Int J Theor Phys (2015) 54:4272–4282
DOI 10.1007/s10773-014-2480-3

# Fuzzy Topology and Geometric Formalism of Quantum Mechanics

**S. Mayburov**

Received: 30 October 2014 / Accepted: 16 December 2014 / Published online: 18 February 2015


**Abstract** Dodson-Zeeman fuzzy topology considered as the possible mathematical framework of quantum geometric formalism. In such formalism the states of massive particle $m$ correspond to elements of fuzzy manifold called fuzzy points. Due to their weak (partial) ordering, $m$ space coordinate $x$ acquires principal uncertainty $\sigma_x$. It's shown that $m$ evolution on such manifold corresponds to quantum dynamics. It's argued also that particle's interactions on such fuzzy manifold should be gauge invariant.

**Keywords** Fuzzy topology · Quantization

## 1 Introduction

There are several serious reasons to try to formulate quantum mechanics (QM) in geometric terms. For instance, general relativity is essentially geometric theory, but the attempts to quantize gravity suffer the serious difficulty already at axiomatics level. Such formalism can be useful also for development of gauge field theory, which is also mainly geometric; its implications can be important for the analysis of QM foundations. Currently, the main impact of QM geometrization studies is done on the exploit of Hilbert manifolds [1], however, the results obtained up to now have quite abstract form, and their applicability to particular problems is questionable. Alternatively, in approach considered here the basic structure is the real manifold equipped with fuzzy topology (FT) [2–4]. In connection with such mathematical framework it's worth to mention the noncommutative fuzzy spaces with finite (sphere, tori) and infinite discrete structure [5]. The general feature of such theories is that the space coordinates turn out to be principally fuzzy, the reason of that is the noncommutativity of coordinate observables $x_{1,2,3}$.

Meanwhile, the similar coordinate fuzziness obtained for the manifolds equipped with dedicated FT [2, 3]. Earlier, it was argued that in its framework the quantization procedure

S. Mayburov (✉)
Lebedev Institute of Physics, Leninsky pr., 53, Moscow, RU-117924, Russia
e-mail: mayburov@sci.lebedev.ru

itself can be defined as the transition from the classical phase space to fuzzy one [6, 7]. Therefore, in such approach the quantum properties of particles and fields are deduced directly from the geometry of phase space induced by underlying FT and don't need to be postulated separately of it. In particular, in such formalism the system evolution can be described as the geometrodynamics which is equivalent to quantum dynamics [6, 7]; as the example, the dynamics of massive particles will be considered. Previously, some phenomenological assumptions were used by the author for its derivation, here the simple formalism which permits to avoid them is described, its main features can be found in [8]. It was argued also that the particle interactions on such fuzzy manifold possess the local gauge invariance [7].

Note that the fuzzy structures were used earlier for the development of QM axiomatic in operator algebra setting [9], yet in such formalism the quantum dynamics is always postulated, no attempts to derive it were published. The important example, illustrating the connection between fuzzy structures and quantum dynamics described in [10].

## 2 Topological Fuzzy Structures

Here the main FT features important for the construction of dynamics on fuzzy manifold are reviewed [2, 4]. For the start we consider the geometry for which its fundamental set is unambiguously defined, later this assumption, in fact, will be dropped. To illustrate FT formalism, let's consider it first for some discrete space. If its fundamental set $S$ is totally ordered set, then for its elements $\{a_i\}$ the ordering relation between the element pairs $a_k \le a_l$ (or vice versa) is fulfilled. But if $S$ is the partially ordered set (Poset), then some its element pairs can enjoy the incomparability relations (IR) between them: $a_j \sim a_k$. If this is the case, then both $a_j \le a_k$ and $a_k \le a_j$ propositions are false, and such structure acquires some nontrivial properties [11]. For instance, consider 2-dimensional discrete plane $D$ with elements $d_{ij} = \{x_i, y_j\}$ where all $x_i, y_i$ are integer numbers. Suppose that the ordering relation is defined from the comparison of both $d$ coordinates, i.e. $d_{ij} \le d_{kl}$ iff $x_i \le x_k.and.y_j \le y_l$. Then if such relation isn't fulfilled or for $x$ coordinate or for $y$ (but not for both of them simultaneously), it means that $d_{ij} \sim d_{kl}$ [11].

As further example, important for our formalism, consider poset $S = A^p \cup B$, which includes the subset of 'incomparable' elements $A^p = \{a_j\}$, and ordered subset $B = \{b_i\}$. For the simplicity suppose that in $B$ the element's indexes grow correspondingly to their ordering, so that $\forall i,\ b_i \le b_{i+1}$. Let's consider $B$ interval $\{b_l, b_n\}$ and suppose that some $A^p$ element $a_j$ is confined in $\{b_l, b_n\}$, i.e. $b_l \le a_j\,;\ a_j \le b_n$, and simultaneously $a_j$ is incomparable with all internal $\{b_l, b_n\}$ elements: $b_i \sim a_j$; $\forall i\,;\ l+1 \le i \le n-1$. In this case $a_j$ can be regarded as 'smeared' over such interval, which is the rough analogue of $a_j$ coordinate uncertainty relative to $B$ 'coordinate', if to consider the sequence of $B$ elements as the analogue of coordinate axe. The generalization of poset structure is the tolerance space for which the ordering relations can be nontransitive, the similar property possesses some quantum structures [3, 12].

It's possible to detalize the described smearing introducing the fuzzy relations, for that purpose one can put in correspondence to each $a_j, b_i$ pair of $S$ set the weight $w_i^j \ge 0$ with the norm $\sum_i w_i^j = 1$. In this case $S$ is fuzzy set, $A^p$ elements $\{a_j\}$ called the fuzzy points (FP) [3, 13]. For the example considered above, one can ascribe $w_i^j = \frac{1}{n-l+1}$ to all $b_i$ inside $\{b_l, b_n\}$ interval, $w_i^j = 0$ for other $b_i$. In the simplest case the continuous fuzzy set $C^F$ is defined analogously to discrete one: $C^F = A^P \cup X$ where $A^P$ is the same discrete subset, $X$ is the continuous ordered subset, which is equivalent to $R^1$ axe of real numbers. Correspondingly, fuzzy relations between elements $a_j, x$ are described by real function $w^j(x) \geq 0$ with the norm $\int w^j dx = 1$. The ordered point $x_a$ is characterized in this framework by $w^a(x) = \delta(x - x_a)$. Remind that in 1-dimensional Euclidean geometry, the elements of its manifold $X$ are the points $x_a$ which constitute the ordered continuum set. Yet in 1-dimensional geometry equipped with FT the position of fuzzy point $a_j$ becomes the positive normalized function $w^j(x)$ on $X$; $w^j$ dispersion $\sigma_x$ characterizes $a_j$ coordinate uncertainty on $X$. If metric is defined on $X$ then $C^f$ is called fuzzy manifold. Note that FT realm incorporates several alternative formalisms in which different FP definitions are exploited, we use here the one given in [13], in fact, it's the geometric analogue of real fuzzy number [12].

We shall suppose that the geometry of physical world corresponds to geometry equipped with FT described here. Note that in such formalism $w^j(x)$ doesn't have any probabilistic meaning but only the algebraic one, characterizing the properties of fuzzy value $\tilde{x}_j$. To describe the distinction between the fuzzy structure and probabilistic one, the correlation $K_0(x, x')$ defined over $w_j$ support can be introduced; thus if $w_j(x_{1,2}) \neq 0$, then $\forall x_1, x_2$; $K_0(x_1, x_2) = 1$ for FP $a_j$ and $K_0(x_1, x_2) = 0$ for probabilistic $a_j$ distribution. Thus $a_j$ state $G$ on $X$ is described by two functions $G = \{w_j(x), K_0(x, x')\}$. As will be shown below, the similar bilocal correlations define, in fact, the dynamical properties of massive particles.

## 3 Linear Model of Fuzzy Dynamics

In the described framework the massive particle of 1-dimensional classical mechanics corresponds to the ordered point $x_a(t) \in X$. By the analogy, we suppose that in 1-dimensional fuzzy mechanics (FM) the particle $m$ corresponds to fuzzy point $a(t)$ in $C^F$ characterized by normalized positive density $w(x, t)$. Beside $w(x)$, $m$ fuzzy state $|g\}$ can also depend on other $m$ degrees of freedom (DFs) characterizing its evolution. To illustrate it, consider $m$ average velocity:

$$\bar{v} = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} x w(x) dx = \int_{-\infty}^{\infty} x \frac{\partial w}{\partial t}(x, t) dx \tag{1}$$

It's reasonable to assume that $\bar{v}(t)$ is independent of $w(x, t)$, below we shall look for such additional DFs in form of real functions $q_{1,\dots,n}(x, t)$. Let's suppose that in FM $m$ evolution is local, i.e.:

$$\frac{\partial w}{\partial t}(x, t) = -\Phi(w, q_1, \dots, q_n) \tag{2}$$

where $\Phi$ is an arbitrary function which depends on $w(x, t), q_{1,\dots,n}(x, t)$ only. From $w$ norm conservation:

$$\int_{-\infty}^{\infty} \Phi(x, t) dx = -\int_{-\infty}^{\infty} \frac{\partial w}{\partial t}(x, t) dx = -\frac{\partial}{\partial t} \int_{-\infty}^{\infty} w(x, t) dx = 0 \tag{3}$$

If to substitute: $\Phi = \frac{\partial J}{\partial x}$ where $J(x)$ is some differentiable function, then (3) demands:

$$J(\infty, t) - J(-\infty, t) = 0 \tag{4}$$

If to suppose that such equality is fulfilled, then $J(x)$ can be regarded as $w$ flow, and (2) is equivalent to 1-dimensional flow continuity equation [16]:

$$\frac{\partial w}{\partial t} = -\frac{\partial J}{\partial x} \tag{5}$$

Meanwhile, $J(x)$ can be decomposed formally as: $J = w(x)v(x)$ where $v(x)$ corresponds to 1-dimensional $w$ flow velocity [16]. In these terms (5) can be written as:

$$\frac{\partial w}{\partial t} = -v\frac{\partial w}{\partial x} - \frac{\partial v}{\partial x}w \tag{6}$$

We shall assume that $v(x)$ can be considered as novel $m$ DF. Note that for normalized density $w(x,t)$ the condition expressed by (4) is trivial, in particular, it's fulfilled if $w$ flow $J(x,t)$ from/to $x = \pm\infty$ is negligible.

FM will be constructed here as minimal theory in a sense that at every step we shall choose the option with minimal number of DFs and theory constants. In such framework one should choose the consistent $|g\}$ ansatz, yet $m$ state representation in form of the line: $|g\} = \{w, v\}$ is asymmetric relative to its norm $w(x)$. Besides, the evolution of its component $w$ described by (6) is nonlinear. Hence it's instructive to look for symmetric $|g\}$ representation $\eta(x)$ for which its evolution would be linear; such ansatz can be complex, quaternionic or some other symmetric form. In general, $\eta(x) = \Upsilon_x(w, v)$ where $\Upsilon_x$ is some $w, v$ functional and $x$ is its parameter. However, $\eta$ norm corresponds to $w(x)$, hence if $w(x) \to 0$ for some $x \to x_0$ , then $\eta(x)$ supposedly also should be negligible in $x_0$ vicinity. Hence $\eta$ can be decomposed also as:

$$\eta(x) = \Upsilon_x(w, v) = f_r[w(x)]F_x(w, v) \tag{7}$$

where $f_r$ is some real function, such that $f_r(\epsilon) \to 0$ for $\epsilon \to 0$; $F_x$ is an arbitrary functional. In this vein $|g\}$ is characterized by two DFs, so it's worth to start $\eta$ ansatz search from complex $F_x$.

Plainly, $m$ evolution as the whole can be characterized by $m$ velocity $u(t)$ with expectation value $\bar{u}(t)$. Yet in FM, alike $m$ coordinate $x$, such $u$ also can be also considered as fuzzy value $\tilde{u}(t)$ with corresponding normalized distribution $w_u(u,t)$ which can be defined in $u$ measurements. Obviously, $\bar{u}(t)$ coincides with $\bar{v}(t)$ of (1), hence:

$$\bar{u} = \int_{-\infty}^{\infty} v(x)w(x)dx \tag{8}$$

i.e. its value is also defined by $w, v$. In place of $u$, below it will be convenient to use the variable $p = \mu u$ where $\mu$ is the theory constant; for the corresponding distribution $w_p(p)$, it gives $w_u(u) = \mu w_p(\mu u)$. If $|g\}$ is physical state then analogously to QM, the expectation value of arbitrary $m$ observable $Q$ in FM is supposedly expressed as some $\eta$ functional. In particular, $w_p(p) = F_p(\eta)$ where $F_p$ is parameter dependent functional. The transformation $\eta \to w_p$ should possess the following properties:

i) Norm conservation: if $\eta(x)$ is normalized, then the same is true for $w_p$.
ii) $p$ expectation value $\bar{p}$ is expressed via $w(x)$, $v(x)$ according to (8).
iii) For free $m$ evolution $w_p$ is independent of $\eta(x) \to \eta(x + x_0)$ space shifts.

For complex $\eta(x)$ its fourier transform satisfies to these conditions [15]. To demonstrate it and calculate $w_p$, let's introduce the auxiliary form: $\varphi(p) = w_p^{\frac{1}{2}} e^{i\beta(p)}$, here $\beta(p)$ is dummy real function on which the final $w_p$ ansatz wouldn't depend. In accordance with equality (7), $\eta(x)$ can be written as:

$$\eta(x) = f_r(w)G_x(w, v)e^{i\lambda_x(w,v)} \tag{9}$$

where $G_x(w, v)$, $\lambda_x(w, v)$ are real functionals. Consider then $\varphi$ fourier decomposition on $X$:

$$\varphi(p) = \int_{-\infty}^{\infty} \eta(x)e^{-ipx}dx = \int_{-\infty}^{\infty} f_r G_x e^{i\lambda_x - ipx}dx \tag{10}$$

$w_p$ is normalized distribution, so the application of Plancherele formulae to that norm gives:

$$\int_{-\infty}^{\infty} w_p(p)dp = \int_{-\infty}^{\infty} \varphi(p)\varphi^*(p)dp = \int_{-\infty}^{\infty} f_r^2(w)G_x^2 dx = 1 \tag{11}$$

The calculation of $\delta w$ variation for the equality:

$$\int_{-\infty}^{\infty} [f_r^2(w)G_x^2 - w]dx = 0 \tag{12}$$

permits to settle $G_x = 1$ and $f_r(w) = \pm w^{\frac{1}{2}}(x)$. Then $\bar{p}$ can be calculated anew from 2-nd Plancherele formulae :

$$\bar{p} = \int_{-\infty}^{\infty} p\varphi(p)\varphi^*(p)dp = \int_{-\infty}^{\infty} \frac{\partial\lambda_x}{\partial x} f_r^2(w)dx \tag{13}$$

From the comparison with (8), since $\bar{p} = \mu\bar{u}$ it follows: $\lambda_x = \gamma(x) + \chi(w)$ where $\gamma$ is the functional:

$$\gamma(x) = \mu \int_{-\infty}^{x} v(\xi)d\xi + c_\gamma \tag{14}$$

here $c_\gamma$ is an arbitrary real number. $\chi(w)$ is an arbitrary real function which obeys to the condition:

$$\int_{-\infty}^{\infty} w\frac{\partial\chi}{\partial x}dx = 0 \tag{15}$$

so it can be regarded as the analogue of $\eta$ gauge. As the result, $w_p$ and $\beta(p)$ can be calculated from (10) as functions of $\chi$. In particular:

$$w_p(p) = |\int_{-\infty}^{\infty} w^{\frac{1}{2}}e^{i\gamma+i\chi-ipx}dx|^2 \tag{16}$$

is independent of $\beta(p)$, so $w_p$ is really $w$, $v$ functional, on the all appearances for minimal theory such $w_p$ ansatz is unique. $\beta(p)$ is, in fact, the analogue of $\gamma(x)$ for $p$ observable. The resulting $m$ state in $x$-representation:

$$\eta(x) = w^{\frac{1}{2}}(x)e^{i\gamma+i\chi} \tag{17}$$

is the vector (ray) of complex Hilbert space $\mathcal{H}$. In this framework, the observable $p$ corresponds to the operator $\hat{p} = -i\frac{\partial}{\partial x}$ acting on $\eta$, i.e. $\bar{p} = \int \eta^*\hat{p}\eta dx$. It means that $x$ and $p$ observables are described by the linear self-adjoint operators, which obey to the commutation relation $[\hat{x}, \hat{p}] = i$. By the analogy, we suppose that all $m$ PV observables $\{Q\}$ are the linear, self-adjoint operators on $\mathcal{H}$. If this is the case, $\eta(x)$ is the plausible candidate for $|g\}$ state ansatz in $X$-representation, because for such $\eta$ the expectation values of all observables $\bar{Q}(t)$ should be expressed as semi-linear $\eta$ functionals.

Note that $\eta = e^{i\chi} g$, where $g(x, t)$ is standard QM wave function, so that $\eta(x, t)$ is its trivial map. Thus we can study first $g(x, t)$ evolution, and then $\eta(x, t)$ properties will be derived basing on obtained results. Evolution equation for $g$ is supposed to be of the first order in time, i.e.:

$$i \frac{\partial g}{\partial t} = \hat{H} g. \tag{18}$$

In general $\hat{H}$ is nonlinear operator, for the simplicity we shall consider first the linear case and turn to nonlinear one in the next section. The free $m$ evolution is invariant relative to $x$ space shifts to arbitrary $x_0$ performed by the operator $\hat{W}(x_0) = \exp(x_0 \frac{\partial}{\partial x})$. Because of it, the corresponding operator $\hat{H}_0$ should commute with $\hat{W}(x_0)$ for the arbitrary $x_0$, i.e. $[\hat{H}_0, \frac{\partial}{\partial x}] = 0$. It holds only if $\hat{H}_0$ is differential polinom of the form:

$$\hat{H}_0 = -\sum_{l=1}^{n} b_l \frac{\partial^l}{\partial x^l} \tag{19}$$

where $b_l$ are arbitrary real constants, $n \geq 2$. From $X-$reflection invariance $b_l = 0$ for noneven $l$. Suppose that the action of external field on $m$ can be accounted in $\hat{H}$ additively: $\hat{H} = \hat{H}_0 + V(x, t)$ where $V$ is real nonsingular function. Let's rewrite (18) separating $w, \gamma$ derivatives:

$$i \frac{\partial g}{\partial t} = \left( i \frac{\partial w^{\frac{1}{2}}}{\partial t} - w^{\frac{1}{2}} \frac{\partial \gamma}{\partial t} \right) e^{i\gamma} = e^{i\gamma} \hat{Z} g \tag{20}$$

where $\hat{Z} = e^{-i\gamma} \hat{H}$. Hence:

$$\frac{\partial w^{\frac{1}{2}}}{\partial t} = im(\hat{Z} g) \tag{21}$$

Yet if to substitute $v(x)$ by $\gamma(x)$ in (6) and transform it to $w^{\frac{1}{2}}$ time derivative, then:

$$\frac{\partial w^{\frac{1}{2}}}{\partial t} = -\frac{1}{\mu} \frac{\partial w^{\frac{1}{2}}}{\partial x} \frac{\partial \gamma}{\partial x} - \frac{1}{2\mu} w^{\frac{1}{2}} \frac{\partial^2 \gamma}{\partial x^2} \tag{22}$$

Plainly, this expression and $im(\hat{Z} g)$ should coincide, then $\hat{H}$ can be obtained from their comparison term by term. In particular, the imaginary part of $\hat{Z} g$ includes the highest $\gamma$ derivative as the term $b_n w^{\frac{1}{2}} \frac{\partial^n \gamma}{\partial x^n}$, yet for (22) the highest $\gamma$ derivative is proportional to $w^{\frac{1}{2}} \frac{\partial^2 \gamma}{\partial x^2}$. Hence it gives: $b_2 = \frac{1}{2\mu}$ and for all $l > 2$ it follows that $b_l = 0$, only in this case both expressions for $\frac{\partial w^{\frac{1}{2}}}{\partial t}$ would coincide. Thus $g$ free evolution is described by the only $\hat{H}_0$ term $b_2 = \frac{1}{2\mu}$, so $\hat{H}$ is Schroedinger Hamiltonian for particle with mass $\mu$. The obtained ansatz gives also $J(\pm\infty, t) = 0$ for $w$ flow of (3), in accordance with our assumptions. Note that in standard QM $m$ evolution equation is, in fact, postulated *ad hoc*; here it's derived from FT premises for particle evolution on fuzzy manifold. The same is true for the commutation relation $[\hat{x}, \hat{p}] = i$.

In this framework the flow velocity $v(x, t)$ isn't observable, but can be formally defined as the ratio of $J(x)$, $w(x)$ observable expectation values, here $w$ observable is described by the projection operator $\hat{\Pi}(x)$; the operator $\hat{J}(x)$ considered in [14]. As was noticed earlier, the particle evolution in QM in some aspects is similar to the motion of continuous media

[16]. This analogy is explored thoroughly in hydrodynamical QM model (QFD) [17, 18], its connection with FM will be discussed in Section 5.

Plainly, $\gamma(x)$ corresponds to $|g\}$ quantum phase, so that:

$$K_1(x, x') = \gamma(x) - \gamma(x')$$

describes the phase correlation between the state components in $x, x'$. Such correlations induce, in fact, the interference effects between $|g\}$ components in $x, x'$. As was noticed above, FM state can be characterized by the density $w(x)$ and and the array of bilocal geometric correlations $\{K_i(x, x')\}$, the first of them: $K_0(x, x')$ was introduced in Section 2, so that $K_1(x, x')$ can be also regarded as $K_i$ component. Until now we've considered only the pure fuzzy states which aren't the probabilistic mixture of several pure states. The mixed states in FM are defined exactly like in QM formalism, i.e. are positive, trace one operators $\rho$ on $\mathcal{H}$. For pure $m$ states:

$$\rho(x, x') = g(x)g^*(x') = [w(x)w(x')]^{\frac{1}{2}} e^{iK_1(x,x')}$$

is equivalent to $g(x)$, yet such $|g\}$ representation demonstrates in the open the correlation structure of $m$ pure states.

## 4 General Fuzzy Dynamics

In the previous Section 1-dimensional FM formalism was derived from FT premises assuming that $|g\}$ evolution is linear and $|g\}$ gauge function $\chi(x)$ can be neglected. Now these assumptions will be dropped one by one and general formalism derived. Concerning with nonlinear evolution, the conditions of QM linearity were reconsidered by Jordan, and turn out to be essentially weaker than Wigner theorem asserts [19]. In particular, it was proved that if the evolution maps the set of all pure states one to one onto itself, and for arbitrary mixture of orthogonal states $\rho(t) = \sum P_i(t)\rho_i(t)$ all $P_i$ are independent of time, then such evolution is linear. Here $\rho_i(x, x', t) = g_i(x, t)g_i^*(x', t)$ are the density matrixes of orthogonal pure states $g_i$. Yet for the considered FM formalism first condition is, in fact, generic: no mixed (i.e. probabilistic) state can appear in the evolution of pure fuzzy state. The second condition involves the probabilistic mixture of such orthogonal states and also seems to be rather weak assumption.

Now let's return to $\eta(x)$ ansatz of (17), it can be shown that Jordan theorem demands also that $\chi(w) = 0$. For $m$ states of (17) and corresponding $g$ ansatz, if $\langle g_i|g_j\rangle = \delta_{ij}$, then $\langle \eta_i|\eta_j\rangle = \delta_{ij}$ and vice versa. As was argued above, in FM any pure state $g(t_0)$ should evolve to pure state $g(t)$ for arbitrary $t$, so the same should be true for any $\eta(t_0)$. Now Jordan theorem can be applied to $\eta$ evolution, to demonstrate it consider $g$ evolution equation:

$$i\frac{\partial g}{\partial t} = i\frac{\partial}{\partial t}(\eta e^{-i\chi}) = i\frac{\partial \eta}{\partial t}e^{-i\chi} + \eta\frac{\partial \chi}{\partial w}\frac{\partial w}{\partial t}e^{-i\chi} = \hat{H}(\eta e^{-i\chi}) \tag{23}$$

From it one can come to the equation for $\eta$, the term containing $\frac{\partial w}{\partial t}$ can be rewritten according to (22). As the result, it gives:

$$i\frac{\partial \eta}{\partial t} = e^{i\chi}\hat{H}(\eta e^{-i\chi}) + \frac{\eta}{\mu}e^{i\chi}\frac{\partial \chi}{\partial w}\frac{\partial}{\partial x}\left(w\frac{\partial \gamma}{\partial x}\right) \tag{24}$$

Resulting equation for $\eta$ is also of first time order, but is openly nonlinear. Therefore, for arbitrary $\chi(w)$, given the initial $\eta(x, t_0)$, the resulting $\eta(x, t)$ can be only the equivalence class of $g(x, t)$ which evolves linearly from:

$$g(x, t_0) = \eta(x, t_0)e^{-i\chi} \tag{25}$$

3-dimensional FM, in fact, doesn't demand any principal modification of described formalism. In this case the fundamental set is $C^F = A^P \cup R^3$ where $A^P$ defined in Section 2, hence for any fuzzy point $a_j \in A^P$ its properties should be defined relative to $X$, $Y$, $Z$ coordinate axes separately. Rotational symmetry implies that FP $a_j$ is described by the positive function $w^j(\vec{r})$ with norm $\int w^j d^3r = 1$. If the particle $m$ corresponds to the fuzzy point $a(t)$ characterized by $w(\vec{r}, t)$, then analogously to Section 2, given $w$ evolution depends on local parameters only, it can be expressed as:

$$\frac{\partial w}{\partial t}(\vec{r}, t) = -\Phi(\vec{r}, t) \tag{26}$$

where $\Phi$ is an arbitrary local function. Then from $w$ norm conservation 3-dimensional flow continuity equation follows:

$$\frac{\partial w}{\partial t} = -div\vec{J} \tag{27}$$

One can decompose $\vec{J} = w\vec{v}$ and consider $w$ flow velocity $\vec{v}(\vec{r})$ as independent $|g\}$ parameter. $m$ state $|g\}$ is supposed to be the complex $w, \vec{v}$ functional $g(\vec{r}) = \Upsilon_{\vec{r}}(w, \vec{v})$. For $m$ as the whole, its velocity is supposedly characterized by fuzzy vector $\tilde{\vec{u}}$ which corresponds to distribution $w_u(\vec{u})$, so that:

$$\langle\vec{u}\rangle = \int \vec{u}w_u(\vec{u})d^3u = \int \vec{v}(\vec{r})w(\vec{r})d^3r \tag{28}$$

$m$ kinematical fuzzy momentum defined as: $\vec{p} = \mu\vec{u}$. From that, analogously to (8 - 17), standard QM ansatz for $m$ state obtained: $g(\vec{r}) = w^{\frac{1}{2}}e^{i\gamma}$. $g$ phase $\gamma(\vec{r})$ obeys to the equality $\mu\vec{v} = grad(\gamma)$. To guarantee the formalism consistency, we assume that the phase correlation value $K_1(\vec{r}, \vec{r'})$ is independent of the path $l$ between $\vec{r}, \vec{r}'$ over which it can be calculated additively :

$$K_1(\vec{r}, \vec{r'}) = \gamma(\vec{r}) - \gamma(\vec{r}') = \int_{\vec{r'}}^{\vec{r}} grad(\gamma)d\vec{l} \tag{29}$$

Considering $g$ linear evolution, for free $m$ evolution its operator $\hat{H}_0$ should be the even polinom of the form:

$$\hat{H}_0 = -\sum_{l=1}^{n} b_{2l}\frac{\partial^{2l}}{\partial\vec{r}^{2l}} \tag{30}$$

If the external field action can be described by the addition of real function $V$ to it:

$$\hat{H} = \hat{H}_0 + V(\vec{r}, t) \tag{31}$$

then from $\frac{\partial g}{\partial t}$ the term $\frac{\partial w^{\frac{1}{2}}}{\partial t}$ can be extracted and expressed via $w, \gamma$ $\vec{r}$-derivatives. From their comparison with corresponding $\hat{H}g$ derivatives the Schroedinger equation is obtained for $m$ evolution. The applicability of Jordan theorem to 3-dimensional $\hat{H}$ is obvious, because the derivation of $\hat{H}$ linearity doesn't depend on the dimensiality of coordinate space. The same is true for the proof of uniqueness of $g(\vec{r}, t)$ ansatz, i.e. that $\chi(w) = 0$.

In our derivation of evolution equation we didn't assume Galilean invariance of FM, rather in our approach it follows from the obtained evolution equation if the reference frame (RF) is regarded as the physical object with mass $\mu \to \infty$ [6]. For the transition to relativistic FM from our ansatz its natural extension for complex scalar state $g$ is Klein-Gordon equation. Yet for such equation it's impossible to define $m$ probability density $\rho(\vec{r})$ which would be nonnegative for all free states [14]. As was noticed in Section 3, in principle, $m$ scalar state can be complex, quaternionic, octonionic, etc.. We find that the minimal consistent $|g\}$ ansatz gives quaternion scalar $\xi(\vec{r})$, so that:

$$\left(\frac{\partial^2}{\partial t^2} - \triangle + \mu^2\right)\xi = 0 \tag{32}$$

For such state the single quantum $g$ phase $\gamma$ is extended to three independent phases: $i\gamma + j\beta + k\alpha$ which correspond to additional geometric DFs. Such DFs can be considered as $m$ phase correlations $K_{1,2,3}$ analogous to (29). To get nonnegative $\rho$ one should broke first $i, j, k$ space symmetry and to choose an arbitrary preferred basis $i', j', k'$. Plainly, in this basis $\xi = \psi_1 + \psi_2 j'$, here $\psi_{1,2} = a_{1,2} + b_{1,2} i'$ where $a_{1,2}, b_{1,2}$ are real functions. Let's rewrite $\psi_{1,2}$ in form of spinor $\psi_u$ and define the auxiliary spinor:

$$\psi_d = -i'\left(I\frac{\partial}{\partial t} + \frac{\partial}{\partial \vec{r}}\vec{\sigma}\right)\psi_u \tag{33}$$

in obvious notations. If to denote $\psi_d$ components as $\psi_{3,4}$, then $\rho(\vec{r}) = \sum_1^4 |\psi_l|^2$ is nonnegative and normalizable for arbitrary $\xi$. If to regard $\psi_{1,\ldots,4}$ as 4-spinor components, it's easy to check that such 4-spinor would obey to Dirac equation in chiral representation [20]. It seems that in FM some geometric DFs can be 'compactified', resulting in the appearance of internal spinor space, so that the particle $m$ acquires spin $\frac{1}{2}$.

Now we shall consider the interaction between fuzzy states in FM framework. Note first that by derivation in FM the free Hamiltonian $\hat{H}_0$ induces, in fact, $\mathcal{H}$ dynamical asymmetry between $|\vec{r}\rangle$ and $|\vec{p}\rangle$ 'axes'. As follows from (19-22) $m$ free dynamics can be described by the system of two equations which define $\frac{\partial w^{\frac{1}{2}}}{\partial t}$ and $\frac{\partial \gamma}{\partial t}$ which for 3-dimensions are equal to:

$$\begin{aligned}\frac{\partial w^{\frac{1}{2}}}{\partial t} &= -\frac{1}{\mu}\frac{\partial w^{\frac{1}{2}}}{\partial \vec{r}}\frac{\partial \gamma}{\partial \vec{r}} - \frac{1}{2\mu} w^{\frac{1}{2}}\frac{\partial^2 \gamma}{\partial \vec{r}^2}\\ \frac{\partial \gamma}{\partial t} &= -\frac{1}{2\mu}\left[\left(\frac{\partial \gamma}{\partial \vec{r}}\right)^2 - \frac{1}{w^{\frac{1}{2}}}\frac{\partial^2 w^{\frac{1}{2}}}{\partial \vec{r}^2}\right]\end{aligned} \tag{34}$$

Yet the first of them is equivalent to (27) which describes just $w(\vec{r})$ balance and so is, in fact, kinematical one and can't depend on any interactions directly. Namely, under some external influence the values of $w, \gamma$ variables can change, but no new terms can appear in that equation. Note that in QM $e\vec{A}$ term formally appears in it, but it's just the part of the expression for kinematic momentum [14]. Hence $m$ interactions can be accounted only via the modification of second equation of system (34). Assuming that analogously to (31) the evolution terms are real and additive, it gives: $\hat{H} = \hat{H}_0 + \hat{H}_{int}$ where $\hat{H}_{int}$ is the interaction term. Let's consider how the interaction of two particles $m, M$ can be described in such approach. Suppose also that $m, M$ interaction is universal in a sense that $\langle \hat{H}_{int}\rangle \neq 0$ for arbitrary relative $m, M$ momentum $\langle \vec{p}_{12}\rangle$, and is induced by the conserved charges $q_1, q_2$. Then the main $\hat{H}_{int}$ term which survives at $\langle \vec{p}_{12}\rangle \to 0$ is equal to $q_1 q_2 U(r_{12})$. as the result, $U(r_{12})$ corresponds to the classical potential. In standard QM such interaction is, in fact, postulated from classical-to-quantum correspondence, whereas here it follows from FM geometric premises. Since $\gamma$ corresponds to the quantum phase, it supposes that in FM $m$ interactions can possess some form of local gauge invariance [21].

## 5 Conclusion

It's well known that QM can be described by several alternative formalisms, of them the most notorious are algebraic QM and Schroedinger or standard formalism. To discuss the possible advantages of FM formalism it's instructive to compare it with the latter one. From the formal side, standard QM exploits two fundamental structures of different nature: space-time manifold $R^3 * T$ and functional space $\mathcal{H}$ defined on $R^3$. In distinction, FM formalism involves only one basic structure, it's fuzzy manifold $\tilde{R}^3 * T$. FM physical states are $\tilde{R}^3$ points, their equivalence to $\mathcal{H}$ Dirac vectors was proved here. In standard QM the evolution equation or postulated *ad hoc* or derived assuming Galilean invariance of object states [14]. In FM the Schroedinger equation is derived assuming only space-time shift invariance which is essentially weaker assumption. Besides, the quantum-classical transition in such theory is essentially more simple, it's just the transition of $\tilde{R}^3$ manifold by $R^3$ one, for which the classical particles correspond to ordered points. As the result, FM formalism possesses simple and logical axiomatics which origin is basically geometrical. It permits, in principle, to explore under the new angles those quantum theories for which geometry is the formalism cornerstone, first of all these are quantum gravity and gauge fields.

Concerning with the connection between FM and QFD noticed in Section 2, in the latter theory all QM axioms are accepted at the initial stage. Then, the postulated Schroedinger equation is rewritten as the system of equations for the motion of classical liquid, similar to (34) [18]. Yet at the next stage, to reach the complete classicality of the theory some *ad hoc* assumptions are added, however the resulting theories contradict to experimental results. In comparison, FM is principally nonclassical theory, this nonclassicality originates from the novel topological structure of space-time, whereas in standard QM formalism the space-time geometry is the same, as in classical mechanics.

In our approach the state space is defined by geometry and corresponding dynamics i.e. is derivable concept. For pure states of free nonrelativistic particle $m$ it obtained to be equivalent to $\mathcal{H}$, but, in principle, it can be different for other systems. The similar features possess the formalism of algebraic QM where the state space is defined by the observable algebra and system dynamics [14]. As was noticed in Section 3, in FM the most consistent state ansatz is given by the density matrix $\rho$. However, the direct derivation of $\rho$ evolution equation is more complicated then for Dirac vector $|g\}$, and because of it, we used $|g\}$ throughout our paper. Planck constant $\hbar = 1$ in our FM ansatz, but the same value ascribed to it in relativistic unit system in which the velocity of light $c = 1$; in FM framework $\hbar$ only connects $x$, $p$ geometric scales and doesn't have any other meaning.

In conclusion, we have shown that the quantization of elementary systems can be derived directly from axiomatic of set theory and topology together with the natural assumptions about system evolution. It allows to suppose that the quantization phenomenon has its roots in foundations of mathematics [14]. Our approach permits to construct QM formalism starting from geometric concepts and structures only, so in these aspects it's analogous to general relativity construction. In the same time the considered fuzzy manifold describes the possible variant of fundamental pregeometry which is basic component of some quantum gravity theories [5]. In this vein, FM provides the interesting opportunities, being generically nonlocal theory which, in the same time, can possess Lorentz covariance and local gauge invariance.

## References


1. Marmo, G., Volkert G.F.: Phys. Scr. **82**, 038117–038129 (2010)
2. Dodson, C.T.: Bull. London Math. Soc. **6**, 191–197 (1974)
3. Zeeman, C. In: Fort, K. (ed.): Topology of 3-manifolds, pp. 240–248. Prentice-Hall, New Jersey (1961)
4. Sostak, A.P.: J. Math. Sci. **6**, 662–697 (1996)
5. Balachandran, A.P., Kurkcuoglu, S., Vaidia, S.: Lectures on Fuzzy and Fuzzy SUSY Physics. World Scientific, Singapore (2007)
6. Mayburov, S.: J. Phys. A **41**, 164071–164080 (2007)
7. Mayburov, S.: Int. J. Theor. Phys. **49**, 3192–3198 (2010)
8. Mayburov, S.: Phys. Part. Nucl. Lett. **11**, 1–9 (2014). Quant-ph:1205.3019
9. Pykaz, J.: Found. Phys. **30**, 1503–1515 (2000)
10. Ali, T., Emch, G.: Journ. Math. Phys. **15**, 176–183 (1974)
11. Schroder, B.: Ordered Sets: An Introduction. Birkhauser, Boston (2003)
12. Dubois, D., Prade, H.: Fuzzy Sets and Systems. Theory and Applications. Academic Press, New York (1980)
13. Bandemer, H., Gottwald, S.: Fuzzy sets, Fuzzy Logics, Fuzzy Methods with Apllications. Wiley, New York (1995)
14. Jauch, J.M.: Foundations of Quantum Mechanics. Addison-Wesly, Reading (1968)
15. Reed, M., Simon, B.: Methods of Modern Mathematical Physics; vol. 2, Fourier Analysis. Academic Press, New York (1972)
16. Landau, L.D., Lifshitz, E.M.: Mechanics of Continuous Media. Pergamon Press, Oxford (1976)
17. Madelung, E.: Z. Phys. **40**, 332–338 (1926)
18. Ghosh, S.K., Deb, B.M.: Phys. Rep. **92**, 1–64 (1982)
19. Jordan, T.: Phys. Rev. A **73**, 022101–22109 (2006)
20. Fayyazuddin and Riazuddin: Quantum Mechanics, pp. 445–461. W. S., Singapore (1990)
21. Cheng, T., Li, L.: Gauge Theory of Elementary Particles. Claredon, Oxford (1984)