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Generalized Invariant Solutions for Spherical Symmetric Non-conformally Flat Fluid Distributions of Embedding Class One

Sachin Kumar · Y.K. Gupta

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Abstract In the present article, using the Lie group of transformations technique all the invariant solutions of Einstein's field equations for non-conformally flat perfect fluid spheres of embedding class one have been derived by considering a 5-flat space. The same problem for conformally flat case was tackled by Thakadiyil and Jasim (Int. J. Theor. Phys. 52:3960, 2013) using the same technique but with the lesser number of symmetries and hence could obtain only lesser number of solutions as compared to the number of solutions in this paper. All the solutions thus obtained have been subjected to reality conditions. As far as the authors are aware some of the solutions are new.

Keywords Spherical symmetric · Perfect fluid spheres · Embedding class one · General relativity

1 Introduction

In the nature by and large all the living or non-living objects are seen to possess symmetry of one or the other kind e.g. axial symmetry, plane symmetry and spherical symmetry. The later is the most attractive and also of great importance. Most of the stellar objects (i.e. sun, moon, planets and stars) are spherically symmetric. The Einstein's field equations are also subjected to this symmetry to describe the behaviour of such objects. The perfect fluid distribution is normally taken as source of various stellar structures. The corresponding Einstein equations reduce to two quasi-linear partial differential equations in three unknown

S. Kumar (✉)
Department of Applied Mathematics, School for Physical Sciences, Babasaheb Bhimrao Ambedkar University (A Central University), Lucknow 226025, India
e-mail: sachinambariya@rediffmail.com

Y.K. Gupta
Department of Applied Science, School of Engineering and Technology, Sharda University, Noida 201306, U.P., India
e-mail: kumar001947@gmail.com

dependent variables of two independent variables for co-moving system or one partial differential equation in two dependent variables depending upon two independent variables for non-comoving system. Therefore one additional condition is sought to obtain a specific model which can be (a) equation of state $p = f(\rho)$, (b) one of the dependent variable or a relation between the two dependent variables is prescribed to facilitate the solution, (c) some geometrical conditions are imposed on the metric describing the concerned Einstein space-time, and (d) some kinematical restrictions such as acceleration-free, shear-free fluids. The researchers have been using all such facilities time to time. For its detailed account one may refer to [26].

In the present article, the geometrical condition which makes the space-time metric of embedding class one is imposed and consequently the Einstein field equations reduced to a single highly non-linear partial differential equation. The relevance of embedding class can be described as follows:

The idea that our space–time can be considered as a four–dimensional space embedded in a higher-dimensional flat space is an old and recursive one [6]. Recently, due to a proposal by Randall and Sundrum [19] and discussions by Anchordoqui and Bergliaffa [1], this idea has again attracted much attention. It is well known that the manifold V_n can always be embedded in Pseudo-Euclidean space E_m of m dimensions, where $m = \frac{n(n+1)}{2}$. The minimum extra dimension p of the pseudo-Euclidean space to be embedded V_n in E_m, is called the class of the manifold V_n and must be less than or equal to the number $(m - n)$ i.e. $\frac{n(n-1)}{2}$. In case of relativistic space time V_4, the embedding class p turns out to be 6. As we have known that the class of spherical, hyperbolic and plane symmetric space-time is 2 and 3 respectively. The famous Friedman-Robertson-Lemaitre space-time, Robertson [21] is of class 1, while the Schwarzschild's exterior and interior solutions are of class 2 and class 1 respectively. Moreover the famous Kerr metric is of class 5 [16].

The postulates of general relativity do not provide any physical meaning to higher dimensional embedding space. However, it provides new characterizations of gravitational fields, which hopefully, can be connected to physics. Some researchers have linked the group of motions of flat embedding space to the internal symmetries of elementary particle physics [20]. Some have utilized the higher dimensions to study the singularity of the space-time. Recently, Pavsic and Tapia [18] have published an article entitled "Resource letter on geometrical results for Embeddings and Brane", where many references regarding the applications of embedding to general relativity, extrinsic gravity, strings and membranes and new brane world are mentioned. Also Andrejs Treibergs [29] has discussed about the embedding diagrams of Schwarzschild space and Misner's wormhole manifold to study the evolution problem for Einstein's equations.

In General Relativity, several authors [2, 11, 17, 22–24, 27] have considered the perfect fluid distributions of class one. They all have concluded that the perfect fluid distributions of class one could be divided into two categories. In the first category, conformally flat (with vanishing conformal curvature tensor Petrov type O) and in the second category, non-conformally flat which possesses non-vanishing Weyl conformal curvature tensor (Petrov type D). Stephani [24] and Barnes [2] have found almost all the perfect fluids belonging to the first category. Barnes [3] also found all those belonging to the second category with vanishing acceleration. Gupta et al. [9] have found all the Zeldovich fluids of second category. Gupta and Gupta [7] obtained a class of non-static analogue of Kohler-Chao solution satisfying the class one conditions. Gupta and Sharma [8], Bhutani and Singh [4], Kumar and Gupta [14], Gupta et al. [10] and S. Kumar et al. [15] derived a new class of solutions satisfying reality conditions and belonging to the second category. Also Stephani [25] has proved that the solutions with the equation of state $\rho = f(p)$ are only those, which describes Zeldovich fluids ($\rho = p$).

In the present article, we have considered spherically symmetric metric of embedding class one in 5-flat form and utilized it to obtain non-conformally flat perfect fluid distributions. As a consequence we come across a non-linear partial differential equation. It is substantially difficult to obtain exact solutions of such equation. Therefore a group theoretic method, in particular, one parametric continuous Lie group of transformation (Similarity transformations method) is used to solve the equation and obtained possible solutions, some of which are new. The solutions are invariant under the Lie group of transformations. The same problem for conformally flat was attempted by the other workers [28] using the similar method but with the lesser number of symmetries. Later could not derive the solutions except Einstein universe and flat solutions.

2 Basic Equations of the Problems

An embedding class one space-time can be expressed in 5-flat form as follows

$$ds^2 = -\left(dz^1\right)^2 - \left(dz^2\right)^2 - \left(dz^3\right)^2 + \left(dz^4\right)^2 - \left(dz^5\right)^2 \tag{1}$$

The spherical symmetry with $\frac{\partial g_{22}}{\partial r} \neq 0$ can be imposed on (1) by means of the transformations

$$\begin{aligned} &z^1 = r\sin\theta\cos\varphi, \qquad z^2 = r\sin\theta\sin\varphi, \qquad z^3 = r\cos\theta, \\ &z^4 = t, \qquad z^5 = u(r,t) \end{aligned} \tag{2}$$

and the embedding surface can be written as

$$z^5 = u\left(\sqrt{\left(z^1\right)^2 + \left(z^2\right)^2\left(z^3\right)^2}, z^4\right) \tag{3}$$

Consequently, (1) reduces to

$$ds^2 = -dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) + dt^2 - du^2 \tag{4}$$

The Einstein's field equations for the perfect fluid distributions can be expressed as

$$8\pi T^i_j = -R^i_j + \frac{1}{2}\delta^i_j R = 8\pi\left[(p+\rho)v^i v_j - \delta^i_j p\right], \tag{5}$$

where $v^i v_i = 1$ and ρ, p and v_i are energy density, pressure and flow vector respectively. The Einstein field equations (5) with reference to the metric (4) can be furnished as below

$$8\pi T^1_1 = \frac{u'^2}{r^2 P} - \frac{2u'}{rP^2}\left[\left(1+u'^2\right)\ddot{u} - \dot{u}u'\dot{u}'\right] = 8\pi(\rho+p)v^1 v_1 - 8\pi p, \tag{6}$$

$$\begin{aligned} 8\pi T^2_2 = 8\pi T^3_3 &= -\frac{1}{P^2}\left(\ddot{u}u'' - \dot{u}'^2\right) - \frac{u'}{rP^2}\left[\left(1+u'^2\right)\ddot{u} - 2D\dot{u}u'\dot{u}' - \left(1-\dot{u}^2\right)u''\right] \\ &= -8\pi p, \end{aligned} \tag{7}$$

$$8\pi T^4_4 = \frac{u'^2}{r^2 P} + \frac{2u'}{rP^2}\left[\left(1-\dot{u}^2\right)u'' + \dot{u}u'\dot{u}'\right] = 8\pi(\rho+p)v^4 v_4 - 8\pi p, \tag{8}$$

$$8\pi T^4_1 = \frac{2u'}{rP^2}\left[\left(1+u'^2\right)\dot{u}' - \dot{u}u'u''\right] = 8\pi(\rho+p)v^4 v_1, \tag{9}$$

$$8\pi T_4^1 = -\frac{2u'}{rP^2}\big[\big(1-\dot{u}^2\big)\dot{u}' + \dot{u}u'\ddot{u}\big] = 8\pi(\rho+p)v^1v_4, \tag{10}$$

$$v^1v_1 + v^4v_4 = 1, \qquad v^2 = v_2 = v^3 = v_3 = 0, \tag{11}$$

where $P = 1 - \dot{u}^2 + u'^2$, $u' = \frac{\partial u}{\partial r}$, $\dot{u} = \frac{\partial u}{\partial t}$.

Eliminating ρ, p and v^i among (6)–(11), we get an equation having the product of two factors, i.e. $w(w+R)=0$ [13], which can be expressed as

$$w\bigg[\big(\ddot{u}u'' - \dot{u}'^2\big) + \frac{u'}{r}\big[\big(1+u'^2\big)\ddot{u} - 2\dot{u}u'\dot{u}' - \big(1-\dot{u}^2\big)u''\big] - \frac{u'^2}{r^2}\big[1-\dot{u}^2+u'^2\big]\bigg] = 0 \tag{12}$$

where

$$8\pi w = \big(\ddot{u}u'' - \dot{u}'^2\big) - \frac{u'}{r}\big[\big(1+u'^2\big)\ddot{u} - 2\dot{u}u'\dot{u}' - \big(1-\dot{u}^2\big)u''\big]$$
$$- \frac{u'^2}{r^2}\big[1-\dot{u}^2+u'^2\big] = 0 \tag{13}$$

Here, the symbol w is used for the eigenvalue of the trace-free Weyl tensor. It has been verified that the vanishing of the first factor w of (13) implies the vanishing of the conformal curvature tensor and the corresponding fluid distribution will be conformally flat. However, the vanishing of the second factor $(w+R)$ corresponds to the non-conformally flat perfect fluid distributions. In the later case, we come across a partial differential equation to be satisfied by $u(r,t)$ as

$$H \equiv \big(\ddot{u}u'' - \dot{u}'^2\big) + \frac{u'}{r}\big[\big(1+u'^2\big)\ddot{u} - 2\dot{u}u'\dot{u}' - \big(1-\dot{u}^2\big)u''\big] - \frac{u'^2}{r^2}\big[1-\dot{u}^2+u'^2\big] = 0 \tag{14}$$

Associated expressions for pressure and density are given by

$$8\pi p = \frac{u'^2}{r^2P}, \qquad 8\pi\rho = 8\pi p + \frac{2(\ddot{u}u'' - \dot{u}')}{P^2} \tag{15}$$

3 Similarity Solutions to the Field Equations

As (14) is highly non-linear, we have chosen the similarity transformation method to solve the same. Following the relevant procedure, which requires the invariance of (14) under the transformations

$$\begin{aligned} u^* &= u + \varepsilon\eta(r,t,u;\varepsilon) + O\big(\varepsilon^2\big) \\ r^* &= r + \varepsilon\xi(r,t,u;\varepsilon) + O\big(\varepsilon^2\big) \\ t^* &= t + \varepsilon\tau(r,t,u;\varepsilon) + O\big(\varepsilon^2\big) \end{aligned} \tag{16}$$

and hence one need to satisfy the following equation

$$\eta_{rr}\frac{\partial H}{\partial u_{rr}} + \eta_{rt}\frac{\partial H}{\partial u_{rt}} + \eta_{tt}\frac{\partial H}{\partial u_{tt}} + \eta_r\frac{\partial H}{\partial u_r} + \eta_t\frac{\partial H}{\partial u_t} + \eta\frac{\partial H}{\partial u} + \xi\frac{\partial H}{\partial r} + \tau\frac{\partial H}{\partial t} = 0 \tag{17}$$

to get the symmetry infinitesimals as

$$\xi = ar, \qquad \eta = kt + au + b, \qquad \tau = ku + at + c, \tag{18}$$

where a, k, b, c are four arbitrary parameters. In this process, we have used (14) once more to eliminate one the derivatives $u', \dot{u}, \dot{u}', u'', \ddot{u}$ etc. in (17) and hence could obtain more general symmetry infinitesimals (ξ, η, τ). However, the said procedure was not followed by Thakadiyil and Jasim [28] in their paper.

Forms of the solution of (14) can be obtained by solving the following with reference to (18)

$$\frac{dr}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}. \tag{19}$$

Now, we are going to find out all the similarity solutions and discuss the following particular cases.

3.1 Case 1

For $(a = 0, k \neq 0)$, (19) are of the form

$$\frac{dr}{0} = \frac{dt}{ku + c} = \frac{du}{kt + b} \tag{20}$$

which immediately yields

$$u = \left[(t + b)^2 + f(r)\right]^{\frac{1}{2}} + \alpha \tag{21}$$

where α being constant of integration.

On inserting the value of u from (21) into (14), we get the second order ordinary differential equation in $f(r)$ as

$$\left[\frac{1}{2} - \frac{1}{4r}\bar{f}(r)\right]\left[f(r)\bar{\bar{f}}(r) - \frac{1}{2}\bar{f}^2(r) + \frac{1}{4r}\bar{f}^3(r) + \frac{1}{r}f(r)\bar{f}(r)\right] = 0, \tag{22}$$

where $\bar{f}(r) = \frac{df}{dr}$.

The corresponding expressions for pressure and density read as

$$\begin{aligned} 8\pi p &= \frac{\bar{f}^2(r)}{r^2[4f(r) + \bar{f}^2(r)]}, \\ 8\pi\rho &= 8\pi p + \frac{8[2f(r)\bar{\bar{f}}(r) - \bar{f}^2(r)]}{[4f(r) + \bar{f}^2(r)]^2}, \end{aligned} \tag{23}$$

where $f(r)$ is given by (22).

3.1.1 Case 1A

Vanishing of first factor of (22) gives rise to

$$f(r) = r^2 + k_1 \tag{24}$$

The expressions for pressure and density can be read as

$$\begin{aligned} 8\pi p &= \frac{1}{[2r^2 + k_1]}, \\ 8\pi\rho &= 8\pi p + \frac{2k_1}{[2r^2 + k_1]^2} \end{aligned} \tag{25}$$

which is well-known Kohler-Chao [12] solution, the only static solution of its class. This solution is accelerating but shear free.

The embedding surface in this case reads as

$$\left(z^5-\alpha\right)^2=\left(z^4+b\right)^2+\left(z^1\right)^2+\left(z^2\right)^2+\left(z^3\right)^2+k_1 \tag{26}$$

3.1.2 Case 1B

Vanishing of second factor of (22) reads as

$$f(r)\bar{\bar{f}}(r)-\frac{1}{2}\bar{f}^2(r)+\frac{1}{4r}\bar{f}^3(r)+\frac{1}{r}f(r)\bar{f}(r)=0 \tag{27}$$

which on integration yields

$$f(r)=e\left[\int\frac{1}{\sqrt{(-e+Ar^2)}}dr+B\right]^2, \tag{28}$$

where A and B being constant of integration.

It has been observed that the u in (21) together with (28) does not represent perfect fluid distributions as in the present case $T_1^1\neq T_2^2=T_3^3=T_4^4$ and $T_4^1=0$ (i.e. isotropic conditions are not satisfied). However the u is capable of representing an anisotropic fluid distributions. Various solutions of (28) depending upon ($A=+m^2,-m^2,0$ and $e=\pm1$) and the corresponding expressions for radial pressure p_r, tangential pressure p_θ and energy-density ρ for various cases are displayed as below:

3.1.2.1. When $A=+m^2$, $e=-1$ or $A=-m^2$, $e=1$ in (28), we get

$$f(r)=-\left[\frac{1}{m}\sinh^{-1}(mr)+B\right]^2 \tag{29}$$

The pressure and density are given by

$$\begin{aligned}&8\pi p_r=-\frac{1}{m^2r^4},\qquad 8\pi p_\theta=-8\pi p_r=\frac{1}{m^2r^4},\\&8\pi\rho=8\pi p_r-\frac{2(1+m^2r^2)^{\frac{1}{2}}}{r^3m[\sinh^{-1}(mr)+mB]}\end{aligned} \tag{30}$$

which is unphysical as $p_r<0$.

3.1.2.2. When $A=-m^2$, $e=-1$ in (28), we get

$$f(r)=-\left[\frac{1}{m}\sin^{-1}(mr)+B\right]^2 \tag{31}$$

The corresponding pressures and density are given by

$$\begin{aligned}&8\pi p_r=\frac{1}{m^2r^4},\qquad 8\pi p_\theta=-8\pi p_r=-\frac{1}{m^2r^4},\\&8\pi\rho=8\pi p_r+\frac{2(1-m^2r^2)^{\frac{1}{2}}}{r^3m[\sinh^{-1}(mr)+mB]}\end{aligned} \tag{32}$$

Tangential pressure turns out to be negative and hence an anisotropic model is unphysical.

3.1.2.3. When $A=m^2$, $e=1$ in (28), we get

$$f(r)=\left[\frac{1}{m}\cosh^{-1}(mr)+B\right]^2 \tag{33}$$

The expressions of pressures and density are given by

$$\begin{aligned} 8\pi p_r &= \frac{1}{m^2r^4}, \qquad 8\pi p_\theta = -8\pi p_r = -\frac{1}{m^2r^4}, \\ 8\pi\rho &= 8\pi p_r - \frac{2(m^2r^2-1)^{\frac{1}{2}}}{r^3m[\cosh^{-1}(mr)+mB]} \end{aligned} \tag{34}$$

Tangential pressure turns out to be negative.

3.1.2.4. When $A=0$, $e=\pm1$ in (28), we get

$$f(r)=-[r+B]^2 \tag{35}$$

This case disturbs the metric conditions i.e. $|g_{ij}|\neq 0$.

3.2 Case 2

For $(a\neq 0, k=0)$, (19) are of the form

$$\frac{dr}{ar}=\frac{dt}{at+c}=\frac{du}{au+b} \tag{36}$$

Integrating (36), we get

$$u=-\frac{b}{a}+\frac{r}{a}f(x) \tag{37}$$

where $x=(\frac{at+c}{r})$ is the similarity variable.

On inserting the value of u from (37) into (14), we get the second order ordinary differential equation in $f(x)$ which can assume the following factorized form

$$\begin{aligned} &\big[f(x)-x\bar{f}(x)\big]\bigg[\bar{\bar{f}}(x)\left(\frac{x^2}{a^2}-1\right)-\frac{1}{a^2}f^2(x)\bar{\bar{f}}(x)+\frac{1}{a^2}f(x)+\frac{1}{a^4}f^3(x)-\frac{1}{a^2}x\bar{f}(x) \\ &\quad -\frac{3}{a^4}xf^2(x)\bar{f}(x)+\frac{1}{a^4}\bar{f}^3(x)\big(a^2x-x^3\big)+\frac{1}{a^4}f(x)\bar{f}^2(x)\big(3x^2-a^2\big)\bigg]=0 \end{aligned} \tag{38}$$

where $\bar{f}(x)=\frac{df}{dx}$.

Expressions for pressure and density are given as

$$8\pi p=8\pi\rho=\frac{(x\bar{f}-f)^2}{r^2[a^2-a^2\bar{f}^2+(x\bar{f}-f)^2]}, \tag{39}$$

where $f(x)$ is given by (38).

Vanishing of the first factor of (38) gives rise the flat space since the Riemannian curvature tensor vanishes identically in the space time, it is said to be flat space and it is devoid of any gravitational significance. While the vanishing of second factor is further solved by the similarity transformation method, we obtain infinitesimals ξ and η as [5]

$$\xi = Kf, \qquad \eta = Kx, \tag{40}$$

Form of the solution can be obtained by solving the following with reference to (40)

$$\frac{dx}{Kf} = \frac{df}{Kx} = \frac{d\bar{f}}{-K(-1+\bar{f}^2)} \tag{41}$$

Integration of first two gives

$$w^2 = x^2 - f^2. \tag{42}$$

While the integration of second and third gives

$$v(x, f, \bar{f}) = \frac{x\bar{f} - f}{x - f\bar{f}} \tag{43}$$

A first order differential equation for $v(w)$ can be found directly as follows

$$a^2\bigl(w^2 - a^2\bigr)w\frac{dv}{dw} - \bigl(w^4 - a^4\bigr)v^3 - a^4 v = 0. \tag{44}$$

On making use of (42) and (43) into (44), we get

$$\frac{x\bar{f} - f}{x - f\bar{f}} = \left[\frac{a^2[-a^2 + (x^2 - f^2)]}{-(x^2 - f^2)^2 - a^4 + k_1 a^2(x^2 - f^2)}\right]^{\frac{1}{2}}, \tag{45}$$

k_1 being a constant.

Equation (45) can easily be sent to the variable separable form by means of the transformation $x = R\cosh\Phi$, $f = R\sinh\Phi$ and the equation yields the following solution in quadrature form

$$\Phi = \frac{1}{R}\int\left(\frac{a^2(-a^2 + R^2)}{-R^4 - a^4 + k_1 a^2 R^2}\right)^{\frac{1}{2}} dR \tag{46}$$

where $R^2 = x^2 - f^2$, $\frac{f}{x} = \tanh\Phi$.

The expressions for pressure and density are given as

$$8\pi p = 8\pi\rho = \frac{(x\bar{f} - f)^2}{r^2[a^2 - a^2\bar{f}^2 + (x\bar{f} - f)^2]} \tag{47}$$

Hence, (47) represent a Zeldovich fluid (Stiff fluid). The solution is already available in non-comoving system [9].

3.3 Case 3

For $(a = 0, k = 0)$, (19) are of the form

$$\frac{dr}{0} = \frac{dt}{c} = \frac{du}{b} \tag{48}$$

which immediately suggests the form of u as

$$u = \alpha t + f(r) \quad \text{where } \alpha = \frac{b}{c}. \tag{49}$$

On inserting the value of u from (49) into (14), we have

$$r\left(1-\alpha^2\right)f'' = -f'\left(1-\alpha^2+f'^2\right) \tag{50}$$

The general solution of (50) yields

$$f'^2 = \left[\frac{\beta(1-\alpha^2)}{(r^2-\beta)}\right] \tag{51}$$

where β being arbitrary constant of integration.

It is observed that (49) with (51) do not satisfy the isotropic conditions $T_1^1 = T_2^2 = T_3^3 \neq T_4^4$. Instead we get the relation $T_1^1 \neq T_2^2 = T_3^3 = T_4^4$ satisfied. Consequently, the present case represents an anisotropic fluid with radial pressure p_r, tangential pressure p_θ and energy-density ρ are given below

$$8\pi p_r = \frac{\beta}{r^4}, \qquad 8\pi p_\theta = -8\pi p_r = -\frac{\beta}{r^4}, \qquad 8\pi\rho = \frac{\beta}{r^4} \tag{52}$$

3.4 Case 4

For ($a \neq 0$, $k \neq 0$), (19) are of the form

$$\frac{dr}{ar} = \frac{dt}{ku+at+c} = \frac{du}{kt+au+b} \tag{53}$$

This case is extremely difficult to solved and unable to provide the similarity variable explicitly. Therefore the fluid solutions could not be derived.

4 Conclusion and Summary of the Results

We have obtained seven invariant solutions for non-conformally flat fluid distributions of embedding class one. Some of the solutions in this paper are found for the first time. Physical implications of the solutions are furnished as below one by one:

Case 1A is famous static solution due to Kohler-Chao [12] which is accelerating but shear free. Case 1B consists one of these is flat space and the remaining three solutions are anisotropic (i.e. $T_1^1 \neq T_2^2 = T_3^3 = T_4^4$ and $T_4^1 = 0$) but unphysical as tangential and normal pressures are not positive. Therefore, all these anisotropic models are unphysical.

Case 2 gives Zeldovich fluid which is accelerating and already available in non-comoving system [9].

Case 3 gives anisotropic fluid distributions but unphysical due to the negative tangential or normal pressures.

Case 4 could not be solved due to the unavailability of similarity variable.

A new class of solutions of Einstein field equations is investigated for nonconformally flat fluid spherically symmetric space-time under the Lie group of transformations. Thus, the plus points of the solutions so obtained are all expressible in 5-flat form and may be used in the various theories named in the introduction.

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