Int J Theor Phys (2015) 54:4068–4085
DOI 10.1007/s10773-014-2468-z

The Spectrum of a Harmonic Oscillator Operator Perturbed by Point Interactions

Boris S. Mityagin

Received: 28 October 2014 / Accepted: 12 December 2014 / Published online: 15 January 2015

Abstract We consider the operator $Ly = -(d/dx)^2 y + x^2 y + w(x)y, \quad y \text{ in} L^2(\mathbb{R})$, where $w(x) = s\delta(x-b) + t\delta(x+b), \quad b \neq 0$ real, $s, t \in \mathbb{C}$. This operator has a discrete spectrum: eventually the eigenvalues are simple. Their asymptotic is given. In particular, if $s = -t$, $\lambda_n = (2n+1) + s^2 \frac{\kappa(n)}{n} + \rho(n)$ where $\kappa(n) = \frac{1}{2\pi}\left[(-1)^{n+1}\sin\left(2b\sqrt{2n}\right) - \frac{1}{2}\sin\left(4b\sqrt{2n}\right)\right]$ and $|\rho(n)| \leq C\frac{\log n}{n^{3/2}}$. If $\overline{s} = -t$, the number $T(s)$ of non-real eigenvalues is finite, and $T(s) \leq (C(1+|s|)\log(e+|s|))^2$. The analogue of the above asymptotic is given in the case of any two-point interaction perturbation.

Keywords Spectral Theory · Harmonic Oscillators · Asymptotics

1 Introduction

The operator

$$L^0 = -\frac{d^2}{dx^2} + x^2, \quad x \in \mathbb{R}^1,$$

is the one-dimensional harmonic oscillator; this is an unbounded self-adjoint operator acting in $L^2(\mathbb{R})$. As one can see in any introductory book on quantum mechanics, L^0 has a discrete spectrum $\Lambda^0 = \{z_n\}_{n=0}^{\infty}$,

$$z_n = 2n+1, \quad n = 0, 1, \ldots$$

and a compact resolvent

$$R^0(z) = (z - L^0)^{-1}, \quad z \notin \Lambda^0. \tag{1.1}$$

A normalized orthogonal system of eigenfunctions can be chosen as the Hermite functions

$$h_n(x) = \left(\pi^{1/2} 2^n n!\right)^{-1/2} e^{-x^2/2} H_n(x), \quad n = 0, 1, \ldots \tag{1.2}$$

B. S. Mityagin (✉)
Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, OH 43210, Ohio, USA

where

$$H_n(x) = e^{x^2/2}\left(e^{-x^2/2}\right)^{(n)} \tag{1.3}$$

are the Hermite polynomials.

Spectral analysis of perturbed operators

$$L = L^0 + W \tag{1.4}$$

with special W, in particular, the point interaction perturbations

$$Wf = wf, \quad w(x) = \sum\nolimits_{j=1}^{J} c_j \delta\left(x - b_j\right), \quad J \text{ finite} \tag{1.5}$$

was studied in many mathematical and physical papers, for example [3, 4, 6–8, 20, 21, 24, 25].

In the series of papers [12–14] S. Fassari, F. Rinaldi and G. Inglese investigate the spectrum of $L \in$(1.4) when the perturbation

$$W = -\tau\left(\delta\left(x - b\right) + \delta\left(x + b\right)\right), \quad \tau, b > 0, \tag{1.6}$$

i.e., L^0 is perturbed by a pair of attractive point interactions of equal strength whose centers are situated at the same distance from the origin. In this case the operator $L = L^0 + W$ is self-adjoint; the techniques used are based on Green's function analysis.

E. Demiralp ([6–8]) found numerically the non-real eigenvalues of (1.6) when

$$-\tau = i\gamma, \quad \gamma \text{ real}$$

for γ large enough.

H. Cartarius, D. Dast, D. Haag, G. Wunner, R. Eichler, and J. Main [5] and [16], motivated by analysis of Bose-Einstein condensates with $\mathcal{PT}$-symmetric loss and gain, focused on the case of non-Hermitian perturbations

$$W = i\gamma\left[\delta\left(x - b\right) - \delta\left(x + b\right)\right].$$

Their numerical estimates showed that for small γ the spectrum of $L = L^0 + W$ is on the real line $\mathbb{R}$, and they gave some predictions on the state of decay of the disk radii where the eigenvalues of the operator L are located. Now we provide a rigorous mathematical analysis of the asymptotics of eigenvalues $\lambda_n = \lambda_n(L^0 + W)$.

We follow the techniques used in [2, 9–11, 19] and based on careful estimates related to the resolvent representation

$$R = R^0 + \sum\nolimits_{j=1}^{\infty} U_j, \tag{1.7}$$

$$U_0 = R^0, \quad U_k = R^0 W U_{k-1} = U_{k-1} W R^0, \quad k \geq 0. \tag{1.8}$$

Moreover, we essentially use the property of perturbations $W \in$ (1.5) to have such a matrix

$$w_{jk} = \left\langle W h_j, h_k \right\rangle, \quad j, k = 0, 1, \ldots \tag{1.9}$$

that for some $\alpha > 0$ there exists $M > 0$ such that

$$|w_{jk}| \leq \frac{M}{(1+j)^\alpha (1+k)^\alpha}, \quad j, k = 0, 1, \ldots, \tag{1.10}$$

Detailed results on the spectrum and convergence of spectral decompositions of $L = L^0 + W$ for a general W under the condition (1.10) were given by B. Mityagin and P. Siegl in [19]. In the case (1.5) of the finite point interaction perturbations, $\alpha = \dfrac{1}{4}$.

2 Preliminaries, Technical Introduction, Review the Results

1 Our main concern is the harmonic oscillator operator (1.4) and its special perturbation W. We will focus on this case, although many constructions are very general and could be performed in analysis of other differential operators — see [1, 9, 19].

Let L^0 be an operator in $\ell^2(\mathbb{Z}_+)$,

$$L^0 e_k = z_k e_k, \quad z_k = (2k+1), \quad k = 0, 1, \dots, \tag{2.1}$$

and $W = (w_{jk})_0^\infty$ a matrix such that for some $\alpha > 0$ and $C_0 > 0$,

$$|w_{jk}| \le \frac{C_0}{(1+j)^\alpha (1+k)^\alpha} \tag{2.2}$$

Then the KLMN Theorem [17, Chapter 6, §§1 – 4] leads to the definition of the closed operator

$$L = L^0 + W \tag{2.3}$$

with a dense domain — see details in [19]. Let us recall some facts, introduce notations and explain a few elementary but important inequalities.

2 To adjust our constructions to the set of eigenvalues of the unperturbed operator (2.1), let us define strips

$$\begin{aligned} H_n &= \{z \in \mathbb{C} : |\Re z - z_n| \le 1\}, \quad n \ge 1 \\ H_0 &= \{z \in \mathbb{C} : \Re z - z_0 \le 1\} \end{aligned} \tag{2.4}$$

and the squares

$$\mathcal{D}_n = \left\{ z \in H_n : |\Re z - z_n| \le \frac{1}{2}, |\Im z| \le \frac{1}{2} \right\}, \quad n \ge 0 \tag{2.5}$$

around eigenvalues $\{z_n\}_{n=0}^\infty = \{2n+1\}_{n=0}^\infty$ in H_n.

The resolvent

$$R(z) = (z - L^0 - W)^{-1} \tag{2.6}$$

of the operator (2.3) is well-defined in the right half-plane

$$\{z : \Re z \ge 2N_*\} \setminus \bigcup_{k=N_*}^\infty \mathcal{D}_k \tag{2.7}$$

outside of the disks $\mathcal{D}_k$, $k \ge N_*$, if N_* is large enough.

It follows from the Neumann-Riesz decomposition

$$R = R^0 + R^0 W R^0 + R^0 W R^0 W R^0 + \cdots = R^0 + \sum\nolimits_{j=1}^\infty U_j, \tag{2.8}$$

where

$$U_0 = R^0 = (z - L^0)^{-1}, \qquad U_j = R^0 W U_{j-1} = U_{j-1} W R^0, \quad j \ge 1. \tag{2.9}$$

Of course, the convergence of the series should be explained at least in (2.7). This is done in [1, 19]; now I will remind only the estimates of N_* because it will be important later (see Theorem 4.4, (4.38) and Theorem 4.1, (4.11)) in accounting for points of the spectrum $\sigma(L)$ outside of the real line.

3 Define a diagonal operator K,

$$K e_j = \frac{1}{\sqrt{z - z_j}} e_j, \; j = 0, 1, \dots, \Im z \ne 0 \tag{2.10}$$

with understanding that

$$\sqrt{\xi} = r^{1/2} e^{i\varphi/2} \text{ if } \xi = re^{i\varphi}, \quad -\pi < \varphi \le \pi. \tag{2.11}$$

Then $K^2 = R^0$, $z \in \mathbb{C} \setminus \mathbb{R}$; maybe, we lose analyticity but rough estimates – when just the absolute values of matrix elements work well – are good enough.

Indeed, (2.8), (2.9) could be rewritten as

$$R^0 = K^2, \quad U_j = K(KWK)^j K, \quad R = R^0 + \sum\nolimits_{j=1}^{\infty} K(KWK)^j K, \tag{2.12}$$

where

$$(KWK)_{km} = \frac{1}{\sqrt{z - z_k}} W_{km} \frac{1}{\sqrt{z - z_m}}, \quad k, m = 0, 1, 2, \ldots, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{2.13}$$

Lemma 2.1 *Under the assumptions* (2.1)*, and* (2.2)*, with* $0 < \alpha < \frac{1}{2}$*, if* $z \in H_n \setminus \mathcal{D}_n$*, then* KWK *is a Hilbert-Schmidt operator, and*

$$\ell \equiv \|KWK\|_{HS} \le \frac{C_0 M(\alpha) \log(en)}{n^{2\alpha}}, \quad M(\alpha) \equiv 6 + \frac{4/3}{1 - 2\alpha} + \frac{1}{3\alpha} \tag{2.14}$$

Proof If $z \in \partial\mathcal{D}_n$, i.e,

$$z = (2n+1) + \xi + i\eta, \quad |\xi| = \frac{1}{2}, \ |\eta| \le \frac{1}{2}, \quad \text{or} \quad |\xi| \le \frac{1}{2}, \ |\eta| = \frac{1}{2}; \quad \xi, \eta \in \mathbb{R}, \tag{2.15}$$

then

$$\frac{1}{2} \le |z - z_j| \le 3, \ j = n, n \pm 1, \tag{2.16}$$

and if $|n - j| \ge 2$,

$$\frac{3}{2}|n - j| \le 2|n - j| - 1 \le |z - z_j| \le 2|n - j| + 1 \le \frac{5}{2}|n - j|. \tag{2.17}$$

Therefore, by (2.2), (2.13),

$$\ell^2 = \sum\nolimits_{j,k=1}^{\infty} \frac{|w_{jk}|^2}{|z - z_j||z - z_k|} \le C_0^2 \mu^2 \tag{2.18}$$

with

$$\mu = \sum\nolimits_{j=0}^{\infty} \frac{1}{(1 + j)^{2\alpha} |z - z_j|}. \tag{2.19}$$

The sum of three terms for $j = n, n \pm 1$ in (2.19) by (2.16) does not exceed

$$3 \cdot \frac{1}{n^{2\alpha}} \cdot 2 = \frac{6}{n^{2\alpha}}, \tag{2.20}$$

and by (2.17), the remaining part of μ, namely, $\sum_{j=0}^{n-2} + \sum_{j=n+2}^{\infty}$, by the integral test does not exceed

$$\frac{2}{3}\left[\frac{1}{n} + \frac{1}{n^{2\alpha}} + \int_0^{n-1} \frac{dx}{(1+x)^{2\alpha}(n-x)}\right] + \frac{2}{3}\left[\frac{1}{2} \cdot \left(\frac{1}{n+3}\right)^{2\alpha} + \int_{n+2}^{\infty} \frac{dy}{(1+y)^{2\alpha}(y-n)}\right]. \tag{2.21}$$

The first integral (after the change of variables $x = n\xi$) is

$$\frac{1}{n^{2\alpha}}\int_0^{1-(1/n)} \frac{n\,d\xi}{n(1-\xi)\left(\frac{1}{n}+\xi\right)^{2\alpha}} \le \frac{1}{n^{2\alpha}}\left[2\int_0^{1/2}\frac{d\xi}{\xi^{2\alpha}} + 2^{2\alpha}\int_{1/2}^{1-(1/n)}\frac{d\xi}{1-\xi}\right] =$$

$$= \left(\frac{2}{n}\right)^{2\alpha}\left[\frac{1}{1-2\alpha} + \log\frac{n}{2}\right]. \tag{2.22}$$

The second integral in (2.21) is equal to

$$\frac{1}{n^{2\alpha}}\int_{1+(2/n)}^{\infty} \frac{d\eta}{(\eta-1)\left(\frac{1}{n}+\eta\right)^{2\alpha}} \le \frac{1}{n^{2\alpha}}\left[\int_{1+(2/n)}^{2}\frac{d\eta}{\eta-1} + \int_2^{\infty}\frac{d\eta}{(\eta-1)^{2\alpha+1}}\right] =$$

$$= \frac{1}{n^{2\alpha}}\left[\log\frac{n}{2} + \frac{1}{2\alpha}\right]. \tag{2.23}$$

If we collect the inequalities (2.20) to (2.23), we get (with $2^{2\alpha} \le 2$)

$$\mu \le \frac{2}{3}\frac{1}{n^{2\alpha}}\left[9 + \frac{3}{2} + \frac{2}{1-2\alpha} + 3\log\frac{en}{2} + \frac{1}{2\alpha}\right] \le$$

$$\le \frac{1}{n^{2\alpha}}[M(\alpha) + 2\log n] \tag{2.24}$$

$$\text{where } M(\alpha) = 6 + \frac{4/3}{1-2\alpha} + \frac{1}{3\alpha}. \tag{2.25}$$

With (2.18) and (2.2) we come to (2.14). □

Of course, the constant factors in the inequalities (2.18) – (2.24) are not sharp but we get some idea on their magnitude. If $\alpha = \frac{1}{4}$ we have

$$M\left(\frac{1}{4}\right) = 6 + \frac{4}{3}\cdot 2 + \frac{2}{3} < 10, \text{ and} \tag{2.26}$$

$$\mu \le \frac{2}{\sqrt{n}}(5 + \log n) \tag{2.27}$$

This case is important in analysis of the harmonic oscillator and its perturbations (1.5). The estimates (2.26) and (2.27) will be used later as well.

Remark 2.2 Let $s \equiv \sum_{\substack{j=0\\ j\neq n}}^{\infty} \frac{1}{(1+j)^\beta}\cdot\frac{1}{|n-j|}$. Then

$$s \le \frac{M(\beta)}{n^\beta}\log en, \quad if\ 0 < \beta \le 1, \tag{2.28.i}$$

$$s \le \frac{M}{n}, \quad if\ \beta > 1. \tag{2.28.ii}$$

Proof The case $\beta = 2\alpha < 1$ is done in the proof of Lemma 2.1. Other cases could be explained in the same way; we omit details. □

4 In this section we use properties of Shatten class operators and related equalities for the norms $\|T\|_p$, $T \in \mathfrak{S}_p$, of compact operators — see details in [15, 22]. By (2.10) the operator K is bounded if $z \in H_n \setminus \mathcal{D}_n$ and by (2.16), (2.17) its norm

$$\|K\| \le \sqrt{2}. \tag{2.29}$$

Therefore, for $U_j \in$ (2.12) if $j \ge 2$

$$\|U_j\|_1 \le 2\|KWK\|_2^j \le 2\ell^j \le 2\left[M(\alpha)\frac{\log en}{n^{2\alpha}}\right]^j. \tag{2.30}$$

But we can claim that U_1 is a trace-class operator as well, and

$$\|U_1\|_1 = \|K(KWK)K\|_1 \le \|K\|_4\|KWK\|_2\|K\|_4 \tag{2.31}$$

because $K \in \mathfrak{S}_4$ [or any $\mathfrak{S}_p$, $p > 2$, as a matter of fact]: just notice that by (2.16), (2.17)

$$\begin{aligned}\|K\|_4^4 &= \sum_{j=0}^{\infty}\frac{1}{|z-z_j|^2} \le \\ &\le 3/4 + 2\sum_{k=2}^{\infty}\left(\frac{2}{3}\right)^2 \cdot \frac{1}{k^2} < 20 < \left(\frac{11}{5}\right)^4,\end{aligned} \tag{2.32}$$

so

$$\|K\|_4 \le \frac{11}{5}; \quad \|K\|_4^2 \le 5. \tag{2.33}$$

Therefore we can claim the following.

Proposition 2.3 *Under the assumptions* (2.1), (2.2), $0 < \alpha < \frac{1}{2}$, *suppose that* $N_* = N_*(\alpha)$ *is chosen in such a way that*

$$M(\alpha)\frac{\log en}{n^{2\alpha}} \le \frac{1}{2} \quad \textit{for all} \quad n \ge N_*. \tag{2.34}$$

Then for $n > N_*(\alpha)$ *if* $z \in \partial\mathcal{D}_n$ *all the operators* $U_j \in$ (2.8) *are of the trace class, their norms satisfy inequalities*

$$\|U_j\|_1 \le 2\left[M(\alpha)\frac{\log en}{n^{2\alpha}}\right]^j, \quad j \ge 2, \tag{2.35}$$

$$\|U_1\|_1 = \|R^0WR^0\|_1 \le \frac{5M(\alpha)\log en}{n^{2\alpha}} \tag{2.36}$$

and the Neumann - Riesz series for the difference of two resolvents

$$R - R^0 = \sum_{j=1}^{\infty} U_j \tag{2.37}$$

converges by the $\mathfrak{S}_1$*-norm and*

$$\|R - R^0\|_1 \le 7M(\alpha)\frac{\log en}{n^{2\alpha}} \tag{2.38}$$

and

$$\left\|\sum_{j=m}^{\infty} U_j\right\|_1 \le 4\left(M(\alpha)\frac{\log en}{n^{2\alpha}}\right)^m, \quad m \ge 2. \tag{2.39}$$

Proof Inequality (2.35) is identical with (proven) line (2.30). (2.36) come if we combine (2.32), (2.31), and (2.14). Therefore, for $m \geq 2$, by (2.35) and (2.34),

$$\left\| \sum_{j=m}^{\infty} U_j \right\|_1 \leq 2 \sum_{j=m}^{\infty} \left(M(\alpha) \frac{\log en}{n^{2\alpha}} \right)^j \leq 4 \left(M(\alpha) \frac{\log en}{n^{2\alpha}} \right)^m . \tag{2.40}$$

Then by (2.36)

$$\begin{aligned} \|R - R^0\|_1 &\leq \|U_1\|_1 + \left\| \sum_{j=2}^{\infty} U_j \right\|_1 \leq \\ &\leq M(\alpha) \frac{\log en}{n^{2\alpha}} \cdot \left(5 + 4M(\alpha) \frac{\log en}{n^{2\alpha}} \right) \leq 7M(\alpha) \frac{\log en}{n^{2\alpha}}. \end{aligned} \tag{2.41}$$

□

3 Deviations of Eigenvalues of The Harmonic Oscillator Operator and its Perturbations

1 Although the constructions and methods of this section are general and applicable to many operators with discrete spectrum and their perturbations, we will focus later in this section on the case of Harmonic Oscillator operator (2.1) and its functional representation

$$L^0 y = -y'' + x^2 y \tag{3.1}$$

in $L^2(\mathbb{R})$.

The Riesz-Neumann Series (2.37), (2.8) and (2.9) — as soon as its convergence in $\mathfrak{S}_1$ is properly justified — can be used to evaluate eigenvalues of the operator $L = L^0 + W$.

Under proper conditions, if $n \geq N_*$, the operator L has the only eigenvalue λ_n in H_n; moreover, λ_n is simple and $\lambda_n \in \mathcal{D}_n$. Therefore, both of the projections

$$P_n^0 = \frac{1}{2\pi i} \int_{\partial \mathcal{D}_n} R^0(z)\, dz = \langle \cdot, h_n \rangle\, h_n \tag{3.2}$$

and

$$P_n = \frac{1}{2\pi i} \int_{\partial \mathcal{D}_n} R(z)\, dz = \langle \cdot, \psi_n \rangle\, \phi_n \tag{3.3}$$

are of rank 1. [In (3.3), ϕ_n is an eigenfunction of L and ψ_n is an eigenfunction of $L^* = L^0 + W^*$, with an eigenvalue $\mu_n = \overline{\lambda}_n$ in $\mathcal{D}_n$. We will not use this specific information so nothing more is explained now.]

Therefore,

$$\text{Trace}\, P_n^0 = \text{Trace}\, P_n = 1, \tag{3.4}$$

$$\text{Trace} \frac{1}{2\pi i} \int_{\partial \mathcal{D}_n} (R(z) - R^0(z))\, dz = 0. \tag{3.5}$$

and

$$z_n = \text{Trace}\, \frac{1}{2\pi i} \int\limits_{\partial \mathcal{D}_n} z R^0(z)\, dz = 2n + 1, \tag{3.6}$$

$$\lambda_n = \text{Trace}\, \frac{1}{2\pi i} \int\limits_{\partial \mathcal{D}_n} z R(z)\, dz, \tag{3.7}$$

So (3.4) to (3.7) imply

$$\lambda_n - z_n = \text{Trace}\, \frac{1}{2\pi i} \int\limits_{\partial \mathcal{D}_n} (z - z_n)[R(z) - R^0(z)]\, dz = \sum_{j=1}^{\infty} T_j(n) \tag{3.8}$$

where we put [with $z_n = \lambda_n^0 = 2n + 1$]

$$T_j(n) = T_j(n; W) = \frac{1}{2\pi i} \text{Trace} \int\limits_{\partial \mathcal{D}_n} (z - z_n) U_j(z)\, dz \tag{3.9}$$

Proposition 2.3 is used in (3.8), (3.9). *Trace* is a linear bounded functional of norm 1, on the space $\mathfrak{S}_1$ of trace-class operators ([15, 22]) . It implies the following.

Corollary 3.1 *Under the assumptions of Proposition 2.3, with $n \geq N_*$, we have*

$$|T_j(n)| \leq \left[M(\alpha) \frac{\log en}{n^{2\alpha}} \right]^j, \quad j \geq 2 \tag{3.10}$$

and

$$|T_1(n)| \leq \frac{9}{4} M(\alpha) \frac{\log en}{n^{2\alpha}} \tag{3.11}$$

Proof With $|\text{Trace}\, A| \leq \|A\|_1$ and

$$\begin{aligned} &|z - z_n| \leq \frac{1}{\sqrt{2}}, z \in \partial \mathcal{D}_n, \\ &\text{length}(\mathcal{D}_n) = 4 \end{aligned} \tag{3.12}$$

rough estimates of integrals (3.9) with $j \geq 2$ and $j = 1$ based on (2.35) and (2.36) lead to (3.10) and (3.11). □

Corollary 3.2 *Under the assumptions of Proposition 2.3, with $n \geq N_*$,*

$$\lambda_n = (2n + 1) + \sum_{j=1}^{q} T_j(n) + r_q(n), \quad q \geq 1, \tag{3.13}$$

where

$$|r_q(n)| \leq 2 \left(M(\alpha) \frac{\log en}{n^{2\alpha}} \right)^{q+1} \tag{3.14}$$

Proof The presentation of λ_n and the inequality follow from (3.8) and (2.39) if we put $m = q + 1$ in (2.39) and notice that $2\sqrt{2} < \pi$ when we multiply the constant factors in inequalities. □

2 Analysis of the function $N_*(\alpha)$. This function is determined by the inequality (2.34). Later we consider potentials with the coupling coefficient s [see (4.2), (4.3)] so it is

useful to know the behavior of $X = X_\beta(t)$, the solution of the equation

$$t\frac{\log eX}{X^\beta} = \frac{1}{2}, \quad \beta = 2\alpha, \text{ for large} t. \tag{3.15}$$

Let us rewrite it as

$$\tau \log Y = Y, \text{ where } Y = (eX)^\beta \tag{3.16}$$

$$\text{and } \tau = \frac{2}{\beta}e^\beta t \tag{3.17}$$

The (3.16) has two solutions

$$y(\tau) = 1 + \frac{1}{\tau} + O\left(\frac{1}{\tau^2}\right), \qquad \tau \to \infty \tag{3.18}$$

and

$$Y(\tau) \to \infty, \qquad \tau \to \infty. \tag{3.19}$$

Lemma 3.3 *The solution $Y \in$ (3.19) has an asymptotic*

$$Y(\tau) = \tau \log \tau \cdot (1 + r(\tau)) \tag{3.20}$$

where

$$r(\tau) = \frac{\log\log\tau}{\log\tau}(1 + o(1)) \tag{3.21}$$

so for any δ we can find τ^* such that

$$Y(\tau) \le \tau\log\tau + \tau(1+\delta)\log\log\tau, \quad \tau \ge \tau^*. \tag{3.22}$$

or $\tau_* < \tau^*$ such that

$$Y(\tau) \le (1+\delta)\tau\log\tau, \quad \tau \ge \tau_*. \tag{3.23}$$

Proof If we look for $r \ge 0$, in (3.20), which solves (3.16) we have:

$$\tau\log\tau[1+r] = \tau[\log\tau + \log\log\tau + \log(1+r)] \tag{3.24}$$

or

$$r = \varphi(r), \quad \varphi(X) = \xi + \eta\log(1+X), \quad r > 0 \tag{3.25}$$

where

$$\xi = \frac{\log\log\tau}{\log\tau}, \quad \eta = \frac{1}{\log\tau} \tag{3.26}$$

For any $0 < \delta \le \frac{1}{2}$ we can choose τ^* such that

$$0 < \xi \le \frac{\delta}{2}, \quad 0 < \eta < \frac{\delta}{2} \quad \text{if} \tau \ge \tau^*. \tag{3.27}$$

Then the function φ, $\varphi : [0, \delta] \to [0, \delta]$ is a contraction mapping, and (3.25) has the unique solution

$$r = r(\tau), \quad 0 < r(\tau) \le \delta. \tag{3.28}$$

Therefore,

$$r = \frac{\log\log\tau}{\log\tau} + \frac{\rho}{\log\tau}, \quad 0 < \rho \le \delta. \tag{3.29}$$

This implies (3.21) with

$$\frac{\rho}{\log\log\tau} = o(1). \tag{3.30}$$

□

Corollary 3.4 *The solution* $X(t)$ *of* (3.15), $0 < \beta \leq 1$, *goes to* ∞ *when* $t \to \infty$ *and*

$$X(t) \leq \left(2t \log \frac{At}{\beta}\right)^{1/\beta}, \quad A \text{ an absolute constant,} \tag{3.31}$$

if t is large enough.

Proof If we put (3.17) into (3.22) or (3.23) elementary simplifications give the inequality (3.31). □

3 Inequalities (2.18) and (2.30) guarantee that we can use the representation (2.8) and eventually "asymptotics" (3.13) if

$$C_0 M(\alpha) \frac{\log en}{n^{2\alpha}} \leq \frac{1}{2}, \quad C_0 \in 2.2 \tag{3.32}$$

and $M(\alpha)$ by (2.25) is chosen as

$$M(\alpha) = \left[6 + \frac{4/3}{1 - 2\alpha} + \frac{1}{3\alpha}\right]. \tag{3.33}$$

Then (3.31), with $\beta = 2\alpha < 1$, $t = 2C_0 M(\alpha)$, implies that N_* can be chosen as

$$N_* = N_*(C_0; \alpha) = \left[2C_0 M(\alpha) \log \left(\frac{A}{2\alpha} 2C_0 M(\alpha)\right)\right]^{1/(2\alpha)} \tag{3.34}$$

Now if α is fixed we are interested in the dependence of N_* on $C_0 \in$ (2.2).

4 Recall that if W is a multiplier-operator

$$Wf = w(x) f(x), \ \text{with} w \in L^p(\mathbb{R}^1), \quad 1 \leq p < \infty, \tag{3.35}$$

then as we observed and used in [19]

$$w_{jk} = \langle Wh_j, h_k \rangle = \int_{-\infty}^{\infty} w(x) h_j(x) h_k(x)\, dx \tag{3.36}$$

so by Hölder inequality

$$|w_{jk}| \leq \|w\|_p \cdot \|h_j\|_{2q} \|h_k\|_{2q}, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{3.37}$$

with

$$q > 1, \quad 2q > 2. \tag{3.38}$$

But

$$|h_k(x)| \leq Ck^{-1/12} \tag{3.39}$$

so

$$\begin{aligned} \int |h_k(x)|^{2q}\, dx &= \int |h_k(x)|^2 |h_k(x)|^{2(q-1)}\, dx2 \\ &\leq C^{2(q-1)} k^{-(q-1)/6} \int |h_k(x)|^2\, dx \end{aligned} \tag{3.40}$$

and

$$\|h_k\|_{2q} \leq C^{1/p} k^{-1/(12p)}, \quad p \geq 1. \tag{3.41}$$

This means that the matrix W satisfies the condition (2.2) with $\alpha = \frac{1}{12p}$ This observation was crucial in [19]; it gives a broad class of operators covered by (2.2) so our claims of this sections are applicable to the operators (3.35).

But there are much better estimates of L^p norms of the Hermite functions than (3.41).

Lemma 3.5 *As $n \to \infty$,*

$$\|h_n\|_r \sim n^{-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{r}\right)}, \quad 1 \le r < 4 \tag{3.42a}$$

$$\|h_n\|_r \sim n^{-\frac{1}{8}} \log n, \quad r = 4 \tag{3.42b}$$

$$\|h_n\|_r \sim n^{-\frac{1}{6}\left(\frac{1}{r}+\frac{1}{2}\right)}, \quad r > 4 \tag{3.42c}$$

See [23, Lemma 1.5.2] for the sketch of the proof and further explanations of these claims.

Therefore, (3.41) could be improved. If $p > 2$ then by (3.37) $q < 2$, $2q < 4$ so

$$\|h_k\|_{2q} \sim k^{-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{2q}\right)} = k^{-\frac{1}{4p}}, \quad p > 2. \tag{3.43}$$

For $p = 2$ we have $2q = 4$ and

$$\|h_k\|_4 \sim k^{-\frac{1}{8}} \log k, \quad p = 2. \tag{3.44}$$

Finally, if $1 \le p < 2$ then $2q > 4$ so

$$\|h_k\|_{2q} \sim k^{-\frac{1}{6}\left(\frac{1}{2q}+\frac{1}{2}\right)} = k^{-\frac{1}{12}\left(2-\frac{1}{p}\right)}, \quad 1 \le p < 2 \tag{3.45}$$

All these estimates are used in Theorem 4.1, — see Section 4 below.

Of course, (3.39) shows that δ-potentials

$$w(x) = \sum\nolimits_{k=1}^{m} c_k \delta(x - b_k), \quad m \le \infty, \quad M \equiv \sum\nolimits_{k=1}^{m} |c_k| < \infty, \tag{3.46}$$

are good for us as well; in this case,

$$\left\langle W h_j, h_i \right\rangle = \sum\nolimits_{k=1}^{m} c_k h_j(b_k) h_i(b_k), \quad \left|W_{ji}\right| \le C M j^{-1/12} i^{-1/12}, \quad i, j \ge 1, \tag{3.47}$$

and we can give a trace-class version of Lemma 2.1.

Remark 3.6 Under the conditions (3.46), (2.10), if $z \in H_n \setminus \mathcal{D}_n$ then KWK is a trace class operator, and

$$\|KWK\|_1 \le C_{12} \frac{M}{n^{1/6}} \tag{3.48}$$

where C_{12} is an absolute constant.

Proof The proof follows if we observe that KWK is a sum of rank-one operators $\langle \cdot, g_k \rangle g_k$ where

$$g_k = \left(h_j(b_k) \frac{1}{\sqrt{z - z_j}} \right)_{j=0}^{\infty}, \quad 1 \le k \le m \tag{3.49}$$

□

With more information on asymptotics of Hermite polynomials and Hermite functions we can be accurate in analysis of point-interaction potentials (3.46) and spectra of operators $L^0 + W$, $L^0 \in$ (3.1), or — equivalently — (2.1). This is the main goal of this paper and its forthcoming extension. Now we go to detailed analysis of these operators.

4 Two-point Interaction Potentials

1 Now we apply general constructions of Sections 2, 3 to the case of the two-point interaction potentials

$$w(x) = c^{+}\delta(x-b) + c^{-}\delta(x+b), \qquad b>0 \tag{4.1}$$

and particular cases of an odd potential

$$sv^{o}(x), \quad s\in\mathbb{C} \quad \text{where} \qquad v^{o}(x) = \delta(x-b) - \delta(x+b) \tag{4.2}$$

and an even potential

$$tv^{e}(x), \quad t\in\mathbb{C} \qquad v^{e}(x) = \delta(x-b) + \delta(x+b) \tag{4.3}$$

— see [5, 14].

Of course, for any *odd* potential (3.35) or (3.46), not just for $v^0 \in (4.2)$, the matrix elements w_{jk} have the property (4.6). Indeed, with parity

$$w_{jk} = \left\langle w(x)h_j(x),\ h_k(x)\right\rangle = \tag{4.4}$$

$$= \left\langle w(-x)h_j(-x),\ h_k(-x)\right\rangle = -(-1)^{j+k}w_{jk} \tag{4.5}$$

so

$$w_{jk} = 0 \text{ if} j+k \text{ even}. \tag{4.6}$$

If, however, w in (3.35) or (3.46) is *even* then we conclude

$$w_{jk} = 0 \text{ if} j+k \text{ odd}. \tag{4.7}$$

These observations lead to information on complex eigenvalues of $L = L^0 + W$.

Theorem 4.1 *Let the potential*

$$w(x)\in L^p,\ 1\le p<\infty, \quad \nu = \|w\|_p\,,\ or \tag{4.8}$$

$$w(x) = \sum_{k=1}^{\infty} c_k\delta(x-b_k)\,, \quad \nu = \sum |c_k| < \infty, \tag{4.9}$$

be $\mathcal{PT}$, *i.e.,* $w(-x) = \overline{w(x)}$. *Then the operator*

$$L = L^0 + W = -\frac{d^2}{dx^2} + x^2 + w \tag{4.10}$$

has at most finitely many non-real eigenvalues, if any, and their number does not exceed N^**, where*

$$N^* = D\left(\nu\log(1+\nu)\right)^{2p}, \qquad p>2 \tag{4.11a}$$

$$N^* = D^*\left(\nu\log^2(1+\nu)\right)^{4}, \qquad p=2 \tag{4.11b}$$

$$N^* = D\left(\nu\log(1+\nu)\right)^{\frac{3}{\left(1-\frac{1}{2p}\right)}}, \qquad 1\le p<2 \tag{4.11c}$$

$$N^* = D_*\left(\nu\log(1+\nu)\right)^{6}, \qquad \textit{in the case (4.9)} \tag{4.11d}$$

D^**,* D_* *are absolute constants, and* $D = D(p)$ *does not depend on the norm* ν*.*

Proof By the estimates in Corollary 3.1 and in (3.34) we can use the series (3.8) to evaluate λ_n if $n\ge N^*$, i. e., all eigenvalues in a half-plane $z: Rez \ge 2N^*$ – see (2.4), (2.5). The lemmas which follows explain that under the assumptions of the theorem every term T_j (3.8), (3.9) is real. But total number of all other eigenvalues in $(z : Rez < 2N^*\}$ *is* N^* as Proposition 2 in [19] explains. So the number of non-real eigenvalues if any does not exceed N^*. Corollary 3.1 and Lemma 3.3 guarantee that N^* could be chosen as (4.11. a – d) indicate because Lemma 3.5 leads good estimates of matrix elements (3.36). □

Lemma 4.2 *Let $w \in$ (4.10) be a $\mathcal{PT}$-potential, i.e.,*

$$w(-x) = \overline{w(x)}, \quad x \in \mathbb{R}^1. \tag{4.12}$$

Then for $n \geq N^* \in$(4.11) all

$$T_j(n; W), \quad 1 \leq j, \quad \textit{are real} \tag{4.13}$$

Proof If $w = p + iq \in$ (4.12), p, q real, then p is even and q is odd so by (4.4) – (4.7),

$$w(k,\ell) = \begin{cases} p(k,\ell), & \text{if} k+\ell \text{ even} \\ iq(k,\ell), & \text{if} k+\ell \text{ odd.} \end{cases}$$

By (3.9)

$$T_j(n) = T_j(n; W) = \frac{1}{2\pi i} \operatorname{Trace} \int_{\partial\mathcal{D}_n} (z - z_n) U_j(z)\, dz \tag{4.14}$$

where

$$U_j = R^0 W R^0 W \cdots W R^0 \tag{4.15}$$

with j “letters” W and $j+1$ “letters” R^0 in this “word” U. All these operators are of trace class [see Corollary 3.1] so Trace U_j is a sum of integrals of the diagonal elements $(U_j)_{mm}$ which in turn are sums of matrix elements $\sum_g u(g,z)$, where $g = \{g_1, g_2, \ldots, g_{j-1}\} \in \mathbb{Z}_+^{j-1}$ and

$$u(g,z) \equiv \frac{1}{z - z_m} \cdot W(m, g_1) \cdot \frac{1}{z - z_{g_1}} \cdot W(g_1, g_2) \cdot \frac{1}{z - z_{g_2}} \cdots W(g_{j-1}, m) \cdot \frac{1}{z - z_m} \tag{4.16}$$

If we put $g_0 = g_j = m$ we have

$$\sum_{t=0}^{j-1} (g_{t+1} - g_t) = g_j - g_0 = 0 \tag{4.17}$$

The sum of all these differences in (4.17) is even (zero): so

$$\gamma^- \equiv \#\Gamma^-, \quad \Gamma^- \equiv \{t : g_{t+1} - g_t \text{ odd}\} \tag{4.18}$$

should be even. Put

$$\gamma^+ \equiv \#\Gamma^+, \quad \Gamma^+ \equiv \{t : g_{t+1} - g_t \text{ even}\}, \tag{4.19}$$

and

$$p(g) \equiv \prod_{t=0}^{j-1} i W(g_t, g_{t+1}). = (-1)^q \cdot (\text{real number}) \tag{4.20}$$

Then

$$p(g) = \left(\prod_{t\in\Gamma^-} W(g_t, g_{t+1})\right) \left(\prod_{t\in\Gamma^+} W(g_t, g_{t+1})\right) = i^{\gamma^-} \cdot (\text{real number}) \tag{4.21}$$

and by (4.18) $p(g)$ is real.

By (4.16)

$$u(g,z) = p(g) \cdot F_g(z), \text{ where} \tag{4.22}$$

$$F_g(z) = \frac{1}{(z - z_m)^2} \prod_{t=1}^{j-1} \frac{1}{z - z_{g_t}} \tag{4.23}$$

and this term brings into (4.14) the number-product $p(g)J(g)$ where

$$J(g) = \frac{1}{2\pi i} \int\limits_{\partial \mathcal{D}_n} (z - z_n) F_g(z)\, dz. \tag{4.24}$$

For any $g \in \mathbb{Z}_+^{(j-1)}$ this integral $J(g)$ is a real number [see the next lemma]. Therefore, $T_j(n)$ — as a sum of (absolutely) convergent series with real terms — is a real number. □

This completes the proof of Proposition 4.1. We will need more specific information about integrals $J(g) \in (4.24)$. The following is true.

Lemma 4.3 *If $m = n$, and $n \geq N^*$ (N^* as defined in (4.11)),*

$$J(g) = 0 \quad \textit{if at least one} \quad g(\tilde{t}) = n, \tag{4.25}$$

and

$$J(g) = \left(\prod_{t=1}^{j-1} 2(n - g_t) \right)^{-1} \quad \textit{otherwise}. \tag{4.26}$$

If $m \neq n$,

$$J(g) = 0 \, \textit{if} \, \# \tau(g) \neq 2, \;\; \textit{where} \tau(g) = \{t : g_t = n\} \tag{4.27}$$

and

$$J(g) = \frac{1}{4(n-m)^2} \left(\prod_{t \notin \tau(g)}^{j-1} 2(n - g_t) \right)^{-1} \textit{if} \# \tau(g) = 2. \tag{4.28}$$

Proof The integrand (4.23) of (4.24) could have a pole inside of $\mathcal{D}_n$ only at $z_n = 2n + 1$. In the cases (4.25) and (4.27) the pole's order ≥ 2 or $F_g(z)$ is analytic on $\overline{\mathcal{D}_n}$, so $J(g) = 0$. In the cases (4.26), (4.28) the pole's order is one and $J(g)$ is the residue of $F_g(z)$ at z_n. □

2 An odd potential v^o in (4.2) As it is noticed in (4.6),

$$\begin{aligned} v_{jk}^0 &= \langle (\delta(x - b) - \delta(x + b)) h_j, h_k \rangle \\ &= [1 - (-1)^{j+k}] a_j a_k \qquad (4.29) \\ &= \begin{cases} 0, & \text{if } j + k \text{ even} \\ 2 a_j a_k, & \text{if } j + k \text{ odd} \end{cases} \qquad (4.30) \end{aligned}$$

where

$$a_k = h_k(b), \quad k = 0, 1, \ldots \tag{4.31}$$

With $b > 0$ fixed, from now on we will use (a_k) as in (4.31). By Lemmas 4.2 and 4.3 for $n \geq N^*$

$$T_j(n; v^0) \equiv T_j(n) = 0 \quad \text{if } j \text{ odd}; \tag{4.32}$$

in particular

$$T_1(n) = 0, \quad T_3(n) = 0. \tag{4.33}$$

To evaluate $T_2(n)$ we sum up (we did it in (4.16) in general setting) Cauchy integrals of functions

$$(z - z_n) \cdot \frac{1}{z - z_m} \cdot v_{mk}^0 \cdot \frac{1}{z - z_k} \cdot v_{km}^0 \cdot \frac{1}{z - z_m} \tag{4.34}$$

If $m \neq n$ it is analytic for any k so Cauchy integral is zero. If $m = n$

$$v^0_{mk} = 0 \quad \text{if} n - k \text{ is even.} \tag{4.35}$$

Therefore, by Lemma 4.3, $j = 2$, with $z_n - z_k = 2(n-k)$,

$$T_2(n; v^0) \equiv T_2(n) = \sum_{\substack{k=0 \\ k-n \text{ odd}}}^{\infty} \frac{v^0_{nk} v^0_{kn}}{z_n - z_k} = \sum_{\substack{k=0 \\ k-n \text{odd}}}^{\infty} \frac{2a_n a_k \cdot 2a_k a_n}{2(n-k)} = 2a_n^2 \widetilde{\sigma}(n) \tag{4.36}$$

where

$$\widetilde{\sigma}(n) = \sum_{\substack{k=0 \\ n-k \text{odd}}}^{\infty} \frac{a_k^2}{n-k} \tag{4.37}$$

The technical analysis of this sequence (4.37) and its variations is the core of our forthcoming paper which complements the present one. All the proofs will be given there. In meantime we can refer a reader to [18], Sections 5–8.

It will bring us the proof of the main result of this paper:

Theorem 4.4 *The operator*

$$L = -\frac{d^2}{dx^2} + x^2 + s[\delta(x-b) - \delta(x+b)], b > 0, s \in \mathbb{C}$$

has a discrete spectrum $\sigma(L)$.

There exists an absolute constant D such that with

$$N^* = (D|s| \log e|s|)^2 \tag{4.38}$$

all eigenvalues $\lambda_n = \lambda_n(L)$ in the half-plane $\{z \in \mathbb{C} : \Re z > N^*\}$ are simple, and

$$\lambda_n = (2n+1) + s^2 \frac{\kappa(n)}{n} + \widetilde{\rho}(n), \quad |\widetilde{\rho}(n)| \leq C \frac{\log n}{n^{3/2}} \tag{4.39}$$

where

$$\kappa_n = \frac{1}{2\pi} \left[(-1)^{n+1} \sin(2b\sqrt{2n}) - \frac{1}{2} \sin(4b\sqrt{2n}) \right] \tag{4.40}$$

The proof of the theorem is based on the following lemma.

Lemma 4.5 *With $\widetilde{\sigma}(n) \in$ (4.37)*

$$\widetilde{\sigma}(n) = (-1)^{n+1} \frac{1}{2} \frac{\sin(2b\sqrt{2n})}{\sqrt{2n}} + \rho(n), \tag{4.41}$$

$$|\rho(n)| \leq C \frac{\log n}{n} \tag{4.42}$$

An even potential $v^e \in$ (4.3) Recall (4.7); now

$$v^e_{jk} = [1 + (-1)^{j+k}] a_j a_k \tag{4.43}$$

$$= \begin{cases} 0, & \text{if} j + k \text{odd} \\ 2a_j a_k, & \text{if} j + k \text{ even} \end{cases} \tag{4.44}$$

Therefore, by Lemma 4.3, $j = 1$,

$$T_1(n; v^e) \equiv T_1(n) = v^e_{nn} = 2a_n^2 \tag{4.45}$$

and [compare (4.32) to (4.46)]

$$T_2(n; v^e) \equiv T_2(n) = 2a_n^2 \sigma'(n), \tag{4.46}$$

$$\sigma'(n) = \sum_{\substack{k=0 \\ n-k \text{even}}}^{\infty}{}' \frac{a_k^2}{n-k}, \tag{4.47}$$

where $\sum'$ means that $k \neq n$.

But for the even potential there is no trivial claim $T_3(n) = 0$. We could make formal references to Lemma 4.3 but let us again look into those terms which form the sum-trace $T_3(n)$. We integrate functions

$$F = (z - z_n) \cdot \frac{1}{z - z_m} \cdot 2a_m a_k \cdot \frac{1}{z - z_k} \cdot 2a_k a_\ell \cdot \frac{1}{z - z_\ell} \cdot 2a_\ell a_m \cdot \frac{1}{z - z_m} \tag{4.48}$$

excluding (by (4.44)) triples (m, k, ℓ) if at least one of the differences $m - k$, $k - \ell$, $\ell - m$ is odd.

If $m = n$ then we can take only $k, \ell \neq n$, otherwise the order of the pole at z_n would be ≥ 2 and Cauchy integral (4.24) be zero. Then the partial sum of (4.14) over triples

$$\{(m, k, \ell) | m = n, k \neq n, \ell \neq n, k - n, \ell - n \text{ even}\}$$

would be

$$2a_n^2 \sum{}'_{\substack{k,\ell \\ n-k, \\ n-\ell \text{even}}} \frac{a_k^2 a_\ell^2}{(n-k)(n-\ell)} = \tag{4.49}$$

$$= 2a_n^2 \left(\sigma'(n)\right)^2. \tag{4.50}$$

If $m \neq n$ Cauchy integral of $F \in$ (4.48) is not zero only if $k = \ell = n$, i.e., if we have two (and only two) zeros in the denominator to balance the factor $(z - z_n)$. This set of triples

$$\{(m, k, \ell) | m \neq n, k = \ell = n, m - n \text{ even}\} \tag{4.51}$$

leads to the subsum in $T_3(n)$ coming from (4.48)

$$2a_n^4 \sum_{\substack{m=0 \\ m-n \text{even}}}^{\infty}{}' \frac{a_m^2}{(n-m)^2} = 2a_n^4 \tau'(n) \tag{4.52}$$

If we combine (4.49) – (4.52) we conclude that

$$T_3(n; v^e) = 2a_n^2 \left[\sigma'(n)^2 + a_n^2 \tau'(n)\right] \tag{4.53}$$

We will analyze the sequences σ', τ' in a forthcoming paper as well.

Acknowledgements The author is indebted to Charles Baker and Petr Siegl for numerous discussions. Without their support this work would hardly be written, at least in a reasonable period of time. I am also thankful to Daniel Elton, Paul Nevai, Günter Wunner, and Miloslav Znojil for valuable comments and information related to topics of this manuscript.

References

1. Adduci, J., Mityagin, B.: Eigensystem of an L^2-perturbed harmonic oscillator is an unconditional basis. Cent. Eur. J. Math. **10**(2), 569–589 (2012). doi:10.2478/s11533-011-0139-3
2. Adduci, J., Mityagin, B.: Root system of a perturbation of a selfadjoint operator with discrete spectrum. Integr. Equ. Oper. Theory **73**(2), 153–175 (2012). doi:10.1007/s00020-012-1967-7
3. Albeverio, S., Fei, S.M., Kurasov, P.: Point interactions: $\mathcal{PT}$-hermiticity and reality of the spectrum. Lett. Math. Phys. **59**(3), 227–242 (2002). doi:10.1023/A:1015559117837
4. Albeverio, S., Gesztesy, F., Hoegh-Krohn, R., Holden, H.: Solvable Models in Quantum Theory, 2nd edn. AMS Chelsea Publishing (2005)
5. Cartarius, H., Dast, D., Haag, D., Wunner, G., Eichler, R., Main, J.: Stationary and dynamical solutions of the Gross-Pitaevskii equation for a Bose-Einstein condensate in a $\mathcal{PT}$-symmetric double well. Acta Polytech. **53**(3), 259–267 (2013)
6. Demiralp, E.: Bound states of n-dimensional harmonic oscillator decorated with Dirac delta functions. J. Phys. A **38**(22), 4783–4793 (2005). doi:10.1088/0305-4470/38/22/003
7. Demiralp, E.: Properties of a pseudo-Hermitian Hamiltonian for harmonic oscillator decorated with Dirac delta interactions. Czechoslovak J. Phys. **55**(9), 1081–1084 (2005). doi:10.1007/s10582-005-0110-2
8. Demiralp, E., Beker, H.: Properties of bound states of the Schrödinger equation with attractive Dirac delta potentials. J. Phys. A **36**(26), 7449–7459 (2003). doi:10.1088/0305-4470/36/26/315
9. Djakov, P., Mityagin, B.: Instability zones of one-dimensional periodic Schrödinger and Dirac operators. Uspekhi Mat. Nauk **61**(4(370)), 77–182 (2006). doi:10.1070/RM2006v061n04ABEH004343
10. Djakov, P., Mityagin, B.: Equiconvergence of spectral decompositions of Hill-Schrödinger operators. J. Differ. Equ. **255**(10), 3233–3283 (2013). doi:10.1016/j.jde.2013.07.030
11. Elton, D.M.: The Bethe-Sommerfeld conjecture for the 3-dimensional periodic Landau operator. Rev. Math. Phys. **16**(10), 1259–1290 (2004)
12. Fassari, S., Inglese, G.: On the spectrum of the harmonic oscillator with a delta-type perturbation. Helv. Phys. Acta **67**(1), 650–659 (1994)
13. Fassari, S., Inglese, G.: On the spectrum of the harmonic oscillator with a delta-type perturbation. ii. Helv. Phys. Acta **70**, 858–865 (1997)
14. Fassari, S., Rinaldi, F.: On the spectrum of the schrdinger hamiltonian of the one- dimensional harmonic oscillator perturbed by two identical attractive point interactions. Rep. Math. Phys. **69**(3), 353–370 (2012)
15. Gohberg, I.C.: Kreĭn, M.G.: Introduction to the theory of linear nonselfadjoint operators. Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, vol. 18. American Mathematical Society, Providence, R.I (1969)
16. Haag, D., Cartarius, H., Wunner, G.: A bose-einstein condensate with $\mathcal{PT}$-symmetric double-delta function loss and gain in a harmonic trap: A test of rigorous estimates (2014). arXiv: 1401.2896v2
17. Kato, T.: Perturbation theory for linear operators, 2nd edn. Springer-Verlag, Berlin-New York (1976). Grundlehren der Mathematischen Wissenschaften, Band 132
18. Mityagin, B.: The spectrum of a harmonic oscillator operator perturbed by point interactions (2014). arXiv: 1407.4153
19. Mityagin, B., Siegl, P.: Root system of singular perturbations of the harmonic oscillator type operators (2013). arXiv: 1307.6245v1

20. Mostafazadeh, A.: Pseudo-hermiticity versus $\mathcal{PT}$ symmetry: The necessary condition for the reality of the spectrum of a non-hermitian hamiltonian. J. Math. Phys. **43**(1), 205–214 (2002). doi:10.1063/1.1418246
21. Mostafazadeh, A.: Exact pt -symmetry is equivalent to hermiticity. J. Phys. A **36**(25), 7081–7091 (2003)
22. Simon, B.: Trace ideals and their applications, London Mathematical Society Lecture Note Series, vol. 35. Cambridge University Press, Cambridge-New York (1979)
23. Thangavelu, S.: Lectures on Hermite and Laguerre expansions, Mathematical Notes, vol. 42. Princeton University Press, Princeton, NJ (1993). With a preface by Robert S. Strichartz
24. Znojil, M.: Solvable simulation of a double-well problem in $\mathcal{PT}$ -symmetric quantum mechanics. J. Phys. A **36**(27), 7639–7648 (2003)
25. Znojil, M., Jakubský, V.t.: Solvability and $\mathcal{PT}$-symmetry in a double-well model with point interactions. J. Phys. A.: Math. Gen. **38**(22), 5041–5056 (2005)