

# Hypergeometric Equation in Modeling Relativistic Isotropic Sphere

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**Abstract** We study the Einstein system of equations in static spherically symmetric spacetimes. We obtained classes of exact solutions to the Einstein system by transforming the condition for pressure isotropy to a hypergeometric equation choosing a rational form for one of the gravitational potentials. The solutions are given in simple form that is a desirable requisite to study the behavior of relativistic compact objects in detail. A physical analysis indicate that our models satisfy all the fundamental requirements of realistic star and match smoothly with the exterior Schwarzschild metric. The derived masses and densities are consistent with the previously reported experimental and theoretical studies describing strange stars. The models satisfy the standard energy conditions required by normal matter.

**Keywords** Einstein field equations · Exact solutions · Strange stars

## 1 Introduction

Static solutions of the Einstein's field equations for spherically symmetric manifolds are important in the description of relativistic spheres in astrophysics. First two exact solutions for the Einstein's field equations were obtained by Schwarzschild [1, 2] about a century ago; one solution describing the geometry of the space-time exterior to a perfect relativistic fluid sphere in hydrostatic equilibrium, and the other corresponding to the interior geometry of a fluid sphere of homogeneous energy-density. There have been considerable efforts since then to find exact solutions to the Einstein's field equations for the interior space-time that smoothly match to the Schwarzschild exterior [1] and such generated models becomes physically admissible to describe compact relativistic spheres with strong gravitational fields

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such as in neutron stars for example. However, only a few of these solutions correspond to non-singular metric functions with physically acceptable energy momentum tensor [3–9]. In fact, the interior Schwarzschild solution provides two important features towards obtaining configurations in hydrostatic equilibrium: (i) an absolute upper limit on compaction parameter (mass-to-radius ratio) for any static spherical solution (provided the density decreases monotonically outwards from the centre) in hydrostatic equilibrium [10], and (ii) an assigned value of the compaction parameter, the minimum central pressure, corresponds to the homogeneous density solution [11].

There have been significant experimental developments in recent years and a number of attempts have been made on measuring the radii and masses of dense astrophysical objects; e.g. observations on double neutron star [12], glitches in radio pulsars [13], thermal emission [14, 15] from accreting neutron star and from millisecond X-ray pulsars, pressure on neutron star matter at supranuclear density [16], etc. Mass-to-radius ratio of astrophysical objects provides a vital clue to distinguish different stars such as white dwarf, neutron stars and strange stars from one another [17]. Stellar objects such as SAX, Her.X-1 and other low-mass X-ray binaries have been differently interpreted at times [18] and hence compaction parameter is suggested to be a good measure in this study to differentiate compact stellar objects.

The study of experimental data of highly dense stars have been reported in literature with various theoretical approaches. Strange quark matter model [19–21] with linear equation of state have been of particular interest, modelling such compact stars [22–28]. There have also been interpretations with quadratic equation of states [29, 30], linear or nonlinear equation of state [31] and polytropic equation of states [32, 33].

In this work we seek new classes of solutions to the Einstein system in static spherically symmetric spacetimes with isotropic matter distribution which satisfy the physical criteria [34]:

- (i) regularity of the gravitational potentials at the origin;
- (ii) positive definiteness of the energy density and the radial pressure at the origin;
- (iii) vanishing of the pressure at some finite radius;
- (iv) monotonic decrease of the energy density and the radial pressure with increasing radius.
- (v) causality condition: speed of sound should be less than the speed of light.

In addition the generated models satisfy the junction condition: interior metric match smoothly with the Schwarzschild exterior metric at the boundary.

## 2 The Field Equations

The internal structure of a dense compact relativistic sphere in static spherically symmetric spacetime can be described by the line element

$$ds^2 = -e^{2\nu(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

in Schwarzschild coordinates  $(x^a) = (t, r, \theta, \phi)$ . The Einstein field equations for a neutral perfect fluid matter distribution can be written in the form

$$\rho = \frac{1}{r^2} [r(1 - e^{-2\lambda})]', \quad (2)$$

$$p = -\frac{1}{r^2}(1 - e^{-2\lambda}) + \frac{2v'}{r}e^{-2\lambda}, \tag{3}$$

$$p = e^{-2\lambda}\left(v'' + v'^2 + \frac{v'}{r} - v'\lambda' - \frac{\lambda'}{r}\right), \tag{4}$$

where primes denote differentiation with respect to the radial coordinate  $r$ . The energy density  $\rho$  and the pressure  $p$  are measured relative to the comoving fluid 4-velocity  $u^a = e^{-\nu}\delta_0^a$ ; and we use the units for the coupling constant  $\frac{8\pi G}{c^4} = 1$  and the speed of light  $c = 1$ . The system of (2)–(4) determines the behaviour of the gravitational field for a neutral perfect fluid source. The mass contained within a radius  $r$  of the sphere is defined as

$$m(r) = \frac{1}{2} \int_0^r \omega^2 \rho(\omega) d\omega. \tag{5}$$

If we introduce the transformation

$$x = Cr^2, \quad Z(x) = e^{-2\lambda(r)}, \quad A^2 y^2(x) = e^{2\nu(r)}, \tag{6}$$

where  $A$  and  $C$  are arbitrary constants, the system (2)–(4) has the equivalent form

$$\frac{\rho}{C} = \frac{1 - Z}{x} - 2\dot{Z}, \tag{7}$$

$$\frac{p}{C} = 4Z\frac{\dot{y}}{y} + \frac{Z - 1}{x}, \tag{8}$$

$$0 = 4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + (\dot{Z}x - Z + 1)y, \tag{9}$$

where dots represent differentiation with respect to  $x$ . The mass function (5) becomes

$$m(x) = \frac{1}{4C^{3/2}} \int_0^x \sqrt{w}\rho(w)dw, \tag{10}$$

in terms of the new variables in (6).

### 3 Choosing Potential

The Einstein system (7)–(9) comprises three equations in the four unknowns  $Z$ ,  $y$ ,  $\rho$  and  $p$ . Therefore we have the freedom to choose one of the gravitational potential to integrate the master equation (9), called the generalised condition of pressure isotropy. In this treatment we specify the gravitational potential  $Z$  on physical ground, so that it is possible to integrate (9). The explicit solution of the Einstein system (7)–(9) then follows. We choose a particular form

$$Z = \frac{1 + ax}{1 + bx}, \tag{11}$$

where  $a$  and  $b$  are real constants. This form of gravitational potential have been previously used to model charged isotropic compact objects [35] and it contains physically acceptable neutron star models as special case: for example, when  $a = -\frac{1}{2}$  and  $b = 1$  we regain the dense neutron star model of Durgapal and Bannerji [3]; and when  $a = -1$  and  $b = 7$  we

regain Tikekar super dense star model [6]. These solutions can be used to model a relativistic compact sphere with desirable physical properties. We attempt to find new classes of solutions with the choice (11) to the Einstein system in closed form.

Substituting (11) into (9) we obtain

$$4(1 + ax)(1 + bx)\ddot{y} + 2(a - b)\dot{y} + b(b - a)y = 0. \tag{12}$$

The differential equation (12) is difficult to solve in the above form; we first introduce a transformation to obtain a more convenient form. We let

$$X = \frac{a}{(a - b)}(1 + bx), \quad Y(X) = y(x), \quad a \neq b \neq 0. \tag{13}$$

With the help of (13), (12) becomes

$$X[X - 1]\frac{d^2Y}{dX^2} + \frac{1}{2}\frac{dY}{dX} - \left[\frac{a - b}{4a}\right]Y = 0 \tag{14}$$

in terms of the new dependent and independent variables  $Y$  and  $X$  respectively, which is a Gaussian type hypergeometric equation.

### 4 Exact Solutions

The general solution of the hypergeometric equation (14) is given by

$$Y = C_1 F\left(\alpha, -(\alpha + 1), -\frac{1}{2}; X\right) + C_2 X^{3/2} F\left(\frac{3}{2} + \alpha, \frac{1}{2} - \alpha, \frac{5}{2}; X\right) \tag{15}$$

in terms hypergeometric functions, where  $C_1, C_2$  are constants of integration and  $\alpha = [-1 \pm \sqrt{(2a - b)/a}]/2$ .

It is noted that in some special cases a hypergeometric function can be expressed in terms of elementary functions [36]. We illustrate some of such cases below as examples.

#### 4.1 Case I: $2a + b = 0$

In this case (15) becomes

$$Y = C_1 F\left(-\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}; X\right) + C_2 X^{3/2} F\left(0, 2, \frac{5}{2}; X\right), \tag{16}$$

which can be expressed as

$$Y = C_1(1 + 2X)\sqrt{X - 1} + C_2 X^{\frac{3}{2}}. \tag{17}$$

Therefore the general solution for (12) becomes

$$y = d_1(5 - 4ax)\sqrt{1 + ax} + d_2(1 - 2ax)^{\frac{3}{2}}, \tag{18}$$

where  $d_1$  and  $d_2$  are new arbitrary constants. It is easy to see that this solution reduces to the Durgapal and Bannerji [3] neutron stars model when  $a = -\frac{1}{2}$  that satisfies all physical

requirements of realistic neutron stars. For the solution (18) the physical quantities takes the form:

$$e^{2\lambda} = \frac{1 - 2ax}{1 + ax}; \tag{19}$$

$$e^{2\nu} = A^2 [d_1(5 - 4ax)\sqrt{1 + ax} + d_2(1 - 2ax)^{\frac{3}{2}}]^2; \tag{20}$$

$$\rho = \frac{-3aC(3 - 2ax)}{(1 - 2ax)^2}; \tag{21}$$

$$p = \frac{9aC[d_1(1 - 4ax)\sqrt{1 + ax} - d_2(1 + 2ax)\sqrt{1 - 2ax}]}{(1 - 2ax)[d_1(5 - 4ax)\sqrt{1 + ax} + d_2(1 - 2ax)^{\frac{3}{2}}]}; \tag{22}$$

and the mass function (10) becomes

$$m(x) = \frac{-3ax^{\frac{3}{2}}}{2\sqrt{C}(1 - 2ax)}. \tag{23}$$

#### 4.2 Case II: $2a - b = 0$

Here (15) reduces to

$$Y = C_1 F\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}; X\right) + C_2 X^{3/2} F\left(1, 1, \frac{5}{2}; X\right), \tag{24}$$

which can be expressed in terms of elementary function

$$Y = C_1 \sqrt{1 - X} + C_2 \sqrt{1 - X} \left( \ln[\sqrt{-X} + \sqrt{1 - X}] - \frac{\sqrt{-X}}{\sqrt{1 - X}} \right). \tag{25}$$

Hence, the general solution for (12) becomes

$$y = g_1 \sqrt{1 + ax} + g_2 (\sqrt{2(1 + ax)} \ln[\sqrt{2(1 + ax)} + \sqrt{1 + 2ax}] - \sqrt{1 + 2ax}), \tag{26}$$

where  $g_1$  and  $g_2$  are new arbitrary constants. For the solution (26) the physical quantities takes the form:

$$e^{2\lambda} = \frac{1 + 2ax}{1 + ax}; \tag{27}$$

$$e^{2\nu} = A^2 [g_1 \sqrt{1 + ax} + g_2 (\sqrt{2(1 + ax)} \ln[\sqrt{2(1 + ax)} + \sqrt{1 + 2ax}] - \sqrt{1 + 2ax})]^2; \tag{28}$$

$$\rho = \frac{-8aC(3 - 7ax)}{(1 - 7ax)^2}; \tag{29}$$

$$p = \left\{ aC [g_1(1 + ax)(\sqrt{1 + 2ax} + \sqrt{2(1 + ax)}) + g_2((1 + 2ax)(\sqrt{1 + ax} + \sqrt{2(1 + 2ax)}) + (1 + ax)(2\sqrt{1 + ax} + \sqrt{2(1 + 2ax)})) \sinh^{-1}[\sqrt{1 + 2ax}]] \right\} / \left\{ \sqrt{1 + ax}(1 + 2ax)(\sqrt{1 + 2ax} + \sqrt{2(1 + ax)}) \right\} \times [g_1 \sqrt{1 + ax} - g_2 (\sqrt{1 + 2ax} + \sqrt{2} \sinh^{-1}[\sqrt{1 + 2ax}])]; \tag{30}$$

and the mass function (10) takes the form

$$m(x) = \frac{ax^{\frac{3}{2}}}{2\sqrt{C}(1+2ax)}. \tag{31}$$

4.3 Case III:  $7a - 4b = 0$

In this case (15) reduces to

$$Y = C_1 F\left(-\frac{1}{4}, -\frac{3}{4}, -\frac{1}{2}; X\right) + C_2 X^{3/2} F\left(\frac{5}{4}, \frac{3}{4}, \frac{5}{2}; X\right), \tag{32}$$

which has the particular form

$$Y = C_1 X^{\frac{3}{2}} [1 + \sqrt{1-X}]^{-\frac{3}{2}} + C_2 [1 + \sqrt{1-X}]^{\frac{3}{2}}. \tag{33}$$

Therefore the general solution for (12) becomes

$$y = D_1 (4 + 7ax)^{\frac{3}{2}} (\sqrt{3} + \sqrt{7(1+ax)})^{-\frac{3}{2}} + D_2 (\sqrt{3} + \sqrt{7(1+ax)})^{\frac{3}{2}}, \tag{34}$$

where  $D_1$  and  $D_2$  are new arbitrary constants. In this case the physical quantities can be written explicitly as:

$$e^{2\lambda} = \frac{4 + 7ax}{4(1 + ax)}; \tag{35}$$

$$e^{2\nu} = A^2 [D_1 (4 + 7ax)^{\frac{3}{2}} (\sqrt{3} + \sqrt{7(1+ax)})^{-\frac{3}{2}} + D_2 (\sqrt{3} + \sqrt{7(1+ax)})^{\frac{3}{2}}]^2; \tag{36}$$

$$\rho = \frac{3aC(12 + 7ax)}{(4 + 7ax)^2}; \tag{37}$$

$$p = \{3aC [D_1 \sqrt{4 + 7ax} (\sqrt{7(1+ax)}(36 + 21ax) + \sqrt{3}(52 + 49ax)) + D_2 (264 + 420ax + 147a^2x^2 + 56\sqrt{21}(1+ax)^{\frac{3}{2}})]\} / \{(4 + 7ax)(\sqrt{3} + \sqrt{7(1+ax)}) [D_1 (4 + 7ax)^{\frac{3}{2}} + D_2 (\sqrt{7(1+ax)}(16 + 7ax) + 3\sqrt{3}(8 + 7ax))]\}; \tag{38}$$

and the mass function (10) becomes

$$m(x) = \frac{3ax^{\frac{3}{2}}}{2\sqrt{C}(4 + 7ax)}. \tag{39}$$

4.4 Case IV:  $7a + b = 0$

In this case (15) becomes

$$Y = C_1 F\left(1, -2, -\frac{1}{2}; X\right) + C_2 X^{3/2} F\left(\frac{5}{2}, -\frac{1}{2}, \frac{5}{2}; X\right), \tag{40}$$

which has a particular form

$$Y = C_1[1 + 4X - 8X^2] + C_2X^{\frac{3}{2}}\sqrt{1-X}. \quad (41)$$

Therefore the general solution for (12) takes the form

$$y = k_1\sqrt{1+ax}(1-7ax)^{\frac{3}{2}} + k_2\left[1 + \frac{1}{2}(1-7ax) - \frac{1}{8}(1-7ax)^2\right], \quad (42)$$

where  $k_1$  and  $k_2$  are new arbitrary constants. It can be shown that the class of solution (42) reduces to the Tikekar super dense star model [6] when  $a = -1$ . The Tikekar super dense models are shown to be useful to study the behaviour of relativistic stars having highly dense matter, cold compact matter.

For the solution (42) the physical quantities takes the form:

$$e^{2\lambda} = \frac{1-7ax}{1+ax}; \quad (43)$$

$$e^{2\nu} = A^2\left[k_1\sqrt{1+ax}(1-7ax)^{\frac{3}{2}} + k_2\left(1 + \frac{1}{2}(1-7ax) - \frac{1}{8}(1-7ax)^2\right)\right]^2; \quad (44)$$

$$\rho = \frac{-8aC(3-7ax)}{(1-7ax)^2}; \quad (45)$$

$$p = \frac{16aC[k_2[2-7ax(5+7ax)] - 8k_1\sqrt{(1+ax)(1-7ax)(2+7ax)}]}{(1-7ax)[8k_1\sqrt{1+ax}(1-7ax)^{\frac{3}{2}} + k_2[11-7ax(2+7ax)]]}; \quad (46)$$

and the mass function (10) becomes

$$m(x) = \frac{-4ax^{\frac{3}{2}}}{\sqrt{C}(1-7ax)}. \quad (47)$$

We should pointed out that in all these cases the physical quantities are expressed in simple elementary functions that is necessary for detailed study of physical properties with ease. The mass functions we obtained are similar to mass functions used by Matese and Whitman [37] to generate equilibrium configurations in general relativity, Finch and Skea [5] to study the behaviour of neutron star and Mak and Harko [38] to analyze anisotropic relativistic stars. It is noted that the solutions given in Case I and Case IV may be regained as special cases of Thirukkanesh and Maharaj [35] charged model in the limit of vanishing electric field intensity. However, the physical significance of these solutions was not shown through a detailed analysis. Moreover, the classes of solutions obtained in Case II and Case III cannot be regained from Thirukkanesh and Maharaj charged model [35] and we believe that these classes are new solutions for the Einstein system of field equations.

## 5 Physical Analysis

In this section, we discuss the physical properties that has to be satisfied by realistic star for the model generated in Case I in detail.

(i) Regularity of the gravitational potentials at the origin.

It is easy to show that  $e^{2\lambda(0)} = 1$ ,  $e^{2\nu(0)} = A^2(5d_1 + d_2)$  and  $[e^{2\lambda(r)}]' = [e^{2\nu(r)}]' = 0$  at the centre  $r = 0$ . This shows that the gravitational potentials are regular at the centre of the star.

(ii) Positive definiteness of the energy density and the radial pressure at the origin.

Since the energy density at the centre is  $\rho(0) = -9aC$ , to satisfy positive definiteness condition we must take  $a < 0$ .

The matter pressure at the centre is  $p(0) = -9aC \frac{(d_2 - d_1)}{(5d_1 + d_2)}$ . Therefore to satisfy the positive definiteness property we have to choose the constants of integration  $d_1$  and  $d_2$  such that  $d_2 > d_1 > -\frac{d_2}{5}$  or  $d_2 < d_1 < -\frac{d_2}{5}$ .

(iii) Vanishing of the radial pressure at some finite radius.

At the boundary of the star  $r = R$  the condition  $p(R) = 0$  implies

$$d_1(1 - 4aCR^2)\sqrt{1 + aCR^2} - d_2(1 + 2aCR^2)\sqrt{1 - 2aCR^2} = 0. \tag{48}$$

This shows that the radius of the star  $R$  depend on the parameters  $a, C$  and the arbitrary constants  $d_1, d_2$ .

Junction condition: Interior metric match smoothly with the Schwarzschild exterior metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

across the boundary  $r = R$ , where  $M$  is the total mass of the sphere. This generate the conditions:

$$\left(1 - \frac{2M}{R}\right)^{-1} = \frac{1 - 2aCR^2}{1 + aCR^2} \tag{49}$$

$$\left(1 - \frac{2M}{R}\right) = A^2[d_1(5 - 4aCR^2)\sqrt{1 + aCR^2} + d_2(1 - 2aCR^2)^{\frac{3}{2}}]^2. \tag{50}$$

Although condition (49) does not impose any restriction on the parameters, condition (50) restricts the parameter  $A$  as

$$A^2 = \frac{1 + aCR^2}{(1 - 2aCR^2)[d_1(5 - 4aCR^2)\sqrt{1 + aCR^2} + d_2(1 - 2aCR^2)^{\frac{3}{2}}]^2}.$$

We shall show the other conditions such as monotonic decrease of the energy density and the radial pressure with increasing radius, and causality property by plotting figures for the relevant parameters for a particular example.

In view of comparing our model with realistic stars, values of model parameters and the relevant physical parameters calculated for various strange star candidates RXJ 1856-37, SAX J1808.4-3658(SS2), Her. X-1, SAX J1808.4-3658(SS1) and 4U 1820-30 in respective order are given in the Table 1. As stated  $a < 0$  above, taking  $a = -1$  as an example, we show that this model satisfy the necessary physical requirements. In the calculation of numerical values for the physical parameters in the Table 1, we fixed values for parameters  $a = -1$  and  $d_2 = 7$  and used values for the parameters  $d_1$  and  $C$  as given in the Table 1 such that the pressure at the boundary  $p(R) = 0$  (i.e. equation (48) satisfied).



**Table 1** Pressure, density and mass for different stellar models

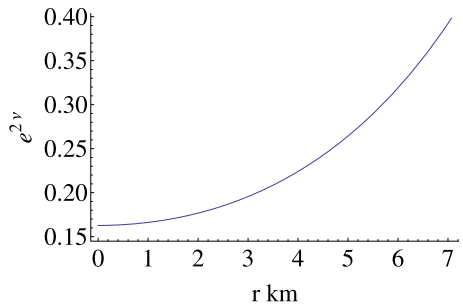
$d_1$	$C$	$R$ [km]	$p(0)$ [ $\times 10^{35}$ dyne $\text{cm}^{-2}$ ]	$\rho(0)$ [ $\times 10^{15}$ $\text{gcm}^{-3}$ ]	$\rho(R)$ [ $\times 10^{15}$ $\text{gcm}^{-3}$ ]	$M$ [ $M_\odot$ ]
2.95980	0.00583	6	9.7205	2.81275	1.59066	0.90044
1.44697	0.00860	6.35	30.1807	4.14917	1.78109	1.31998
3.05488	0.00451	6.7	7.1882	2.17537	1.25105	0.98001
1.56540	0.00670	7.07	22.0922	3.23250	1.41817	1.43951
0.95158	0.00398	10	18.4182	1.92020	0.75325	2.24968

Our calculated masses  $1.44M_\odot$  and  $1.32M_\odot$  for radii 7.07 km and 6.35 km respectively, well corroborates with theoretical model [18] reported analysing pulsars SAX J1808.4-3658 (SS1 & SS2), which is also shown to be consistent with observational data and remarkable accord with the strange star models [18]. Our calculated values for surface densities for SAX J1808.4-3658 (SS1 & SS2) in Table 1 are more than five times of nuclear saturation density  $\rho_n$ , suggesting the chargeless beta-stable (u,d,s) quarks may form the surface of the compact star with central core density in the order of  $10\rho_n$ , which substantiate the reported claim for strange star. Moreover, Mass-radius relation studies using different theoretical models to analyze the original experimental observations associated with cyclotron line data from the X-ray pulsar Her X-1 [39, 40], and with the X-ray burst spectra of 4U 1820-1830 [41], have shown that they are good strange star candidates. Our calculations of surface densities show  $\rho(R) = 4.6\rho_n$  and  $\rho(0) = 8\rho_n$  for Her X-1, and  $\rho(R) = 2.8\rho_n$  and  $\rho(0) = 7.1\rho_n$  for 4U 1820-30. Moreover, our calculated masses for respective strange star candidates coincide with the values of Tikekar and Jotania [17]. Therefore we compare the compaction parameter (mass-to-radius ratio) in [17]:  $u > 0.3$  for pulsars SAX J1808.4-3658 (SS1 & SS2) and 4U 1820-30 suggests they are strange stars of type I, and Her X-1 and RXJ 1856-37 are of type II ( $0.2 < u < 0.3$ ), and the  $u$  value for neutron star counterparts are considered to be still lower.

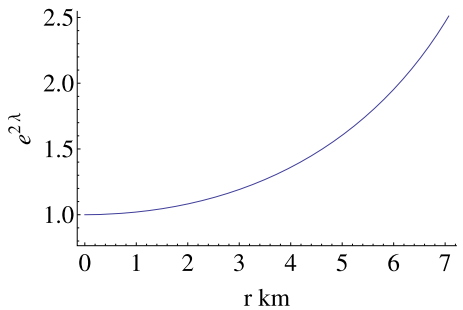
To illustrate the radial dependence of physical quantities of the model, Figs. 1–6 were plotted for the same parameter values utilised in Table 1 corresponding for SAX J1808.4-3658 (SS1) with radius  $R = 7.07$  km. Figures 1 and 2 shows that the gravitational potential is finite and continuous at the interior and increases radially. Figure 3 illustrates that the energy density of the star decreases continuously from a value  $\rho(0) = 12\rho_n$  at the centre to  $\rho(7.07) = 5.25\rho_n$  at the surface where the isotropic interior matter pressure vanishes as illustrated in Fig. 4. The isotropic pressure increases continuously towards the origin from zero at the surface and reaches a maximum value  $p(0) = 22.09 \times 10^{35}$  dyne  $\text{cm}^{-2}$  at the centre. Figure 5 shows that  $0 < \frac{dp}{d\rho} < 1$  throughout the interior of the star and hence satisfy the causality condition: the speed of sound is less than the speed of light throughout the interior of star. The combinations  $\frac{dp}{d\rho}$  and  $\frac{p}{\rho}$  are related to speed of sound  $v_s$  as  $v_s^2 = \frac{dp}{d\rho} = \gamma \frac{p}{\rho}$ , and the bulk modulus  $\kappa = \gamma p$  (where  $\gamma$  is the adiabatic index) gives a measure of the stiffness of the substance. Hence, the marginal increase and decrease of  $v_s^2$  (8 %) shown in Fig. 5 suggests adiabatic perturbations in the properties of the interior substance of the star. The observed decrease in  $\rho$  in Fig. 3 should increase  $v_s$  towards the surface, but the  $v_s^2$  behaviour observed in Fig. 5 (i.e. a marginal increase of about 2.3 % and decrease) shows the fluctuation in the decreasing trend of the stiffness  $\kappa$  in the interior material as  $r$  increases.

Note that Figs. 3 and 4 respectively show  $\rho > 0$  and  $p \geq 0$  throughout the interior of the star, and from the physical analysis it is observed that the models satisfy the null energy

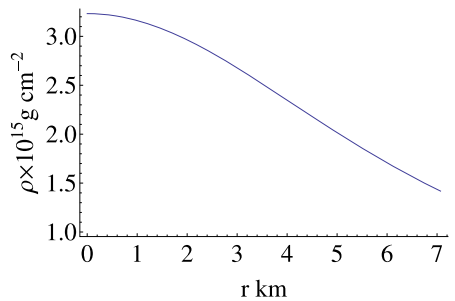
**Fig. 1** Gravitational potential  $e^{2\nu}$



**Fig. 2** Gravitational potential  $e^{2\lambda}$



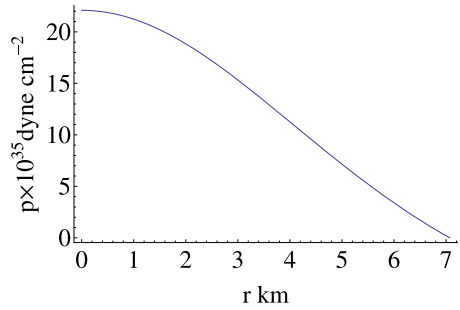
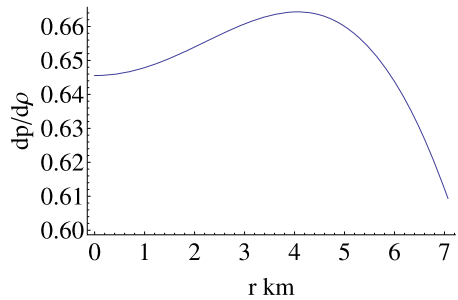
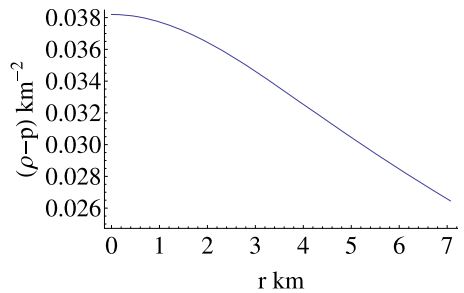
**Fig. 3** Energy density



condition ( $\rho + p \geq 0$ ), weak energy condition ( $\rho \geq 0$  and  $\rho + p \geq 0$ ) and strong energy condition ( $\rho + 3p \geq 0$  and  $\rho + p \geq 0$ ). Moreover, Fig. 6 shows  $\rho - p > 0$  throughout the interior of the star, implying the model also satisfy the dominant energy condition ( $\rho \geq 0$  and  $\rho \pm p \geq 0$ ), and hence collectively satisfying the standard point-wise energy condition [42] of normal matter. The above detailed study is for a particular example from Case I. Given the simplicity of the models generated in subsequent cases, similar analysis can be performed without much difficulty.

### 6 Conclusion

We have obtained four classes of solutions to the Einstein system in static spherically symmetric spacetimes by transforming the pressure isotropic condition into a hypergeometric equation by choosing a generalised rational form for one of the gravitational potentials. In general the solution is obtained in terms of hypergeometric functions and we extract four

**Fig. 4** Matter pressure**Fig. 5** The derivative  $dp/d\rho$ **Fig. 6** The difference  $\rho - p$ 

classes of solutions in terms of simple elementary functions as examples which would be desirable to study the physical behavior of relativistic compact objects in detail. A physical analysis show that our models satisfy all the fundamental requirements of a realistic star and match smoothly with the exterior Schwarzschild metric. We have demonstrated that our models yield stellar structures with masses and densities consistent with the previously reported experimental and theoretical studies in the description of strange stars. The models satisfy the standard point-wise energy condition that is required by normal matter. Therefore it is likely that our solutions may be helpful in the gravitational description of dense and ultradense stellar bodies such as SAX J1808.4-3658, 4U 1820-30, RXJ 1856-37, Her. X-1, etc. We believe that the classes of exact solutions found in this paper may assist in more detailed studies of relativistic compact stellar bodies and allow for varied matter distributions.

## References

1. Schwarzschild, K.: Über das Gravitationsfeld eines Massenpunktes nach der Einstein Theorie. Sitz. Deut. Akad. Wiss. Math.-Phys. Berlin **23**, 189–196 (1916)
2. Schwarzschild, K.: Über das Gravitationsfeld einer Kugel aus inkompressibler Flüssigkeit nach der Einstein Theorie. Sitz. Deut. Akad. Wiss. Math.-Phys. Berlin **24**, 424–434 (1916)
3. Durgapal, M.C., Bannerji, R.: New analytical stellar model in general relativity. Phys. Rev. D **27**, 328–331 (1983)
4. Durgapal, M.C., Fuloria, R.S.: Analytical relativistic model for a superdense star. Gen. Relativ. Gravit. **17**, 671–681 (1985)
5. Finch, M.R., Skea, J.E.F.: A realistic stellar model based on the ansatz of Duorah and Ray. Class. Quantum Gravity **6**, 467–476 (1989)
6. Tikekar, R.: Exact model for a relativistic star. J. Math. Phys. **31**, 2454–2458 (1990)
7. Maharaj, S.D., Leach, P.G.L.: Exact solutions for the Tikekar super dense star. J. Math. Phys. **37**, 430–437 (1996)
8. Sharma, R., Mukherjee, S.: Compact stars: a core-envelope model. Mod. Phys. Lett. A **17**, 2535–2544 (2002)
9. Thirukkanesh, S., Maharaj, S.D.: Exact models for isotropic matter. Class. Quantum Gravity **23**, 2697–2709 (2006)
10. Buchdahl, H.A.: General relativistic fluid spheres. Phys. Rev. **116**, 1027–1034 (1959)
11. Weinberge, S.: Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity. Wiley, New York (1972)
12. Thorsett, S.E., Chakrabarty, D.: Neutron star mass measurement. I. Radio pulsars. Astrophys. J. **512**, 288–299 (1999)
13. Link, B., Epstein, R.I., Lattimer, J.M.: Pulsar constrains on neutron star structure and equation of state. Phys. Rev. Lett. **83**, 3362–3365 (1999)
14. Heinke, C.O., Rybicki, G.B., Narayan, R., Grindlay, J.E.: A hydrogen atmosphere spectral model applied to the neutron star X7 in the Globular cluster 47 tucanae. Astrophys. J. **644**, 1090–1103 (2006)
15. Ho Wynn, C.G., Heinke, C.O.: A neutron star with a carbon atmosphere in the cassiopeia a supernova remnant. Nature **462**, 71–73 (2009)
16. Özel, F., Üver, T.G., Psaltis, D.: The mass and radius of neutron star in EXO 1745-248. Astrophys. J. **693**, 1775–1779 (2009)
17. Tikekar, R., Jotania, K.: On relativistic models of strange stars. Pramana J. Phys. **68**, 397–406 (2007)
18. Li, X.-D., Bombaci, I., Dey, M., van den Heuvel, E.P.J.: Is SAX J1808.4-3658 a strange star? Phys. Rev. Lett. **83**, 3776–3779 (1999)
19. Written, E.: Cosmic separation of phase. Phys. Rev. D **30**, 272–285 (1984)
20. Hansel, P., Zdunik, J.L., Schaefer, R.: Strange quark stars. Astron. Astrophys. **160**, 121–128 (1986)
21. Alcock, C., Farhi, E., Olinto, A.: Strange stars. Astrophys. J. **310**, 261–272 (1986)
22. Lobo, F.S.N.: Stable dark energy stars. Class. Quantum Gravity **23**, 1525–1542 (2006)
23. Mak, M.K., Harko, T.: Quark stars admitting a one-parameter group of conformal motions. Int. J. Mod. Phys. D **13**, 149–156 (2004)
24. Komathiraj, K., Maharaj, S.D.: Analytical models for quark stars. Int. J. Mod. Phys. D **16**, 1803–1811 (2007)
25. Sharma, R., Maharaj, S.D.: A class of relativistic stars with a linear equation of state. Mon. Not. R. Astron. Soc. **375**, 1265–1268 (2007)
26. Thirukkanesh, S., Maharaj, S.D.: Charged anisotropic matter with linear equation of state. Class. Quantum Gravity **25**, 235001 (2008)
27. Thirukkanesh, S., Ragel, F.C.: A class of exact strange quark star model. Pramana J. Phys. **81**, 275–286 (2013)
28. Mafa Takisa, P., Maharaj, S.D.: Compact models with regular charge distributions. Astrophys. Space Sci. **343**, 569–577 (2013)
29. Feroze, T., Siddiqui, A.A.: Charged anisotropic matter with quadratic equation of state. Gen. Relativ. Gravit. **43**, 1025–1035 (2011)
30. Maharaj, S.D., Mafa Takisa, P.: Regular models with quadratic equation of state. Gen. Relativ. Gravit. **44**, 1419–1432 (2012)
31. Varela, V., Rahaman, F., Ray, S., Chakraborty, K., Kalam, M.: Charged anisotropic matter with linear or nonlinear equation of state. Phys. Rev. D **82**, 044052 (2010)
32. Thirukkanesh, S., Ragel, F.C.: Exact anisotropic sphere with polytropic equation of state. Pramana J. Phys. **78**, 687–696 (2012)
33. Mafa Takisa, P., Maharaj, S.D.: Some charged polytropic models. Gen. Relativ. Gravit. **45**, 1951–1961 (2013)

34. Delgaty, M.S.R., Lake, K.: Physical acceptability of isolated, static, spherically symmetric, perfect fluid solutions of Einstein's equations. *Comput. Phys. Commun.* **115**, 395–415 (1998)
35. Thirukkanesh, S., Maharaj, S.D.: Charged relativistic sphere with generalised potentials. *Math. Methods Appl. Sci.* **32**, 684–701 (2009)
36. Polyanina, A.D., Zaitsev, V.F.: *Handbook of Exact Solutions for Ordinary Differential Equations*. Chapman & Hall/CRC, New York (2003)
37. Matese, J.J., Whitman, P.G.: New method for extracting static equilibrium configurations in general relativity. *Phys. Rev. D* **22**, 1270–1275 (1980)
38. Mak, M.K., Harko, T.: Anisotropic stars in general relativity. *Proc. R. Soc. Lond. A* **459**, 393–408 (2003)
39. Li, X.-D., Dai, Z.-G., Wang, Z.-R.: Is HER X-1 a strange star? *Astron. Astrophys.* **303**, L1 (1995)
40. Dey, M., Bombaci, I., Dey, J., Ray, S., Samanta, B.C.: Strange stars with realistic quark vector interaction and phenomenological density-dependent scalar potential. *Phys. Lett. B* **438**, 123–128 (1998)
41. Bombaci, I.: Observational evidence for strange matter in compact objects from the X-ray burster 4U 1820-30. *Phys. Rev. C* **55**, 1587–1590 (1997)
42. Delsate, T., Steinhoff, J.: New insights on the matter-gravity coupling paradigm. *Phys. Rev. Lett.* **109**, 021101 (2012)