Int J Theor Phys (2014) 53:1710–1716
DOI 10.1007/s10773-013-1968-6

Hawking Radiation of Black Hole in Einstein-Proca Theory

Shuzheng Yang · Kai Lin · Jin Li

Received: 6 October 2013 / Accepted: 17 December 2013 / Published online: 9 January 2014

Abstract The Hawking radiation of black hole in Einstein-Proca theory is discussed in this paper. The Einstein-Proca black hole is more general than Reissner-Nordström black hole, because Proca field is massive vector field. We calculate several quantum perturbations in this spacetime, and obtain the Hawking radiation at the horizon in Einstein-Proca theory.

Keywords Einstein-Proca theory · Black-hole thermodynamics · Hawking radiation

1 Introduction

General relativity predicted the black hole, which can capture any particle insider the event horizon, so the observer at infinity is impossible to get any information and particle inside the horizon of classical black hole. However, in 1976, Hawking introduced the quantum effect into black hole physics, and proved the black hole could emit quantum thermal radiation [1–3]. Basing on this theory, people constructed the black hole thermodynamic, which can connect with gravity, quantum field theory and thermodynamic.

In 2000, Parikh and Wilczek et al. proposed that the Hawking radiation can be researched by the quantum tunneling theory [4–7], and this work quickly attracted the interests of many physicists [8–22]. Subsequently, Kernel and Mann studied the Dirac particle tunneling radiation of black hole in 2008 [23, 24], and then Li, Chen, Jiang et al. used this method to

S. Yang (✉)
Institute of Theoretical Physics, China West Normal University, Nanchong 637002, China
e-mail: szyangphys@126.com

K. Lin
Department of Physics, University of Electronic Science and Technology of China, Chengdu 610054, China
e-mail: lk314159@126.com

J. Li
Department of Physics, Chongqing University, Chongqing 400030, China
e-mail: cqstarv@hotmail.com

research the Hawking tunneling radiation from Kerr black hole, $2+1$ dimensional BTZ black hole, and $4+1$ dimensional black holes [25–35]. We propose a new method to derive the semi-classical Hamilton-Jacobi Equation from Dirac equation in curved spacetime, and finally get the Dirac Hawking tunneling radiation of black holes [36–39]. What's more, we also prove that other quantum equations, such as vector equation, spin 3/2 equation and gravitational perturbation equation, could be simplified to Hamilton-Jacobi equation, so that the Hamilton-Jacobi method can be used to uniformly calculate all kinds of tunneling radiation from black hole. In this paper, we will use the Hamilton-Jacobi method to study the Hawking radiation of black hole in Einstein-Proca theory.

2 Static Black Hole Solution in Einstein-Proca Theory

The Proca field is massive vector field, which action in curve spacetime can be given by [40]

$$S=\int d^4x\sqrt{-g}\left(R+2\Lambda-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}+\frac{\mu^2}{2}A_\nu A^\nu\right), \tag{1}$$

where $F_{\alpha\beta}=\partial_\alpha A_\beta-\partial_\beta A_\alpha$ is electromagnetic tensor, and μ is the mass of Proca particle. The static spherical symmetrical spacetime metric is

$$ds^2=-N(r)^2f(r)dt^2+\frac{dr^2}{f(r)}+r^2d\theta^2+r^2\sin^2\theta d\varphi^2. \tag{2}$$

Substituting this metric into the action of Einstein-Proca theory, considering the four-potential in static spacetime $A_\nu=\{A_0(r),0,0,0\}$, we can get the differential equations

$$r^2A_0'^2+4N^2\left(rf'+f-1-\Lambda r^2\right)-\frac{\mu^2r^2A_0^2}{f}=0, \tag{3}$$

$$\frac{\mu^2rA_0^2}{f^2}+4NN'=0, \tag{4}$$

$$\mu^2rA_0+f\left(rA_0''+2A_0'\right)-\frac{rfA_0'N'}{N}=0. \tag{5}$$

It is very difficult to get the analytic solution from above equations, but fortunately, the research about Hawking radiation just needs to study the property near the horizon of black hole.

The definition of the horizon $r=r_0$ of static spacetime requires

$$f(r_0)=0. \tag{6}$$

Therefore, at the event horizon of spacetime, we find $A_0(r_0)=0$, which means that the spacetime structure in Einstein-Proca spacetime cannot be reduced to Reissner-Nordström spacetime as $\mu\longrightarrow 0$. Now, let's expand the f and A_0 as follows

$$f=f'(r_0)(r-r_0)+\frac{1}{2}f''(r_0)(r-r_0)^2+\mathcal{O}(r-r_0)^3, \tag{7}$$

$$A=A_0'(r_0)(r-r_0)+\frac{1}{2}A_0''(r_0)(r-r_0)^2+\mathcal{O}(r-r_0)^3, \tag{8}$$

where setting $a=f'(r_0)$, $b=A_0'(r_0)$ and $c=A_0''(r_0)$. Because we discuss non-extreme spacetime case in this paper, the condition $a\neq 0$ is satisfied. Near the event horizon $r-r_0\ll 1$, Eq. (4) can be expand, and then derived

$$N(r)=\sqrt{1-\frac{\mu^2b^2}{4a^2}r^2}, \tag{9}$$

note that we choose the integral constant as 1, since $N=1$ when μ is very small. Subsequently, combing the Eqs. (3), (5), (7) and (8) near the horizon, we obtain

$$b^2r_0^2+4\left(1-\frac{\mu^2b^2}{4a^2}r^2\right)\left(ar_0-1-\Lambda r_0^2\right)=0, \tag{10}$$

$$\frac{\mu^2br_0}{a}+2b+cr_0+\frac{b^3\mu^2r_0^2}{4a^2-\mu^2b^2r_0^2}=0, \tag{11}$$

a and b can be solved by above two equations, where r_0 and c are undetermined constants which depend on the mass and charge of black hole. As $c=0$, the solution is given by

$$a=\frac{1+\Lambda r_0^2}{r_0}, \qquad b=0, \tag{12}$$

as $\mu=0$, the solution is

$$a=\frac{16-c^2r_0^4+16\Lambda r_0^2}{16r_0}, \qquad b=-\frac{c}{2}r_0. \tag{13}$$

However, it is difficult to get a general solution for Eqs. (10) and (11), so we have to use diagrams to show the dynamical property of the Einstein-Proca black hole at horizon in this paper.

3 Quantum Perturbation at Horizon and Hamilton-Jacobi Equation

Hawking radiation is quantum radiation, since the semi-classical Hamilton-Jacobi equation will be derived in this section. First of all, in black hole spacetime, the scalar field equation with mass μ called as Klein-Gordon equation, is given by

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial}{\partial x^\alpha}-\frac{iq}{\hbar}A_\alpha\right)\left[\sqrt{-g}g^{\alpha\beta}\left(\frac{\partial}{\partial x^\beta}-\frac{iq}{\hbar}A_\beta\right)\right]\Phi-\frac{\mu^2}{\hbar^2}\Phi=0. \tag{14}$$

We set $\Phi=Ce^{\frac{i}{\hbar}S}$, and consider the $\hbar$ is very small, so that we can keep the 0 order term by expanding as $\hbar$. Therefore, we can get the Hamilton-Jacobi equation

$$g^{\alpha\beta}\left(\frac{\partial S}{\partial x^\alpha}+qA_\alpha\right)\left(\frac{\partial S}{\partial x^\beta}+qA_\beta\right)+\mu^2=0. \tag{15}$$

Because the Hamilton-Jacobi equation comes from Klein-Gordon equation, it depicts the dynamical property of scalar particle in principle, but we will prove that this equation can also be derived by other quantum field equations.

Dirac equation can depict the spin-$1/2$ particle field, such as neutrinos and electrons. The Dirac equation in curved spacetime is

$$\gamma^\nu D_\nu\Psi+\frac{\mu}{\hbar}\Psi=0, \tag{16}$$

where

$$D_\nu=\partial_\nu+\frac{i}{2}\Gamma_\nu^{\alpha\beta}\Pi_{\alpha\beta}, \quad \Pi_{\alpha\beta}=\frac{i}{4}\left[\gamma^\alpha,\gamma^\beta\right], \tag{17}$$

and the gamma matrixes satisfy the relationship $\{\gamma^\alpha,\gamma^\beta\}=2g^{\alpha\beta}I$. In semi-classical approximation, we can set [39]

$$\Psi=j\left(t,r,\cdots x^\eta\cdots\right)e^{\frac{i}{\hbar}S(t,r,\cdots x^\eta\cdots)}=\begin{bmatrix}A_{\frac{m}{2}\times 1}(t,r,\cdots x^\eta\cdots)\\ B_{\frac{m}{2}\times 1}(t,r,\cdots x^\eta\cdots)\end{bmatrix}e^{\frac{i}{\hbar}S(t,r,\cdots x^\eta\cdots)} \tag{18}$$

where j is a column matrix. Dividing by the exponential terms and multiplying by $\hbar$, the resulting equations to leading order in $\hbar$ are

$$i\gamma^{\nu}\frac{\partial S}{\partial x^{\nu}}j+\mu j=0. \tag{19}$$

Multiplying both sides of above equation by the matrix $-i\gamma^{\nu}\frac{\partial S}{\partial x^{\nu}}j$, we obtain

$$\gamma^{\mu}\frac{\partial S}{\partial x^{\mu}}\gamma^{\nu}\frac{\partial S}{\partial x^{\nu}}j-i\gamma^{\nu}\frac{\partial S}{\partial x^{\nu}}\mu j=0. \tag{20}$$

Using the relationship $-i\gamma^{\nu}\frac{\partial S}{\partial x^{\nu}}j=\mu j$ to simplify Eq. (20), we get

$$\gamma^{\mu}\frac{\partial S}{\partial x^{\mu}}\gamma^{\nu}\frac{\partial S}{\partial x^{\nu}}j+\mu^{2}j=0. \tag{21}$$

Exchanging μ and ν in Eq. (21), it is

$$\gamma^{\nu}\frac{\partial S}{\partial x^{\nu}}\gamma^{\mu}\frac{\partial S}{\partial x^{\mu}}j+\mu^{2}j=0, \tag{22}$$

(Eqs. (34) + Eq. (35))/2, it is given by

$$\frac{1}{2}\left\{\gamma^{\mu},\gamma^{\nu}\right\}\frac{\partial S}{\partial x^{\nu}}\frac{\partial S}{\partial x^{\mu}}j+\mu^{2}j=0. \tag{23}$$

Finally, considering the relationship $\{\gamma^{\mu},\gamma^{\nu}\}=2g^{\mu\nu}I$, the matrix equation becomes

$$\left(g^{\mu\nu}\frac{\partial S}{\partial x^{\nu}}\frac{\partial S}{\partial x^{\mu}}+\mu^{2}\right)j=0, \tag{24}$$

so

$$\det\left(g^{\mu\nu}\frac{\partial S}{\partial x^{\nu}}\frac{\partial S}{\partial x^{\mu}}+\mu^{2}\right)=0. \tag{25}$$

It is no other than the Hamilton-Jacobi equation (14). Therefore, it is proven that Hamilton-Jacobi equation can depict the Dirac particle in black hole spacetime.

In Einstein-Proca spacetime background, the radiation could also be Proca particle, and vector field equation in curved spacetime is given by [41, 42]

$$\tilde{F}^{\beta\alpha}_{;\alpha}-\frac{\mu^{2}}{\hbar^{2}}=0. \tag{26}$$

Note that the μ is the mass of radiation of Proca particle. We can choose the gauge $\tilde{A}^{\alpha}_{;\alpha}=0$, so equation becomes

$$\tilde{A}^{;\alpha}_{\beta;\alpha}-\frac{\mu^{2}}{\hbar^{2}}\tilde{A}_{\beta}=0. \tag{27}$$

So we let $\tilde{A}_{\beta}=a_{\beta}e^{\frac{i}{\hbar}S}$. After dividing by the exponential terms and multiplying by $\hbar$, the resulting equations to leading order in $\hbar$ are Hamilton-Jacobi equation.

Another famous perturbation field is gravitational perturbation which can be derived by classical gravity field equation. Consider the metric can be rewritten as $g_{\alpha\beta}=\bar{g}_{\alpha\beta}+h_{\alpha\beta}$, where the $\bar{g}_{\alpha\beta}$ and $h_{\alpha\beta}$ are spacetime background part and perturbation part respectively. The gauge usually is $\nabla^{\alpha}h_{\alpha\beta}=0$, and the gravitational wave equation in curved spacetime is given by [43]

$$h^{;\nu}_{ab;\nu}-R_{adbc}h^{ac}=0, \tag{28}$$

where, the R_{adbc} is Riemann tensor of background spacetime. Next, we let $h_{ab} = b_{ab} e^{\frac{i}{\hbar} S}$, and ignore all small terms with $\hbar$, we can get a massless Hamilton-Jacobi equation

$$g^{\alpha\beta} \frac{\partial S}{\partial x^\alpha} \frac{\partial S}{\partial x^\beta} = 0, \tag{29}$$

because graviton in general relativity is massless, and gravitational wave radiate at speed of light. Recently, some works also show that the graviton could have mass in several modify gravity.

4 Hawking Temperature and Tunneling Rate of Einstein-Proca Black Hole

In Sect. 3, we have proven that all kinds of quantum field equation can become as Hamilton-Jacobi equation, so the semi-classical equation can depict Hawking radiation as any quantum particle. Now let's separate the variable for the action S as

$$S = -\omega t + W(r) + Y(\theta, \varphi), \tag{30}$$

and substituting the black hole metric (1), we can get the radial Hamilton-Jacobi equation

$$-\frac{(\omega - qA_0)^2}{N^2 f} + f\left(\frac{dW}{dr}\right)^2 + \mu^2 = \frac{\lambda}{r^2}, \tag{31}$$

where the λ is constant, and W in above equation can be solved as

$$W_\pm(r) = \int dr \frac{\sqrt{(\omega - qA_0)^2 + N^2(r_0) f'(r_0)(r - r_0)(\frac{\lambda}{r^2} - \mu^2)}}{N(r_0) f'(r_0)(r - r_0)} = \pm i\pi \frac{\omega - \omega_0}{N(r_0) f'(r_0)}, \tag{32}$$

where $\omega_0 = qA_0(r_0)$. Note the condition $A_0(r_0) = 0$ in Sect. 2, we get

$$W_\pm(r) = \pm \frac{i\pi\omega}{N(r_0) f'(r_0)} = \pm \frac{i\pi\omega}{N(r_0)a}. \tag{33}$$

Therefore, the tunneling rate at the event horizon of black hole is

$$\Gamma = e^{-2(\Im W_+ - \Im W_-)} = \exp\left(-4\pi \frac{\omega}{N(r_0)a}\right), \tag{34}$$

and the Hawking radiation is given by

$$T_h = \frac{a}{4\pi} N(r_0). \tag{35}$$

It is evident that charged particle in Einstein-Proca spacetime doesn't depend on the potential A_ν. According to Eq. (12), as $c = 0$, we have

$$T_h = \frac{1 + \Lambda r_0^2}{4\pi r_0} \sqrt{1 - \frac{\mu^2 b^2}{4a^2} r_0^2}. \tag{36}$$

If ignore the cosmology constant Λ, we find the temperature will be lower as black hole is bigger. As $\mu = 0$, we get

$$T_h = \left(\frac{1 + \Lambda r_0^2}{4\pi r_0} - \frac{c^2 r_0^4}{64\pi r_0}\right) \sqrt{1 - \frac{\mu^2 b^2}{4a^2} r_0^2}. \tag{37}$$

The diagram (Fig. 1) shows the temperature property of Einstein-Proca black hole, where $r_0 = 1$. It's evident that the c can result in the temperature decrease, but the mass of Proca mass m can cause the temperature increase.

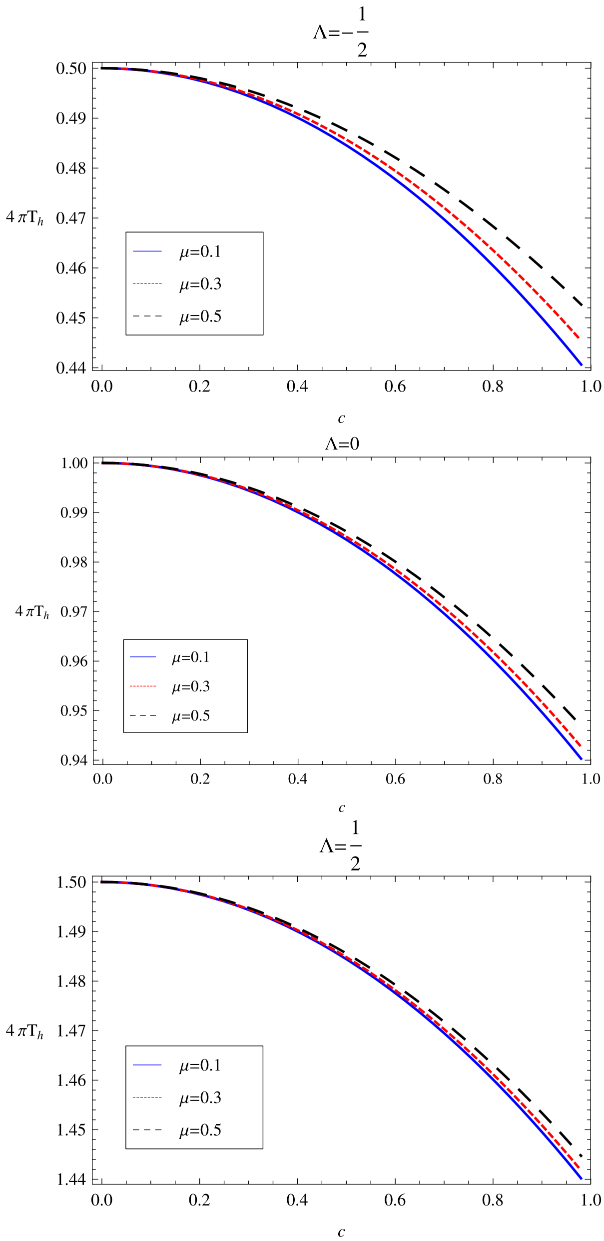

Fig. 1 The Relationship of c, T_h and μ

5 Conclusion

We study the Hawking tunneling radiation of black hole in Einstein-Proca theory. Our results show that the potential A_ν at horizon is 0, so the temperature of black hole doesn't depend on the potential. Our work also shows that the massive black hole has low temperature but the mass of Proca field could cause the temperature of black hole increase. Of course, the effect of Proca field mass is very weak.

In quantum field theory, the mass of field particle can be produced by the Higgs mechanism, which comes from the $U(1)$ symmetry breaking. Recently, LHC have gotten several results to support the Higgs particle, so the research about Proca field could be intriguing. On the other hand, the experiment of Luo et al. proved that the mass of photon should be lower than $(0.9 \pm 1.5) \times 10^{-52}$ g, but it still can not rule out the probability of massive photon [44, 45], and it should use Proca theory to depict the property of electromagnetic field if photon has rest mass.

Acknowledgements This work is supported by National Natural Science Foundation of China Nos. 11178018, 11075224 and 11205254, and the Natural Science Foundation Project of CQ CSTC, 2011BB0052.

References

1. Hawking, S.W.: Nature **248**, 30 (1974)
2. Hawking, S.W.: Commun. Math. Phys. **43**, 199 (1975)
3. Damoar, T., Ruffini, R.: Phys. Rev. D **14**, 332 (1976)
4. Kraus, P., Wilczek, F.: Nucl. Phys. B **433**, 403 (1995). arXiv:gr-qc/9408003
5. Robinson, S.P., Wilczek, F.: Phys. Rev. Lett. **95**, 011303 (2005). arXiv:gr-qc/0502074
6. Iso, S., Umetsu, H., Wilczek, F.: Phys. Rev. D **74**, 044017 (2006). arXiv:hep-th/0606018
7. Parikh, M.K., Wilczek, F.: Phys. Rev. Lett. **85**, 5042 (2000). arXiv:hep-th/9907001
8. Zhang, J.Y., Zhao, Z.: Phys. Lett. B **638**, 110 (2006). arXiv:gr-qc/0512153
9. Hemming, S., Keski-Vakkuri, E.: Phys. Rev. D **64**, 044006 (2001). arXiv:gr-qc/0005115
10. Wu, S.Q., Jiang, Q.Q.: J. High Energy Phys. **0603**, 079 (2006)
11. Vagenas, E.C.: Mod. Phys. Lett. A **17**, 609 (2002). arXiv:hep-th/0108147
12. Medved, A.J.M.: Phys. Rev. D **66**, 124009 (2002). arXiv:hep-th/0207247
13. Srinivasan, K., Padmanabhan, T.: Phys. Rev. D **60**, 24007 (1999). arXiv:gr-qc/9812028
14. Akhmedov, E.T., Akhmedova, V., Singleton, D.: Phys. Lett. B **642**, 124 (2006)
15. Angheben, M., Nadalini, M., Vanzo, L., Zerbini, S.: J. High Energy Phys. **0505**, 014 (2005)
16. Shankaranarayanan, S., Padmanabhan, T., Srinivasan, K.: Class. Quantum Gravity **19**, 2671 (2002)
17. Cai, R.G., Cao, L.M., Hu, Y.P.: J. High Energy Phys. **0808**, 090 (2008). arXiv:0807.1232 [hep-th]
18. Banerijee, R., Majhi, B.R.: J. High Energy Phys. **0806**, 095 (2008). arXiv:0805.2220
19. Majhi, B.R.: Phys. Rev. D **79**, 044005 (2009). arXiv:0809.1508
20. Modak, S.K.: Phys. Lett. B **671**, 167 (2009). arXiv:0807.0959
21. Lin, K., Yang, S.Z.: Europhys. Lett. **82**, 20006 (2009)
22. Lin, K., Yang, S.Z.: Int. J. Theor. Phys. **48**, 2920 (2009)
23. Kerner, R., Mann, R.B.: Class. Quantum Gravity **25**, 095014 (2008). arXiv:0710.0612 [hep-th]
24. Kerner, R., Mann, R.B.: Phys. Lett. B **665**, 277 (2008). arXiv:0803.2246 [hep-th]
25. Chen, D.Y., Jiang, Q.Q., Zu, X.T.: Class. Quantum Gravity **25**, 205022 (2008). arXiv:0803.3248 [hep-th]
26. Chen, D.Y., Jiang, Q.Q., Zu, X.T.: Phys. Lett. B **665**, 106 (2008). arXiv:0804.0131 [hep-th]
27. Li, R., Ren, J.R., Wei, S.W.: Class. Quantum Gravity **25**, 125016 (2008). arXiv:0803.1410 [gr-qc]
28. Li, R., Ren, J.R.: Phys. Lett. B **661**, 370 (2008). arXiv:0802.3954 [gr-qc]
29. Li, H.L., Yang, S.Z., Zhou, T.J., Lin, R.: Europhys. Lett. **84**, 20003 (2008)
30. Criscienzo, R.D., Vanzo, L.: Europhys. Lett. **82**, 60001 (2008). arXiv:0803.0435 [hep-th]
31. Yang, S.Z., Lin, K., Yang, J.: Mod. Phys. Lett. A **24**, 2187 (2009)
32. Lin, K., Yang, S.Z.: Chin. Phys. B **18**, 2154 (2009)
33. Lin, K., Yang, S.Z.: Chin. Phys. Lett. **26**, 010401 (2009)
34. Jiang, Q.Q.: Phys. Rev. D **78**, 044009 (2008)
35. Jiang, Q.Q.: Phys. Lett. B **666**, 517 (2008)
36. Lin, K., Yang, S.Z.: Phys. Rev. D **79**, 064035 (2009)
37. Lin, K., Yang, S.Z.: Phys. Lett. B **674**, 127 (2009)
38. Yang, S.Z., Lin, K.: Sci. China **40**, 507 (2010) (in Chinese)
39. Lin, K., Yang, S.Z.: Chin. Phys. B **20**, 110403 (2011)
40. Torii, T., Maeda, K.-I., Tachizawa, T.: Phys. Rev. D **51**, 1510 (1995). arXiv:gr-qc/9406013
41. Rosa, J.G., Dolan, S.R.: Phys. Rev. D **85**, 044043 (2012). arXiv:1110.4494
42. Konopya, R.A.: Phys. Rev. D **73**, 024009 (2009). arXiv:gr-qc/0509026
43. Flanagan, E.E., Hughes, S.A.: New J. Phys. **7**, 204 (2005). arXiv:gr-qc/0501041
44. Tu, L.C., Shao, C.C., Luo, J.: Phys. Lett. A **352**, 267 (2006)
45. Tu, L.C., Luo, J.: Physics **35**, 779 (2006) (in Chinese)