisms into a complete lattice* $\mathcal{L}$, $M = L^S$. *Then the map* $i_{\mathcal{A}}^S : A \to L^S$ *given by* $i_{\mathcal{A}}^S(x)(s) = s(x)$ *for all* $x \in A$ *and all* $s \in S$ *is an order reflecting morphism of bounded posets such that* $i_{\mathcal{A}}^S(A)$ *is a sub-poset of* $\mathcal{M} = \mathcal{L}^S$.

Proof Since S is a full set S of morphisms we have from Observation 1 that $i_{\mathcal{A}}^S$ is an injective order-reflecting morphism of bounded posets. It follows that $i_{\mathcal{A}}^S(A)$ is a bounded sub-poset of L^S. □

We immediately obtain from Theorem 3 and Proposition 1 the following.

Corollary 2 *Let $\mathcal{A} = (A; \leq, 0, 1)$ be a bounded poset equipped with a full set S of morphisms into a complete lattice $\mathcal{L}$, $M = L^S$ and (T, R) be a frame. Define partial mappings G, P of A^T into itself as follows: For all $p \in A^T$, $G(p)$ is defined if and only if, for all $s \in T$, $\bigwedge_M \{p(t) \mid sRt\} \in A$ in which case*

$$G(p)(s) = \bigwedge_M \{p(t) \mid sRt\}.$$

and for all $p \in A^T$, $P(p)$ is defined if and only if, for all $s \in T$, $\bigvee_M \{p(t) \mid tRs\} \in A$ in which case

$$P(p)(s) = \bigvee_M \{p(t) \mid tRs\}.$$

Then G, P are dynamic order operators on $\mathcal{A}^T$, i.e. $\mathcal{D} = (\mathcal{A}^T; G, P)$ is a partial dynamic order algebra and $i_{\mathcal{A}}^{S,T} : \mathcal{A}^T \to \mathcal{M}^T$ defined by $i_{\mathcal{A}}^{S,T}((x_t)_{t \in T}) = ((i_{\mathcal{A}}^S(x_t))_{t \in T})$ is an order reflecting morphism of bounded posets into the complete dynamic order algebra $(\mathcal{M}^T; \widehat{G}, \widehat{P})$ given by Corollary 1.

In the previous section, we carried out the construction of dynamic operators using the given time frame. As announced above, we would like to solve the converse problem, i.e., to find a suitable time frame for given dynamic operators which are given in the dynamic algebra. The aim of the following Lemma is to develop a method using Galois connection (P, G) for a construction of a certain relation, denoted by R_G. The resulting relation will be used successfully in the next section for solving the representation problem provided the given poset is equipped with a full set of morphisms.

We start with this construction of R_G.

Lemma 1 *Let $(\mathcal{A}; G, P)$ be a dynamic order algebra and T a set of morphisms from $\mathcal{A}$ into a bounded poset $\mathcal{C}$. Let us put $R_G = \{(s, t) \in T \times T \mid (\forall x \in A)(s(G(x)) \leq t(x))\}$. Then (i)-(v) hold:*

(i) *If G is contractive then R_G is reflexive.*

(ii) *If G is transitive then R_G is transitive.*

(iii) *Let $s, s \circ G \in T$. Then $(s, s \circ G) \in R_G$ and, for all $x \in A$,*

$$s(G(x)) = \min_C \{t(x) \mid sR_Gt\}.$$

(iv) *Let $s, s \circ P \in T$. Then $(s \circ P, s) \in R_G$ and, for all $x \in A$,*

$$s(P(x)) = \max_C \{t(x) \mid tR_Gs\}.$$

(v)

$$R_G = \{(s, t) \in T \times T \mid (\forall x \in A)(t(P(x)) \geq s(x))\}.$$

Proof

(i): $G(x) \leq x$ yields $s(G(x)) \leq s(x)$ for all $x \in A$ and all $s \in T$. Hence sR_Gs.

(ii): Let $s, t, u \in T$, sR_Gt and tR_Gu. Let $x \in A$. Then $s(G(x)) \leq s(GG(x)) \leq t(G(x)) \leq u(x)$. Hence sR_Gu.

(iii): Since, for all $x \in A$, $s(G(x)) = (s \circ G)(x)$ we have that $(s, s \circ G) \in R_G$ and clearly, for all $x \in A$, $(s \circ G)(x) = \min_C \{t(x) \mid sR_Gt\}$.

(iv): From the definition of a dynamic order algebra we get that $s(PG(x)) \leq s(x)$ for all $x \in A$. Hence $(s \circ P, s) \in R_G$. Evidently, for all $x \in A$, tR_Gs yields $t(x) \leq t(GP(x)) \leq s(P(x))$. It follows that $(s \circ P)(x) = \max_C \{t(x) \mid tR_Gs\}$.

(v): Assume first that $(s, t) \in R_G$, $x \in A$. Then $s(x) \leq s(G(P(x))) \leq t(P(x))$. Conversely, let $s(x) \leq t(P(x))$ for all $x \in A$. It follows, for any $x \in A$, that $s(G(x)) \leq t(P(G(x))) \leq t(x)$, i.e., $(s, t) \in R_G$. Hence,

$$R_G = \{(s, t) \in T \times T \mid (\forall x \in A)(t(P(x)) \geq s(x))\}.$$

$\square$

The relation R_G introduced in Lemma 1 will be called the *G-induced relation*.

Theorem 3 *Let $(\mathcal{A}; G, P)$ be a dynamic order algebra with a full set T of morphisms into a complete lattice $\mathcal{L}$ such that*

1. *for all $x \in A$ and for all $s \in T$, $s(G(x)) = \bigwedge_L \{t(x) \mid s R_G t\}$,*
2. *for all $x \in A$ and for all $s \in T$, $s(P(x)) = \bigvee_L \{t(x) \mid t R_G s\}$.*

Then the map $i_{\mathcal{A}}^T$ is an order reflecting morphism of dynamic order algebras into the complete dynamic order algebra $(\mathcal{M}; \widehat{G}, \widehat{P})$, $M = L^T$, given by the frame (T, R_G).

Moreover, if G is contractive (transitive) then $\widehat{G}$ is contractive (transitive) and if $(\mathcal{A}; G, P)$ is a modal algebra then $(\mathcal{M}; \widehat{G}, \widehat{P})$ is a complete modal algebra.

The dynamic order algebra $(\mathcal{A}; G, P)$ is then said to be *representable in $\mathcal{L}$ with respect to T*.

Proof Recall first that since $\mathcal{L}$ is a complete lattice we have from Corollary 1 that $(M; \widehat{G}, \widehat{P})$ is a complete dynamic order algebra. Here $\widehat{G}(m)(s) = \bigwedge_L \{m(t) \mid s R_G t\}$ and $\widehat{P}(m)(s) = \bigvee_L \{m(t) \mid t R_G s\}$ for all $m \in M$ and for all $s \in T$. By Proposition 1, $i_{\mathcal{A}}^T$ is an order reflecting morphisms into $\mathcal{M}$. Since $s(G(x)) = \bigwedge_L \{t(x) \mid s R_G t\}$ we get that $i_{\mathcal{A}}^T(G(x))(s) = \bigwedge_L \{t(x) \mid s R_G t\}$. Then $i_{\mathcal{A}}^T(G(x)) = \widehat{G}(i_{\mathcal{A}}^T(x))$. Similarly $i_{\mathcal{A}}^T(P(x)) = \widehat{P}(i_{\mathcal{A}}^T(x))$.

The remaining part of the theorem follows immediately from Lemma 1 and Theorem 3. Namely, if G is contractive (transitive) then R_G is reflexive (transitive) by Lemma 1. Since R_G is reflexive (transitive) then $\widehat{G}$ is contractive (transitive) by Theorem 3. $\square$

Now, let us prove a representation theorem for dynamic order algebras with a full set of morphisms.

Theorem 4 *(Representation theorem for dynamic order algebras)* *For any dynamic order algebra $\mathcal{B} = (\mathcal{A}; G, P)$ with a full set S of morphisms into a complete lattice $\mathcal{L}$, there exists a full set T of morphisms into $\mathcal{L}$ including S such that $\mathcal{B}$ is representable in $\mathcal{L}$ with respect to T. In particular, $i_{\mathcal{A}}^T$ is an order reflecting morphism of dynamic order algebras.*

Proof Let $T = \{s \circ G^{n_1} \circ P^{m_1} \cdots \circ G^{n_k} \circ P^{m_k} \mid s \in S, k, n_1, m_1, \ldots, n_k, m_k \in \mathbb{N}_0\}$. Then T is the smallest set of morphisms into $\mathcal{L}$ including S such that $s \in T$ implies that $s \circ G, s \circ P \in T$. Since T includes S it is again a full set of morphisms.

It is enough to verify that for all $x \in A$ and for all $s \in T$, $s(G(x)) = \bigwedge_L \{t(x) \mid s R_G t\}$ and $s(P(x)) = \bigvee_L \{t(x) \mid t R_G s\}$ where R_G is the *G-induced relation*. But this follows from Lemma 1. $\square$

5 Representation of Tense Algebras

In this section, we establish representation and approximation results for tense algebras taking into account our results from Section 4.

Definition 2 Let $(\mathcal{A}; G, P)$ and $(\mathcal{A}; H, F)$ be partial dynamic order algebras such that, for all $p \in A$,

$$G(p) \leq F(p) \quad \text{and} \quad H(p) \leq P(p)$$

whenever the respective sides of the relation $\leq$ are defined.

The quintuple $\mathcal{T}(A) = (\mathcal{A}; G, P, H, F)$ is called a *partial tense algebra*. If all G, P, H, F are total maps we speak about a *tense algebra*. If $(\mathcal{A}; G, P)$ and $(\mathcal{A}; H, F)$ are partial modal algebras (modal algebras) then $\mathcal{T}(A)$ is called a *partial tense modal algebra* (*tense modal algebra*).

If $(\mathcal{A}_1; G_1, P_1, H_1, F_1)$ and $(\mathcal{A}_2; G_2, P_2, H_2, F_2)$ are partial tense algebras and $f : A_1 \to A_2$ is a map such that

(i) $f : (\mathcal{A}_1; G_1, P_1) \to (\mathcal{A}_2; G_2, P_2)$ is a morphism of partial dynamic order algebras,
(ii) $f : (\mathcal{A}_1; H_1, F_1) \to (\mathcal{A}_2; H_2, F_2)$ is a morphism of partial dynamic order algebras,
  then we say that $f : (\mathcal{A}_1; G_1, P_1, H_1, F_1) \to (\mathcal{A}_2; G_2, P_2, H_2, F_2)$ is a *morphism of partial tense algebras*.

For the formulation of our main results, we need the following lemma.

Lemma 2 *Let $\mathcal{B} = (\mathcal{A}; G, P, H, F)$ be a tense algebra and let T be a set of morphisms from $\mathcal{A}$ into a bounded poset $\mathcal{C}$. Let us put*

$$R_{G,H} = \{(s, t) \in T \times T \mid (\forall x \in A)(s(G(x)) \leq t(x) \text{ and } t(H(x)) \leq s(x))\}.$$

and

$$R_{H,G} = \{(s, t) \in T \times T \mid (\forall x \in A)(s(H(x)) \leq t(x) \text{ and } t(G(x)) \leq s(x))\}$$

where R_G is the G-induced relation and R_H is the H-induced relation. Then $R_{G,H} = R_G \cap R_H^{op}$, $R_{H,G} = R_H \cap R_G^{op}$, $R_{G,H} = R_{H,G}^{op}$ and

(i) *If G and H are contractive then $R_{G,H}$ and $R_{H,G}$ are reflexive.*
(ii) *If G and H are transitive then $R_{G,H}$ and $R_{H,G}$ are transitive.*
(iii) *Let $s, s \circ G \in T$. Then $(s, s \circ G) \in R_{G,H}$ and, for all $x \in A$,*

$$s(G(x)) = \min_C \{t(x) \mid s R_{G,H} t\}.$$

(iv) *Let $s, s \circ P \in T$. Then $(s \circ P, s) \in R_{G,H}$ and, for all $x \in A$,*

$$s(P(x)) = \max_C \{t(x) \mid t R_{G,H} s\}.$$

(v) *Let $s, s \circ H \in T$. Then $(s, s \circ H) \in R_{H,G}$ and, for all $x \in A$,*

$$s(H(x)) = \min_C \{t(x) \mid s R_{H,G} t\}.$$

(vi) *Let $s, s \circ F \in T$. Then $(s \circ F, s) \in R_{H,G}$ and, for all $x \in A$,*

$$s(F(x)) = \max_C \{t(x) \mid t R_{H,G} s\}.$$

Proof
  (i): From Lemma 1, (i) we know that R_G and R_H are reflexive. It follows that $R_{G,H}$ and $R_{H,G}$ are reflexive.

(ii): From Lemma 1, (ii) we know that R_G and R_H are transitive. We get that $R_{G,H}$ and $R_{H,G}$ are transitive.

(iii): Since, for all $x \in A$, $(s \circ G)(H(x)) \leq s(FH(x)) \leq s(x)$ we have that $(s, s \circ G) \in R_H^{op}$. From Lemma 1, (iii) we obtain that $(s, s \circ G) \in R_G$, i.e., $(s, s \circ G) \in R_{G,H}$. Clearly, for all $x \in A$, $(s \circ G)(x) = \min_C \{t(x) \mid s R_{G,H} t\}$.

(iv): From the definition of a tense algebra we get that $s(H(x)) \leq s(P(x))$ for all $x \in A$. Hence $(s \circ P, s) \in R_H^{op}$. From Lemma 1, (iv) we know that $(s \circ P, s) \in R_G$. It follows that $(s \circ P, s) \in R_{G,H}$ and, for all $x \in A$, $(s \circ P)(x) = \max_C \{t(x) \mid t R_{G,H} s\}$. This follows from the fact that $t R_{G,H} s$ implies $t R_G s$ which implies, for all $x \in A$, $t(x) \leq t(GP(x)) \leq s(P(x))$.

(v), (vi): By interchanging the role of G with H and of P with F we get by (iii) and (iv) the statements.

$\square$

In what follows, we show that the full set of morphisms of a given tense algebra into a complete lattice can serve as a time scale which, together with the introduced relation $R_{G,H}$, already forms a time frame such that the given tense operators can be reached by our construction from Theorem 3.

Theorem 5 *Let $\mathcal{B} = (\mathcal{A}; G, P, H, F)$ be a tense algebra with a full set T of morphisms into a complete lattice $\mathcal{L}$ such that*

1. *for all $x \in A$ and for all $s \in T$, $s(G(x)) = \bigwedge_L \{t(x) \mid s R_{G,H} t\}$,*
2. *for all $x \in A$ and for all $s \in T$, $s(P(x)) = \bigvee_L \{t(x) \mid t R_{G,H} s\}$,*
3. *for all $x \in A$ and for all $s \in T$, $s(H(x)) = \bigwedge_L \{t(x) \mid t R_{G,H} s\}$,*
4. *for all $x \in A$ and for all $s \in T$, $s(F(x)) = \bigvee_L \{t(x) \mid s R_{G,H} t\}$.*

Then the map $i_{\mathcal{A}}^T$ is an order reflecting morphism of tense algebras into the complete tense algebra $(\mathcal{M}; \widehat{G}, \widehat{P}, \widehat{H}, \widehat{F})$, $M = L^T$, given by the frame $(T, R_{G,H})$. $\mathcal{B} = (\mathcal{A}; G, P, H, F)$ is then said to be representable in $\mathcal{L}$ with respect to T.

Moreover, if G and H are contractive (transitive) then $\widehat{G}$ and $\widehat{H}$ are contractive (transitive) and if $(\mathcal{A}; G, P, H, F)$ is a tense modal algebra then $(\mathcal{M}; \widehat{G}, \widehat{P}, \widehat{H}, \widehat{F})$ is a complete tense modal algebra.

Proof It follows by the same arguments as in Theorem 4. $\square$

The previous result can be extended into a representation theorem, i.e., we are going to show that any full set S of morphisms into a complete lattice $\mathcal{L}$ can be inserted in a possibly larger set T such that our tense algebra is representable in the power $\mathcal{L}^T$.

Theorem 6 *(Representation theorem for tense algebras)* *For any tense (tense modal) algebra $\mathcal{B} = (\mathcal{A}; G, P, H, F)$ with a full set S of morphisms into a complete lattice $\mathcal{L}$, there exists a full set T of morphisms into $\mathcal{L}$ including S such that $\mathcal{B}$ is representable in $\mathcal{L}$ with respect to T. In particular, $i_{\mathcal{A}}^T$ is an order reflecting morphism of tense (tense modal) algebras.*

Proof Let $T = \{s \circ G^{n_1} \circ P^{m_1} \circ H^{p_1} \circ F^{q_1} \cdots \circ G^{n_k} \circ P^{m_k} \circ H^{p_k} \circ F^{q_k} \mid s \in S, k, n_1, m_1, p_1, q_1, \ldots, n_k, m_k, p_k, q_k \in \mathbb{N}_0\}$. Then T is the smallest set of morphisms into $\mathcal{L}$ including S such that $s \in T$ implies that $s \circ G, s \circ P \in T, s \circ H, s \circ F \in T$. Since T includes S it is again a full set of morphisms. The remaining conditions on T

from Theorem 6 are satisfied by the same arguments as in the proof of Theorem 5 using Lemma 2.   □

6 Dynamic Order Algebras and Their Dedekind-MacNeille Completion

In this section, we will show that any dynamic order algebra is representable in its Dedekind-MacNeille completion. The terminology and symbols used here coincide in general with those used in [14].

If $\mathcal{A} = (A; \leq)$ is an ordered set then there always exists a lattice $\mathcal{L} = (L; \vee, \wedge)$ with the induced order $\leq$ such that $A \subseteq L$ and $x \leq y$ in $(A; \leq)$ implies $x \leq y$ in $\mathcal{L}$. One possible construction of $\mathcal{L}$ for a given ordered set $(A; \leq)$ is that using so-called cuts. Let us describe this construction.

Let $\mathcal{A}$ be an ordered set, $X \subseteq A$. The set

$$\mathbf{L}(X) := \{y \in A \mid (\forall x \in X)(y \leq x)\}$$

is called the *left polar* of X in $\mathcal{A}$. Similarly, the set

$$\mathbf{U}(X) := \{y \in A \mid (\forall x \in X)(y \geq x)\}$$

is called the *right polar* of X in $\mathcal{A}$.

Recall that the operators $\mathbf{L}$ and $\mathbf{U}$ define an antitone Galois connection on $\mathcal{P}(A)$, i.e., $Y \subseteq \mathbf{L}(X)$ if and only if $X \subseteq \mathbf{U}(Y)$. In particular, $\mathbf{L}$ and $\mathbf{U}$ are antitone mappings such that $\mathbf{L}(X) = \bigcap \{\mathbf{L}(x) \mid x \in X\}$, $\mathbf{U}(Y) = \bigcap \{\mathbf{U}(y) \mid y \in Y\}$, and their compositions $\mathbf{L} \circ \mathbf{U}$ and $\mathbf{U} \circ \mathbf{L}$ are monotone mappings.

A subset $X \subseteq A$ such that $\mathbf{L}(\mathbf{U}X) = X$ is said to be a *cut*. The complete lattice of all cuts in $\mathcal{A}$ will be denoted by $\mathbf{MC}(\mathcal{A})$. We speak about the *Dedekind-MacNeille completion* or *completion by cuts*.

Define a mapping $e_{\mathcal{A}} : \mathcal{A} \to \mathbf{MC}(\mathcal{A})$ via $e_{\mathcal{A}}(x) = \mathcal{L}(\{x\}) = \{y \mid y \leq x\}$. It is easy to see that $e_{\mathcal{A}}$ is an order reflecting morphism of bounded posets. Hence $\{e_{\mathcal{A}}\}$ is a full set of morphisms into $\mathbf{MC}(\mathcal{A})$. Moreover, the set $e_{\mathcal{A}}(A)$ is both join-dense and meet-dense in $\mathbf{MC}(\mathcal{A})$; that is, every element of the completion is a join of some set of elements of $e_{\mathcal{A}}(A)$, and is also a meet of some set of elements of $e_{\mathcal{A}}(A)$.

Recall that Corollary 2 allows us to introduce, for any bounded poset $\mathcal{A}$ with the Mac-Neille completion $\mathbf{MC}(\mathcal{A})$ and for any frame (T, R), a partial dynamic order structure on $\mathcal{A}$ that can be fully reconstructed from $(\mathbf{MC}(\mathcal{A})^T; \widehat{G}, \widehat{P})$. Note that T is not a set of morphisms but we may identify some elements of T with the canonical embedding $e_{\mathcal{A}} : \mathcal{A} \to \mathbf{MC}(\mathcal{A})$. For a dynamic order algebra $(\mathcal{A}; G, H)$ we can show a converse. To do this we will need the following lemma.

Lemma 3 ([8]) *Let $\mathcal{A}$ be an ordered set, $G, P : A \to A$ order-preserving mappings such that (P, G) is an adjunction. Then*

(i) *The map $\widetilde{P} : \mathbf{MC}(\mathcal{A}) \to \mathbf{MC}(\mathcal{A})$ given via $\widetilde{P}(X) = \bigwedge \{e_{\mathcal{A}}(y) \mid X \subseteq e_{\mathcal{A}}(G(y))\}$ for all $X \in \mathbf{MC}(\mathcal{A})$ preserves arbitrary joins and $e_{\mathcal{A}} \circ P = \widetilde{P} \circ e_{\mathcal{A}}$.*

(ii) *The map $\widetilde{P} : \mathbf{MC}(\mathcal{A}) \to \mathbf{MC}(\mathcal{A})$ has a right adjoint $\widetilde{G} : \mathbf{MC}(\mathcal{A}) \to \mathbf{MC}(\mathcal{A})$ and $e_{\mathcal{A}} \circ G = \widetilde{G} \circ e_{\mathcal{A}}$.*

(iii) *For all $Y \in \mathbf{MC}(\mathcal{A})$, $\widetilde{G}(Y) = \bigvee \{e_{\mathcal{A}}(y) \mid e_{\mathcal{A}}(P(y)) \subseteq Y\}$ and*

$$\widetilde{G}(Y) = \bigwedge \{e_{\mathcal{A}}(G(z)) \mid Y \subseteq e_{\mathcal{A}}(z)\}.$$

Theorem 7 *Let $(\mathcal{A}; G, P)$ be a dynamic order algebra. Then the MacNeille completion $(\mathbf{MC}(\mathcal{A}); \widetilde{G}, \widetilde{P})$ of $(\mathcal{A}; G, P)$ is a complete dynamic order algebra and $e_{\mathcal{A}} : \mathcal{A} \to \mathbf{MC}(\mathcal{A})$ is an order reflecting morphism of dynamic order algebras.*

Proof By Theorem 2 it is enough to check the conditions (D1) and (D2). Evidently, $(\widetilde{P}, \widetilde{G})$ is an adjunction by Lemma 3. That (D2) is satisfied follows immediately from

$$\widetilde{P}(A) = \left(\widetilde{P} \circ e_{\mathcal{A}}\right)(1) = (e_{\mathcal{A}} \circ P)(1) = e_{\mathcal{A}}(1) = A$$

and

$$\widetilde{G}(\{0\}) = \left(\widetilde{G} \circ e_{\mathcal{A}}\right)(0) = (e_{\mathcal{A}} \circ G)(0) = e_{\mathcal{A}}(0) = \{0\}.$$

$\square$

Remark 4 The preceding Theorem 8 yields a representation of any dynamic order algebra $(\mathcal{A}; G, P)$ in a complete one via the canonical embedding $e_{\mathcal{A}}$.

Following Theorem 5 and Theorem 7 we obtain that all dynamic order algebras $(\mathcal{A}; G, P)$ and tense algebras $(\mathcal{A}; G, P, H, F)$ are representable in $\mathbf{MC}(\mathcal{A})$ with respect to a full countable set of morphisms from $\mathcal{A}$ to $\mathbf{MC}(\mathcal{A})$.

Hence, we can state the following theorem.

Theorem 8 *Let $(\mathcal{A}; G, P)$ be a dynamic order algebra and $\mathbf{MC}(\mathcal{A})$ the MacNeille completion of $\mathcal{A}$. Then there is a countable time frame (T, R) such that $G = \widehat{G}_{|A}$ and $P = \widehat{P}_{|A}$ with $\left(\mathbf{MC}(\mathcal{A})^T; \widehat{G}, \widehat{P}\right)$ given as in Corollary 1.*

Proof Let $e_{\mathcal{A}} : \mathcal{A} \to \mathbf{MC}(\mathcal{A})$ be the corresponding embedding of bounded posets. Let $T = \{e_{\mathcal{A}} \circ G^{n_1} \circ P^{m_1} \cdots \circ G^{n_k} \circ P^{m_k} \mid k, n_1, m_1, \ldots, n_k, m_k \in \mathbb{N}_0\}$. Then T is the smallest set of morphisms containing $e_{\mathcal{A}}$ that is closed under composition with G and P. Evidently, T is countable and we have an order reflecting morphism $i_{\mathcal{A}}^T : \mathcal{A} \to \mathbf{MC}(\mathcal{A})^T$ of dynamic order algebras. $\square$

Using the fact that every tense algebra is composed by means of two dynamic order algebras, we can prove easily the following result.

Theorem 9 *Let $(\mathcal{A}; G, P, H, F)$ be a tense (tense modal) algebra and $\mathbf{MC}(\mathcal{A})$ the MacNeille completion of $\mathcal{A}$. Then there is a countable time frame $(T, R_{G,H})$ such that $G = \widehat{G}_{|A}$, $P = \widehat{P}_{|A}$, $H = \widehat{H}_{|A}$ and $F = \widehat{F}_{|A}$ with $(\mathbf{MC}(\mathcal{A})^T; \widehat{G}, \widehat{P}, \widehat{H}, \widehat{F})$ given as in Corollary 1.*

Proof Let $e_{\mathcal{A}} : \mathcal{A} \to \mathbf{MC}(\mathcal{A})$ be the corresponding embedding of bounded posets. Let $T = \{e_{\mathcal{A}} \circ G^{n_1} \circ P^{m_1} \circ H^{p_1} \circ F^{q_1} \cdots \circ G^{n_k} \circ P^{m_k} \circ H^{p_k} \circ F^{q_k} \mid k, n_1, m_1, p_1, q_1, \ldots, n_k, m_k, p_k, q_k \in \mathbb{N}_0\}$. Then T is the smallest set of morphisms containing $e_{\mathcal{A}}$ that is closed under composition with G, P, H and F. Evidently, T is a countable set and we have an order reflecting morphism $i_{\mathcal{A}}^T : \mathcal{A} \to \mathbf{MC}(\mathcal{A})^T$ of tense (tense modal) algebras. $\square$

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