

A Quantum Computational Semantics for Epistemic Logical Operators. Part II: Semantics

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Abstract By using the abstract structures investigated in the first Part of this article, we develop a semantics for an epistemic language, which expresses sentences like “Alice knows that Bob does not understand that π is irrational”. One is dealing with a *holistic* form of quantum computational semantics, where entanglement plays a fundamental role; thus, the *meaning* of a global expression determines the *contextual meanings* of its parts, but generally not the other way around. The epistemic situations represented in this semantics seem to reflect some characteristic limitations of the real processes of acquiring information. Since knowledge is not generally closed under logical consequence, the unpleasant phenomenon of *logical omniscience* is here avoided.

Keywords Contextual meaning · Holistic model · Epistemic semantic

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1 The Epistemic Quantum Computational Syntax

The structures, investigated in the first Part of this article, provide the mathematical basis for the development of our epistemic semantics.¹ The basic intuitive idea can be sketched as follows: pieces of quantum information (qumixes) can be denoted by the sentences of a formal language, whose logical connectives correspond to some quantum gates, while the two fundamental epistemic operators (*to understand* and *to know*) are interpreted as epistemic operations living in semantic models based on convenient epistemic quantum computational structures. Accordingly, sentences like “At time t Alice knows that Bob does not understand that π is irrational” turn out to denote particular examples of qumixes, representing possible states of quantum objects. Let us first introduce the epistemic language that will be used. This language, indicated by \mathcal{L}^{EpQC} , contains atomic sentences (say, “the spin-value in the x -direction is up”), including two privileged sentences \mathbf{t} and \mathbf{f} that represent the truth-values *Truth* and *Falsity*, respectively. We will use $\mathbf{q}, \mathbf{r}, \dots$ as metavariables for atomic sentences, and α, β, \dots as metavariables for sentences. The quantum computational connectives of \mathcal{L}^{EpQC} are: the negation \neg (which corresponds to the gate *Negation*), the square root of the identity \sqrt{id} (which corresponds to *Hadamard*-gate), a ternary connective \top (which corresponds to the *Toffoli*-gate), the exclusive disjunction \oplus (which corresponds to *XOR*). The epistemic sub-language of \mathcal{L}^{EpQC} contains: a finite set of names for epistemic agents ($\mathbf{a}, \mathbf{b}, \dots$); a set of names (t_1, t_2, \dots) for the elements of a given time-sequence; the epistemic operators \mathcal{U} (*to understand*) and \mathcal{K} (*to know*).

For any sentences α, β, γ , the expressions $\neg\alpha, \sqrt{id}\alpha, \top(\alpha, \beta, \gamma), \alpha \oplus \beta$ are sentences. For any sentence α , for any agent-name \mathbf{a} (say, Alice), for any time-name t , the expressions $\mathcal{U}\mathbf{a}_t\alpha$ (*at time t agent \mathbf{a} understands the sentence α*) and $\mathcal{K}\mathbf{a}_t\alpha$ (*at time t agent \mathbf{a} knows the sentence α*) are sentences. Notice that nested epistemic operators are allowed: an expression like $\mathcal{K}\mathbf{a}_t\neg\mathcal{U}\mathbf{b}_t\alpha$ is a well-formed sentence. The connectives $\neg, \sqrt{id}, \top, \oplus$ are called *gate-connectives*. Any subexpression $\mathcal{U}\mathbf{a}_t$ or $\mathcal{K}\mathbf{a}_t$ of an epistemic sentence will be called an *epistemic connective*.

We recall that, for any truth-perspective \mathfrak{T} , the Toffoli-gate permits one to define a reversible conjunction $\text{AND}_{\mathfrak{T}}$ (for any $\rho \in \mathfrak{D}(\mathcal{H}^{(m)})$ and for any $\sigma \in \mathfrak{D}(\mathcal{H}^{(1)})$):

$$\text{AND}_{\mathfrak{T}}(\rho, \sigma) := \mathfrak{D}\mathbb{T}_{\mathfrak{T}}^{(m,n,1)}(\rho \otimes \sigma \otimes P_0^{(1)}).$$

Accordingly, from a syntactical point of view, it is reasonable to define (metalinguistically) the logical conjunction \wedge as follows (for any sentences α and β):

$$\alpha \wedge \beta := \top(\alpha, \beta, \mathbf{f}).$$

We will now introduce some syntactical notions that will be used in our semantics.

Definition 1.1

- α is called a gate-sentence iff either α is atomic or the principal connective of α is a gate-connective.
- α is called an epistemic sentence iff α has the form $\mathcal{K}\mathbf{a}_t\beta$.

Definition 1.2 (The Atomic Complexity of a Sentence) The atomic complexity $At(\alpha)$ of a sentence α is the number of occurrences of atomic sentences in α .

¹See [1, 3, 5–7].

For instance, $At(\top(\mathbf{q}, \mathbf{q}, \mathbf{f})) = 3$. We will also indicate by $\alpha^{(n)}$ a sentence whose atomic complexity is n . The notion of atomic complexity plays an important semantic role. As we will see, the meaning of any sentence whose atomic complexity is n shall live in the domain $\mathfrak{D}(\mathcal{H}^{(n)})$. For this reason, $\mathcal{H}^{(At(\alpha))}$ (briefly indicated by \mathcal{H}^α) will be also called the *semantic space* of α .

Any sentence α can be naturally decomposed into its parts, giving rise to a special configuration called the *syntactical tree* of α (indicated by $STree^\alpha$).

Roughly, $STree^\alpha$ can be represented as a finite sequence of *levels*:

$$\begin{aligned} &Level_k(\alpha) \\ &\quad \vdots \\ &Level_1(\alpha), \end{aligned}$$

where:

- each $Level_i(\alpha)$ (with $1 \leq i \leq k$) is a sequence $(\beta_1, \dots, \beta_m)$ of subformulas of α ;
- the *bottom level* $Level_1(\alpha)$ is α ;
- the *top level* $Level_k(\alpha)$ is the sequence $(\mathbf{q}_1, \dots, \mathbf{q}_r)$, where $\mathbf{q}_1, \dots, \mathbf{q}_r$ are the atomic occurrences in α ;
- for any i (with $1 \leq i < k$), $Level_{i+1}(\alpha)$ is the sequence obtained by dropping the *principal gate-connective* in all molecular gate-sentences occurring at $Level_i(\alpha)$, by dropping the epistemic connectives $(\mathcal{U}\mathbf{a}_r, \mathcal{K}\mathbf{a}_r)$ in all epistemic sentences occurring at $Level_i(\alpha)$ and by repeating all the atomic sentences that occur at $Level_i(\alpha)$.

By *Height* of α (indicated by $Height(\alpha)$) we mean the number of levels of the syntactical tree of α .

As an example, consider the following sentence:

$$\alpha = \mathcal{K}\mathbf{a}_r \neg(\mathbf{q} \wedge \neg\mathbf{q}) = \mathcal{K}\mathbf{a}_r \neg(\top(\mathbf{q}, \neg\mathbf{q}, \mathbf{f}))$$

(say, “At time t Alice knows the non-contradiction principle”, instantiated by means of the atomic sentence \mathbf{q}).

The syntactical tree of α is the following sequence of levels:

$$\begin{aligned} Level_5(\alpha) &= (\mathbf{q}, \mathbf{q}, \mathbf{f}) \\ Level_4(\alpha) &= (\mathbf{q}, \neg\mathbf{q}; \mathbf{f}) \\ Level_3(\alpha) &= (\top(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})) \\ Level_2(\alpha) &= (\neg \top(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})) \\ Level_1(\alpha) &= (\mathcal{K}\mathbf{a}_r \neg(\top(\mathbf{q}, \neg\mathbf{q}, \mathbf{f}))). \end{aligned}$$

Clearly, $Height(\top(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})) = 5$.

More precisely, the syntactical tree of a sentence (whose atomic complexity is r) is defined as follows.

Definition 1.3 (The Syntactical Tree of α) The syntactical tree of α is the following sequence of sentence-sequences:

$$STree^\alpha = (Level_1(\alpha), \dots, Level_k(\alpha)),$$

where:

- $Level_1(\alpha) = (\alpha)$;
- $Level_{i+1}$ is defined as follows for any i such that $1 \leq i < k$. The possible cases are:

- (1) $Level_i(\alpha)$ does not contain any connective. Hence, $Level_i(\alpha) = (\mathbf{q}_1, \dots, \mathbf{q}_r)$ and $Height(\alpha) = i$;
- (2) $Level_i(\alpha) = (\beta_1, \dots, \beta_m)$, and for at least one j , β_j has a (principal) connective. Consider the following sequence of sentence-sequences:

$$J'_1, \dots, J'_m,$$

where

$$J'_h = \begin{cases} (\beta_h), & \text{if } \beta_h \text{ is atomic;} \\ J_h^*, & \text{otherwise,} \end{cases}$$

with

$$J_h^* = \begin{cases} (\delta), & \text{if } \beta_h = \neg\delta \text{ or } \beta_h = \sqrt{id}\delta; \\ (\gamma, \delta, \theta), & \text{if } \beta_h = \top(\gamma, \delta, \theta); \\ (\gamma, \delta), & \text{if } \beta_h = \gamma \uplus \delta; \\ (\delta), & \text{if } \beta_h = \mathcal{U}\mathbf{a}_r\delta \text{ or } \beta_h = \mathcal{K}\mathbf{a}_r\delta. \end{cases}$$

Then,

$$Level_{i+1}(\alpha) = J'_1 \bullet \dots \bullet J'_m,$$

where \bullet represents the sequence-composition.

2 The Epistemic Quantum Computational Semantics

We will now give the basic definitions of our semantics. We will apply to epistemic situations a *holistic* version of quantum computational semantics (which has been naturally inspired by the characteristic holistic features of the quantum theoretic formalism).² In this semantics any model assigns to any sentence a *global meaning* that determines the contextual meanings of all its parts (from the whole to the parts!). It may happen that one and the same sentence receives different meanings in different contexts.

Before defining holistic models, we will first introduce the weaker notion of *quasi-model* of the language \mathcal{L}^{EpQC} .

Definition 2.1 (Quasi-model) A *quasi-model* of the language \mathcal{L}^{EpQC} is a system

$$\mathcal{M}^q = (T, Ag, \mathbf{EpSit}, den)$$

where:

²See [4] and [2].

- (1) (T, Ag, \mathbf{EpSit}) is an epistemic quantum computational structure;³
- (2) den is a function that interprets the individual names of the language. By simplicity, we put:

$$den(\mathbf{a}) = \mathbf{a}; \quad den(t) = \mathbf{t}.$$

Apparently, quasi-models represent partial interpretations of the language: while names of times and of agents receive an interpretation in the framework of a given epistemic situation, meanings of sentences are not determined.

In the first Part of this article we have seen that knowledge operations cannot be generally represented as qumix gates. At the same time, once fixed an epistemic quantum computational structure $S = (T, Ag, \mathbf{EpSit})$, one can naturally define the following notion of *pseudo-gate* with respect to S .

Definition 2.2 (Pseudo-gate) Let $S = (T, Ag, \mathbf{EpSit})$ be an epistemic quantum computational structure. A *pseudo-gate* of S is an operator-product

$$\mathbf{X}_1^{(n_1)} \otimes \dots \otimes \mathbf{X}_m^{(n_m)},$$

where any $\mathbf{X}_i^{(n_i)}$ (with $1 \leq i \leq m$) is either a qumix gate ${}^{\mathfrak{T}}G_{\mathfrak{T}}^{(n_i)}$ with respect to a truth-perspective \mathfrak{T} or an epistemic operation $(\mathbf{U}_{\mathbf{a}_t}^{(n_i)}$ or $\mathbf{K}_{\mathbf{a}_t}^{(n_i)})$ of S .

One can show that for any choice of a truth-perspective \mathfrak{T} and of a quasi-model $\mathcal{M}^g = (T, Ag, \mathbf{EpSit}, den)$, the syntactical tree of a sentence α uniquely determines a sequence of pseudo-gates, that will be called the $(\mathfrak{T}, \mathcal{M}^g)$ -*pseudo-gate tree* of α .

As an example, consider again the sentence

$$\alpha = \mathcal{K}\mathbf{a}_t \neg(\mathbf{q} \wedge \neg\mathbf{q}) = \mathcal{K}\mathbf{a}_t \neg(\top(\mathbf{q}, \neg\mathbf{q}, \mathbf{f}))$$

and its syntactical tree.

Apparently, $Level_4(\alpha)$ is obtained from $Level_5(\alpha)$ by repeating the first occurrence of \mathbf{q} , by negating the second occurrence of \mathbf{q} and by repeating \mathbf{f} . Hence the pseudo-gate that corresponds to $Level_4(\alpha)$ will be ${}^{\mathfrak{T}}I^{(1)} \otimes {}^{\mathfrak{T}}NOT_{\mathfrak{T}}^{(1)} \otimes {}^{\mathfrak{T}}I^{(1)}$. $Level_3(\alpha)$ is obtained from $Level_4(\alpha)$ by applying to the three sentences occurring at $Level_4(\alpha)$ the connective \top . Hence the pseudo-gate that corresponds to $Level_3(\alpha)$ will be ${}^{\mathfrak{T}}T_{\mathfrak{T}}^{(1,1,1)}$. $Level_2(\alpha)$ is obtained from $Level_3(\alpha)$ by applying to the sentence occurring at $Level_3(\alpha)$ the connective \neg . Hence the pseudo-gate that corresponds to $Level_2(\alpha)$ will be ${}^{\mathfrak{T}}NOT_{\mathfrak{T}}^{(3)}$. Finally, $Level_1(\alpha)$ is obtained from $Level_2(\alpha)$ by applying to the sentence occurring at $Level_2(\alpha)$ the epistemic connective $\mathcal{K}\mathbf{a}_t$. Hence the pseudo-gate that corresponds to $Level_1(\alpha)$ will be $\mathbf{K}_{\mathbf{a}_t}^{(3)}$.

On this basis, the $(\mathfrak{T}, \mathcal{M}^g)$ -*pseudo-gate tree* of the sentence

$$\alpha = \mathcal{K}\mathbf{a}_t \neg(\mathbf{q} \wedge \neg\mathbf{q}) = \mathcal{K}\mathbf{a}_t \neg(\top(\mathbf{q}, \neg\mathbf{q}, \mathbf{f}))$$

can be identified with the following sequence consisting of four pseudo-gates:

$$({}^{\mathfrak{T}}I^{(1)} \otimes {}^{\mathfrak{T}}NOT_{\mathfrak{T}}^{(1)} \otimes {}^{\mathfrak{T}}I^{(1)}, {}^{\mathfrak{T}}T_{\mathfrak{T}}^{(1,1,1)}, {}^{\mathfrak{T}}NOT_{\mathfrak{T}}^{(3)}, \mathbf{K}_{\mathbf{a}_t}^{(3)}).$$

Notice that the truth-perspectives \mathfrak{T} and $\mathfrak{T}_{\mathbf{a}_t}$ may be different.

The general definition of $(\mathfrak{T}, \mathcal{M}^g)$ -*pseudo-gate tree* is the following:

³See Sect. 3 of the first Part of this article.

Definition 2.3 ($(\mathfrak{T}, \mathcal{M}^q)$ -pseudo-gate tree) Let α be a sentence such that $Height(\alpha) = k$. The $(\mathfrak{T}, \mathcal{M}^q)$ -pseudo-gate tree of α is the sequence of pseudo-gates

$$PsTree_{\mathfrak{T}}^{\alpha} = (\alpha \mathbf{O}_{\mathfrak{T}}^{(k-1)}, \dots, \alpha \mathbf{O}_{\mathfrak{T}}^{(1)}),$$

that is defined as follows. Suppose that

$$Level_{i-1}(\alpha) = (\beta_1^{(r_1)}, \dots, \beta_m^{(r_m)}),$$

(where $1 < i \leq k$). We put:

$$\alpha \mathbf{O}_{\mathfrak{T}}^{(i-1)} = \alpha \mathbf{X}_{\mathfrak{T}}^{(r_1)} \otimes \dots \otimes \alpha \mathbf{X}_{\mathfrak{T}}^{(r_m)},$$

where any $\alpha \mathbf{X}_{\mathfrak{T}}^{(r_j)}$ is a pseudo-gate defined on $\mathcal{H}^{(r_j)}$ such that:

$$\alpha \mathbf{X}_{\mathfrak{T}}^{(r_j)} = \begin{cases} \mathfrak{D} \mathbf{I}_{\mathfrak{T}}^{(r_j)}, & \text{if } \beta_j^{(r_j)} \text{ is atomic;} \\ \mathfrak{D} \mathbf{NOT}_{\mathfrak{T}}^{(r_j)}, & \text{if } \beta_j^{(r_j)} = \neg \delta; \\ \mathfrak{D} \sqrt{\mathbf{I}_{\mathfrak{T}}^{(r_j)}}, & \text{if } \beta_j^{(r_j)} = \sqrt{id} \delta; \\ \mathfrak{D} \mathbf{T}_{\mathfrak{T}}^{(u,v,w)}, & \text{if } \beta_j^{(r_j)} = \mathbf{T}(\gamma^{(u)}, \delta^{(v)}, \theta^{(w)}); \\ \mathfrak{D} \mathbf{XOR}_{\mathfrak{T}}^{(u,v)}, & \text{if } \beta_j^{(r_j)} = \gamma^{(u)} \uplus \delta^{(v)}; \\ \mathbf{U}_{\mathfrak{at}}^{(r_j)}, & \text{if } \beta_j^{(r_j)} = \mathcal{U} \mathfrak{a}_t \delta; \\ \mathbf{K}_{\mathfrak{at}}^{(r_j)}, & \text{if } \beta_j^{(r_j)} = \mathcal{K} \mathfrak{a}_t \delta. \end{cases}$$

Consider now a sentence α and let $(\alpha \mathbf{O}_{\mathfrak{T}}^{(k-1)}, \dots, \alpha \mathbf{O}_{\mathfrak{T}}^{(1)})$ be the $(\mathfrak{T}, \mathcal{M}^q)$ -pseudo-gate tree of α . Any choice of a qumix ρ in \mathcal{H}^{α} determines a sequence (ρ_k, \dots, ρ_1) of qumixes of \mathcal{H}^{α} , where:

$$\begin{aligned} \rho_k &= \rho \\ \rho_{k-1} &= \alpha \mathbf{O}_{\mathfrak{T}}^{(k-1)}(\rho_k) \\ &\vdots \\ \rho_1 &= \alpha \mathbf{O}_{\mathfrak{T}}^{(1)}(\rho_2). \end{aligned}$$

The qumix ρ_k can be regarded as a possible *input-information* concerning the atomic parts of α , while ρ_1 represents the *output-information* about α , given the input-information ρ_k . Each ρ_i corresponds to the *information* about $Level_i(\alpha)$, given the input-information ρ_k .

How to determine an information about the parts of α under a given input? It is natural to apply the *reduced state function* that determines for any state ρ of a composite system $S = S_1 + \dots + S_n$ the state $Red^{i_1, \dots, i_m}(\rho)$ of any subsystem $S_{i_1} + \dots + S_{i_m}$ (where $1 \leq i_1 \leq n, \dots, 1 \leq i_m \leq n$). Consider the syntactical tree of α and suppose that:

$$Level_i(\alpha) = (\beta_{i_1}, \dots, \beta_{i_r}).$$

We know that the $(\mathfrak{T}, \mathcal{M}^q)$ -pseudo-gate tree of α and the choice of an input ρ_k (in \mathcal{H}^α) determine a sequence of qumixes:

$$\begin{aligned} \rho_k &\rightsquigarrow Level_k(\alpha) = (\mathbf{q}_1, \dots, \mathbf{q}_r) \\ &\vdots \\ \rho_i &\rightsquigarrow Level_i(\alpha) = (\beta_{i_1}, \dots, \beta_{i_r}) \\ &\vdots \\ \rho_1 &\rightsquigarrow Level_1(\alpha) = (\alpha). \end{aligned}$$

We can consider $Red^j(\rho_i)$, the *reduced information of ρ_i with respect to the j -th part*. From a semantic point of view, this object can be regarded as a *contextual information* about β_{i_j} (the subformula of α occurring at the j -th position at $Level_i(\alpha)$) under the input ρ_k .

We can now define the notion of *holistic model*, which assigns meanings to all sentences of the language, for any choice of a truth-perspective \mathfrak{T} .

Definition 2.4 (Holistic Model) A *holistic model* of the language \mathcal{L}^{EpQC} is a system

$$\mathcal{M} = (T, Ag, \mathbf{EpSit}, den, \mathcal{M}_{Hol})$$

where:

- (1) $(T, Ag, \mathbf{EpSit}, den)$ is a quasi-model \mathcal{M}^q of the language.
- (2) \mathcal{M}_{Hol} is a map that associates to any truth-perspective \mathfrak{T} a map $\mathcal{M}_{Hol_{\mathfrak{T}}}$ representing a *holistic interpretation* of the sentences of the language. The following conditions are required.
 - (2.1) For any sentence α , the interpretation $\mathcal{M}_{Hol_{\mathfrak{T}}}$ associates to each level of the syntactical tree of α a *meaning*, represented by a qumix living in \mathcal{H}^α (the semantic space of α).
 - (2.2) Let $({}^\alpha \mathbf{O}_{\mathfrak{T}}^{(k-1)}, \dots, {}^\alpha \mathbf{O}_{\mathfrak{T}}^{(1)})$ be the $(\mathfrak{T}, \mathcal{M}^q)$ -pseudo-gate tree of sentence α and let $1 \leq i < Height(\alpha)$. Then,

$$\mathcal{M}_{Hol_{\mathfrak{T}}}(Level_i(\alpha)) = {}^\alpha \mathbf{O}_{\mathfrak{T}}^{(i)}(\mathcal{M}_{Hol_{\mathfrak{T}}}(Level_{i+1}(\alpha))).$$

In other words, the global meaning of each level (different from the top level) is obtained by applying the corresponding pseudo-gate to the meaning of the level that occurs immediately above.

- (2.3) Let $Level_i(\alpha) = (\beta_1, \dots, \beta_r)$. Then: $\beta_j = \mathbf{f} \Rightarrow Red^j(\mathcal{M}_{Hol_{\mathfrak{T}}}(Level_i(\alpha))) = {}^{\mathfrak{T}}P_0^{(1)}$; $\beta_j = \mathbf{t} \Rightarrow Red^j(\mathcal{M}_{Hol_{\mathfrak{T}}}(Level_i(\alpha))) = {}^{\mathfrak{T}}P_1^{(1)}$, for any j ($1 \leq j \leq r$). In other words, the contextual meanings of \mathbf{f} and of \mathbf{t} are always the \mathfrak{T} -Falsity and the \mathfrak{T} -Truth, respectively.

On this basis, we put:

$$\mathcal{M}_{Hol_{\mathfrak{T}}}(\alpha) := \mathcal{M}_{Hol_{\mathfrak{T}}}(Level_1(\alpha)),$$

for any sentence α .

As an example, consider again the sentence

$$\alpha = \mathcal{K}a_t \neg(\mathbf{q} \wedge \neg\mathbf{q}) = \mathcal{K}a_t \neg(\top(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})).$$

As we have seen, any $(\mathfrak{T}, \mathcal{M}^q)$ -pseudo-gate-tree of α will have the following form:

$$(\mathfrak{D}_{\mathbb{I}^{(1)}} \otimes \mathfrak{D}_{\text{NOT}_{\mathfrak{T}}^{(1)}} \otimes \mathfrak{D}_{\mathbb{I}^{(1)}} \otimes \mathfrak{D}_{\mathbb{T}_{\mathfrak{T}}^{(1,1,1)}} \otimes \mathfrak{D}_{\text{NOT}_{\mathfrak{T}}^{(3)}} \otimes \mathbf{K}_{a_t}^{(3)}).$$

Take a model

$$\mathcal{M} = (T, Ag, \mathbf{EpSit}, den, \mathcal{M}_{\text{HOL}})$$

such that:

- $\mathcal{M}_{\text{HOL}_{\mathbb{I}}}(\text{Level}_{\text{Height}(\alpha)}(\alpha)) = \mathcal{M}_{\text{HOL}_{\mathbb{I}}}((\mathbf{q}, \mathbf{q}, \mathbf{f})) = P_{|\psi\rangle}$, where

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle.$$

- **EpSit** assigns to agent a_t the epistemic situation

$$(\mathfrak{T}_{a_t}, \text{EpD}_{a_t}, \mathbf{U}_{a_t}, \mathbf{K}_{a_t}),$$

where $\text{EpD}_{a_t} = \mathfrak{D}$ and $\mathbf{K}_{a_t}(\rho) = \rho$, for any $\rho \in \mathfrak{D}$. In other words, a_t has a *maximal epistemic capacity*.⁴

We obtain: $\mathcal{M}_{\text{HOL}_{\mathbb{I}}}(\mathcal{K}a_t \neg(\mathbf{q} \wedge \neg\mathbf{q})) = P_{|\varphi\rangle}$, where: $|\varphi\rangle = \mathbf{K}_{a_t}^{(3)} \text{NOT}_{\mathbb{I}}^{(3)} \mathbb{T}_{\mathbb{I}}^{(1,1,1)} (\mathbb{I}^{(1)} \otimes \text{NOT}_{\mathbb{I}}^{(1)} \otimes \mathbb{I}^{(1)}) (\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle) = \frac{1}{2}(|0, 1, 1\rangle + |0, 0, 1\rangle + |1, 1, 0\rangle + |1, 0, 1\rangle)$. Hence, $p_{\mathbb{I}}(\mathcal{M}_{\text{HOL}_{\mathbb{I}}}(\mathcal{K}a_t \neg(\mathbf{q} \wedge \neg\mathbf{q}))) = \frac{3}{4} \neq 1$. This example clearly shows how even an agent with a maximal epistemic capacity does not necessarily know a very simple instance of the non-contradiction principle!

Unlike standard *compositional* semantics, any $\mathcal{M}_{\text{HOL}_{\mathfrak{T}}}(\alpha)$ represents a kind of autonomous semantic context that is not necessarily correlated to the meanings of other sentences. At the same time, given a sentence γ , $\mathcal{M}_{\text{HOL}_{\mathfrak{T}}}$ determines the *contextual meaning*, with respect to the context $\mathcal{M}_{\text{HOL}_{\mathfrak{T}}}(\gamma)$, of any *occurrence of a subformula* β in the syntactical tree of γ .

Definition 2.5 (Contextual Meaning) Consider a sentence γ such that

$$\text{Level}_i(\gamma) = (\beta_{i_1}, \dots, \beta_{i_r}).$$

The *contextual meaning* of the occurrence β_{i_j} with respect to the context $\mathcal{M}_{\text{HOL}_{\mathfrak{T}}}(\gamma)$ is defined as follows:

$$\mathcal{M}_{\text{HOL}_{\mathfrak{T}}}^{\gamma}(\beta_{i_j}) := \text{Red}^j(\mathcal{M}_{\text{HOL}_{\mathfrak{T}}}(\text{Level}_i(\gamma))).$$

Hence, in particular, we have for any sentence γ

$$\mathcal{M}_{\text{HOL}_{\mathfrak{T}}}^{\gamma}(\gamma) = \mathcal{M}_{\text{HOL}_{\mathfrak{T}}}(\text{Level}_1(\gamma)) = \mathcal{M}_{\text{HOL}_{\mathfrak{T}}}(\gamma).$$

⁴See Sect. 3 of the first Part of this article.

Generally, different occurrences β_{i_j} and β_{h_k} of one and the same subformula β in the syntactical tree of γ may receive different contextual meanings. In other words, we may have:

$$\mathcal{M}_{\text{Hol}\perp_{\mathfrak{T}}}^{\gamma}(\beta_{i_j}) \neq \mathcal{M}_{\text{Hol}\perp_{\mathfrak{T}}}^{\gamma}(\beta_{h_k}).$$

When this is not the case, we will say that one is dealing with a *normal model*.

Definition 2.6 (Normal Holistic Model) A *normal holistic model* of the language \mathcal{L}^{EPQC} is a holistic model \mathcal{M} such that for any truth-perspective \mathfrak{T} and for any sentence γ , the interpretation $\mathcal{M}_{\text{Hol}\perp_{\mathfrak{T}}}$ determines for any occurrence β_{i_j} of a subformula β of γ in the syntactical tree of γ the same contextual meaning, which will be uniformly indicated by $\mathcal{M}_{\text{Hol}\perp_{\mathfrak{T}}}^{\gamma}(\beta)$.

In the following we will always refer to normal holistic models.

Suppose that β is a subformula of two different formulas γ and δ . Generally, we have:

$$\mathcal{M}_{\text{Hol}\perp_{\mathfrak{T}}}^{\gamma}(\beta) \neq \mathcal{M}_{\text{Hol}\perp_{\mathfrak{T}}}^{\delta}(\beta).$$

In other words, sentences may receive different contextual meanings in different contexts also in the case of the normal holistic semantics.

To what extent do contextual meanings and gates (associated to the logical connectives) commute? An answer to this question is given by the following theorem.

Theorem 2.1 Consider a holistic model $\mathcal{M} = (T, \text{Ag}, \text{EpSit}, \text{den}, \mathcal{M}_{\text{Hol}})$ and a truth-perspective \mathfrak{T} .

1. Let $\neg\alpha$ be a subformula of γ . Suppose that $\neg\alpha = \beta_{i_j}$ (the formula occurring at the j -th position of the i -th level in the syntactical tree of γ), while $\alpha = \beta_{(i+1)_k}$. We have:

$$\begin{aligned} \mathcal{M}_{\text{Hol}\perp_{\mathfrak{T}}}^{\gamma}(\neg\alpha) &= \text{Red}^j(\mathcal{M}_{\text{Hol}\perp_{\mathfrak{T}}}(\text{Level}_i(\gamma))) \\ &= \mathfrak{D}_{\text{NOT}_{\mathfrak{T}}}^{(\text{Ar}(\alpha))}(\text{Red}^k(\mathcal{M}_{\text{Hol}\perp_{\mathfrak{T}}}(\text{Level}_{i+1}(\gamma)))) \\ &= \mathfrak{D}_{\text{NOT}_{\mathfrak{T}}}^{(\text{Ar}(\alpha))}(\mathcal{M}_{\text{Hol}\perp_{\mathfrak{T}}}^{\gamma}(\alpha)). \end{aligned}$$

2. Let $\sqrt{id}\alpha$ be a subformula of γ . Suppose that $\sqrt{id}\alpha = \beta_{i_j}$, while $\alpha = \beta_{(i+1)_k}$. We have:

$$\begin{aligned} \mathcal{M}_{\text{Hol}\perp_{\mathfrak{T}}}^{\gamma}(\sqrt{id}\alpha) &= \text{Red}^j(\mathcal{M}_{\text{Hol}\perp_{\mathfrak{T}}}(\text{Level}_i(\gamma))) \\ &= \mathfrak{D}_{\sqrt{\text{I}}_{\mathfrak{T}}}^{(\text{Ar}(\alpha))}(\text{Red}^k(\mathcal{M}_{\text{Hol}\perp_{\mathfrak{T}}}(\text{Level}_{i+1}(\gamma)))) \\ &= \mathfrak{D}_{\sqrt{\text{I}}_{\mathfrak{T}}}^{(\text{Ar}(\alpha))}(\mathcal{M}_{\text{Hol}\perp_{\mathfrak{T}}}^{\gamma}(\alpha)). \end{aligned}$$

3. Let $\top(\alpha_1, \alpha_2, \alpha_3)$ be a subformula of γ . Suppose that in the syntactical tree of γ : $\top(\alpha_1, \alpha_2, \alpha_3) = \beta_{i_j}$, while $\alpha_1 = \beta_{(i+1)_{k_1}}$, $\alpha_2 = \beta_{(i+1)_{k_2}}$, $\alpha_3 = \beta_{(i+1)_{k_3}}$. We have:

$$\begin{aligned} \mathcal{M}_{\text{Hol}\perp_{\mathfrak{T}}}^{\gamma}(\top(\alpha_1, \alpha_2, \alpha_3)) &= \text{Red}^j(\mathcal{M}_{\text{Hol}\perp_{\mathfrak{T}}}(\text{Level}_i(\gamma))) \\ &= \mathfrak{D}_{\top_{\mathfrak{T}}}^{(\text{Ar}(\alpha_1), \text{Ar}(\alpha_2), \text{Ar}(\alpha_3))}(\text{Red}^{k_1, k_2, k_3}(\mathcal{M}_{\text{Hol}\perp_{\mathfrak{T}}}(\text{Level}_{i+1}(\gamma)))). \end{aligned}$$

4. Let $\alpha_1 \uplus \alpha_2$ be a subformula of γ . Suppose that in the syntactical tree of γ : $\alpha_1 \uplus \alpha_2 = \beta_{ij}$, while $\alpha_1 = \beta_{(i+1)k_1}$, $\alpha_2 = \beta_{(i+1)k_2}$. We have:

$$\begin{aligned} \mathcal{M}_{\text{HOL}_{\mathcal{T}}^{\gamma}}(\alpha_1 \uplus \alpha_2) &= \text{Red}^j(\mathcal{M}_{\text{HOL}_{\mathcal{T}}}(\text{Level}_i(\gamma))) \\ &= \mathcal{D}_{\text{XOR}_{\mathcal{T}}}^{(A_1(\alpha_1), A_1(\alpha_2))}(\text{Red}^{k_1, k_2}(\mathcal{M}_{\text{HOL}_{\mathcal{T}}}(\text{Level}_{i+1}(\gamma)))) \end{aligned}$$

5. Let $\mathcal{U}_{\mathbf{a}}, \alpha$ be a subformula of γ . Suppose that $\mathcal{U}_{\mathbf{a}}, \alpha = \beta_{ij}$, while $\alpha = \beta_{(i+1)k}$. We have:

$$\begin{aligned} \mathcal{M}_{\text{HOL}_{\mathcal{T}}^{\gamma}}(\mathcal{U}_{\mathbf{a}}, \alpha) &= \text{Red}^j(\mathcal{M}_{\text{HOL}_{\mathcal{T}}}(\text{Level}_i(\gamma))) \\ &= \mathbf{U}_{\mathbf{a}_t}^{(A_1(\alpha))}(\text{Red}^k(\mathcal{M}_{\text{HOL}_{\mathcal{T}}}(\text{Level}_{i+1}(\gamma)))) \\ &= \mathbf{U}_{\mathbf{a}_t}^{(A_1(\alpha))}(\mathcal{M}_{\text{HOL}_{\mathcal{T}}^{\gamma}}(\alpha)). \end{aligned}$$

6. Let $\mathcal{K}_{\mathbf{a}}, \alpha$ be a subformula of γ . Suppose that $\mathcal{K}_{\mathbf{a}}, \alpha = \beta_{ij}$, while $\alpha = \beta_{(i+1)k}$. We have:

$$\begin{aligned} \mathcal{M}_{\text{HOL}_{\mathcal{T}}^{\gamma}}(\mathcal{K}_{\mathbf{a}}, \alpha) &= \text{Red}^j(\mathcal{M}_{\text{HOL}_{\mathcal{T}}}(\text{Level}_i(\gamma))) \\ &= \mathbf{K}_{\mathbf{a}_t}^{(A_1(\alpha))}(\text{Red}^k(\mathcal{M}_{\text{HOL}_{\mathcal{T}}}(\text{Level}_{i+1}(\gamma)))) \\ &= \mathbf{K}_{\mathbf{a}_t}^{(A_1(\alpha))}(\mathcal{M}_{\text{HOL}_{\mathcal{T}}^{\gamma}}(\alpha)). \end{aligned}$$

Proof By definition of syntactical tree, of pseudo-gate tree, of normal holistic model and of contextual meaning. □

Notice that, generally, the contextual meaning of a conjunction is not the conjunction of the contextual meanings of the two members. As a counterexample, consider the following contradictory sentence:

$$\gamma = \mathbf{q} \wedge \neg \mathbf{q} = \top(\mathbf{q}, \neg \mathbf{q}, \mathbf{f}),$$

whose syntactical tree is:

$$\begin{aligned} \text{Level}_3(\gamma) &= (\mathbf{q}, \mathbf{q}, \mathbf{f}) \\ \text{Level}_2(\gamma) &= (\mathbf{q}, \neg \mathbf{q}, \mathbf{f}) \\ \text{Level}_1(\gamma) &= \top(\mathbf{q}, \neg \mathbf{q}, \mathbf{f}). \end{aligned}$$

Consider a model \mathcal{M} such that:

$$\mathcal{M}_{\text{HOL}_{\mathcal{T}}}(\text{Level}_3(\gamma)) = P_{\frac{1}{\sqrt{2}}(|(0,1,0)\rangle + |1,0,0\rangle)}$$

(which is a maximally entangled quregister with respect to the first and to the second part.)⁵
Hence:

$$\mathcal{M}_{\text{HOL}_{\mathcal{T}}}(\text{Level}_1(\gamma)) = \mathcal{M}_{\text{HOL}_{\mathcal{T}}}(\gamma) = \mathcal{M}_{\text{HOL}_{\mathcal{T}}^{\gamma}}(\gamma) = P_{\frac{1}{\sqrt{2}}(|(0,0,0)\rangle + |1,1,1\rangle)}$$

⁵See Sect. 2 of the first Part of this article.

(which is a maximally entangled quregister). At the same time, we have:

$$\mathcal{M}_{\text{HOL}}^\gamma(\mathbf{q}) = \mathcal{M}_{\text{HOL}}^\gamma(\neg\mathbf{q}) = \frac{1}{2}P_0^{(1)} + \frac{1}{2}P_1^{(1)},$$

which is a proper mixture. Consequently:

$$\mathfrak{D}_{\text{T}^{(1,1,1)}}(\mathcal{M}_{\text{HOL}}^\gamma(\mathbf{q}) \otimes \mathcal{M}_{\text{HOL}}^\gamma(\neg\mathbf{q}) \otimes \mathcal{M}_{\text{HOL}}^\gamma(\mathbf{f})) \neq \mathcal{M}_{\text{HOL}}^\gamma(\mathfrak{T}(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})).$$

Notice that: $p_{\text{I}}(\mathcal{M}_{\text{HOL}}^\gamma(\mathbf{q} \wedge \neg\mathbf{q})) = \frac{1}{2}$; while: $p_{\text{I}}(\mathfrak{D}_{\text{T}^{(1,1,1)}}(\mathcal{M}_{\text{HOL}}^\gamma(\mathbf{q}) \otimes \mathcal{M}_{\text{HOL}}^\gamma(\neg\mathbf{q}) \otimes \mathcal{M}_{\text{HOL}}^\gamma(\mathbf{f}))) = \frac{1}{4}$.

Definition 2.7 (Harmonic and Sound Models) Let $\mathcal{M} = (T, Ag, \mathbf{EpSit}, den, \mathcal{M}_{\text{HOL}})$ be a model.

- \mathcal{M} is called *harmonic* iff the epistemic structure of \mathcal{M} is harmonic, i.e. all agents of the structure share the same truth-perspective \mathfrak{T} . Furthermore, the interpretation-function \mathcal{M}_{HOL} is only defined for the truth-perspective \mathfrak{T} .
- \mathcal{M} is called *sound* iff all agents a_i of \mathcal{M} have a sound epistemic capacity (i.e. assign the “right” probability-values to the truth-values of their truth-perspectives).⁶

By *harmonic epistemic quantum computational semantics (sound epistemic quantum computational semantics)* we will mean the semantics that only refers to harmonic models (sound models).

We can now define the notions of *truth, validity* and *logical consequence*.

Definition 2.8 (Contextual Truth) Let α be a subformula of γ . $\models_{(\gamma, \mathcal{M}, \mathfrak{T})} \alpha$ (the sentence α is *true* with respect to the context γ , the model \mathcal{M} and the truth-perspective \mathfrak{T}) iff $p_{\mathfrak{T}}(\mathcal{M}_{\text{HOL}}^\gamma(\alpha)) = 1$.

Definition 2.9 (Truth) $\models_{(\mathcal{M}, \mathfrak{T})} \alpha$ (the sentence α is *true* with respect to the model \mathcal{M} and the truth-perspective \mathfrak{T}) iff $\models_{(\alpha, \mathcal{M}, \mathfrak{T})} \alpha$.

Hence, the concept of truth turns out to be a special case of the concept of contextual truth.

Definition 2.10 (Contextual Validity) Let α be a subformula of γ .

- $\models_{(\gamma, \mathfrak{T})} \alpha$ (the sentence α is *valid* with respect to the context γ and the truth-perspective \mathfrak{T}) iff for any model \mathcal{M} , $\models_{(\gamma, \mathcal{M}, \mathfrak{T})} \alpha$.
- $\models_\gamma \alpha$ (the sentence α is *valid* with respect to the context γ) iff for any truth- perspective \mathfrak{T} , $\models_{(\gamma, \mathfrak{T})} \alpha$.

Definition 2.11 (Validity)

- $\models_{\mathfrak{T}} \alpha$ (the sentence α is *valid* with respect to the truth-perspective \mathfrak{T}) iff $\models_{(\alpha, \mathfrak{T})} \alpha$.
- $\models \alpha$ (the sentence α is *valid*) iff $\models_\alpha \alpha$.

⁶See Sect. 3 of the first Part of this article.

Definition 2.12 (Consequence with Respect to a Quasi-model \mathcal{M}^q) Let γ be a sentence such that α and β are subformulas of γ and let \mathfrak{T} be a truth-perspective.

- $\alpha \models_{(\gamma, \mathcal{M}^q, \mathfrak{T})} \beta$ (the sentence β is a *consequence* of the sentence α with respect to the context γ , the quasi-model \mathcal{M}^q and the truth-perspective \mathfrak{T}) iff for any model \mathcal{M} based on \mathcal{M}^q :

$$\models_{(\gamma, \mathcal{M}, \mathfrak{T})} \alpha \Rightarrow \models_{(\gamma, \mathcal{M}, \mathfrak{T})} \beta.$$

- $\alpha \models_{(\gamma, \mathcal{M}^q)} \beta$ (the sentence β is a *consequence* of the sentence α with respect to the context γ and the quasi-model \mathcal{M}^q) iff for any truth-perspective \mathfrak{T} , $\alpha \models_{(\gamma, \mathcal{M}^q, \mathfrak{T})} \beta$.

Definition 2.13 (Logical Consequence) Let γ be a sentence such that α and β are subformulas of γ and let \mathfrak{T} be a truth-perspective.

- $\alpha \models_{(\gamma, \mathfrak{T})} \beta$ (β is a *logical consequence* of α with respect to the context γ and the truth-perspective \mathfrak{T}) iff for any quasi-model \mathcal{M}^q , $\alpha \models_{(\gamma, \mathcal{M}^q, \mathfrak{T})} \beta$.
- $\alpha \models_{\gamma} \beta$ (β is a *logical consequence* of α with respect to the context γ) iff for any truth-perspective \mathfrak{T} , $\alpha \models_{(\gamma, \mathfrak{T})} \beta$.
- $\alpha \models \beta$ (β is a *logical consequence* of α iff for any context γ such that α and β are subformulas of γ , $\alpha \models_{\gamma} \beta$).

The concepts of *consequence* and of *logical consequence*, defined above, correspond to *weak* concepts, being defined in terms of \mathfrak{T} -*Truth*, and not in terms of the preorder relation $\leq_{\mathfrak{T}}$ (as one could expect). From an intuitive point of view, however, such weak notions turn out to be more interesting in the case of epistemic situations described in the framework of a holistic semantics.

Notice that only the contextual notion of logical consequence turns out to satisfy transitivity ($\alpha \models_{\gamma} \beta$ and $\beta \models_{\gamma} \delta \Rightarrow \alpha \models_{\gamma} \delta$). Full transitivity ($\alpha \models \beta$ and $\beta \models \delta \Rightarrow \alpha \models \delta$) is naturally violated in the holistic semantics.

As expected, in the particular case of the harmonic epistemic semantics (where all agents share the same truth-perspective) the definitions of truth, validity and logical consequence can be simplified, since the reference to \mathfrak{T} is no longer necessary. Accordingly, in such a case we will write:

$$\begin{aligned} &\models_{(\gamma, \mathcal{M})}^{Harm} \alpha; && \models_{\mathcal{M}}^{Harm} \alpha && \text{(harmonic truth);} \\ &\models_{\gamma}^{Harm} \alpha; && \models^{Harm} \alpha && \text{(harmonic validity);} \\ &\alpha \models_{(\gamma, \mathcal{M}^q)}^{Harm} \beta; && \alpha \models_{\gamma}^{Harm} \beta; && \alpha \models^{Harm} \beta && \text{(harmonic logical consequence).} \end{aligned}$$

3 Some Epistemic Situations

We will now illustrate some significant examples of epistemic situations that arise in this semantics. We will always refer to models

$$\mathcal{M} = (T, Ag, \mathbf{EpSit}, den, {}^{\mathcal{M}}\text{HOL})$$

such that $den(\mathbf{a}) = \mathbf{a}$; $den(t) = t$.

- (1) $\mathcal{K}_{\mathbf{a}_t}\alpha \models^{Harm} \alpha$. In the harmonic semantics, sentences that are known by a given agent at a given time are true.
 (1) is an immediate consequence of the definition of logical consequence and of Theorem 2.1.
- (2) As a particular case of (1) we obtain:

$$\mathcal{K}_{\mathbf{a}_t}\alpha \mathcal{K}_{\mathbf{a}_t}\alpha \models^{Harm} \mathcal{K}_{\mathbf{a}_t}\alpha.$$

Knowing of knowing implies knowing. But not the other way around!

- (3) In the non-harmonic semantics only the two following conditions (which are weaker than (1) and (2)) hold for any quasi-model \mathcal{M}^q and any agent \mathbf{a}_t of \mathcal{M}^q :
 (3.1) $\mathcal{K}_{\mathbf{a}_t}\alpha \models_{(\mathcal{M}^q, \mathfrak{T}_{\mathbf{a}_t})} \alpha$;
 (3.2) $\mathcal{K}_{\mathbf{a}_t}\alpha \mathcal{K}_{\mathbf{a}_t}\alpha \models_{(\mathcal{M}^q, \mathfrak{T}_{\mathbf{a}_t})} \mathcal{K}_{\mathbf{a}_t}\alpha$.
- (4) $\mathcal{K}_{\mathbf{a}_t}\mathcal{K}_{\mathbf{b}_t}\alpha \models^{Harm} \alpha$. In the harmonic semantics, knowing that another agent knows a given sentence implies that the sentence in question holds. At the same time, we will have:

$$\mathcal{K}_{\mathbf{a}_t}\mathcal{K}_{\mathbf{b}_t}\alpha \not\models^{Harm} \mathcal{K}_{\mathbf{a}_t}\alpha.$$

Alice might know that Bob knows a given sentence, without knowing herself the sentence in question!

- (5) In the harmonic sound semantics (where for any agent \mathbf{a}_t , $\mathbf{K}_{\mathbf{a}_t} \mathfrak{T}_{\mathbf{a}_t} P_1^{(1)} = \mathfrak{T}_{\mathbf{a}_t} P_1^{(1)}$ and $\mathbf{K}_{\mathbf{a}_t} \mathfrak{T}_{\mathbf{a}_t} P_0^{(1)} = \mathfrak{T}_{\mathbf{a}_t} P_0^{(1)}$) we have:

$$\models^{Harm} \mathcal{K}_{\mathbf{a}_t}\mathbf{t}; \quad \models^{Harm} \mathcal{K}_{\mathbf{a}_t}\neg\mathbf{f}.$$

Hence, there are sentences that every agent knows.

- (6) $\mathcal{K}_{\mathbf{a}_t}(\alpha \wedge \beta) \not\models \mathcal{K}_{\mathbf{a}_t}\alpha; \mathcal{K}_{\mathbf{a}_t}(\alpha \wedge \beta) \not\models \mathcal{K}_{\mathbf{a}_t}\beta$. Knowing a conjunction does not generally imply knowing its members.
- (7) $\models_{(\mathcal{Y}, \mathcal{M})} \mathcal{K}_{\mathbf{a}_t}\alpha$ and $\models_{(\mathcal{Y}, \mathcal{M})} \mathcal{K}_{\mathbf{a}_t}\beta \not\Rightarrow \models_{(\mathcal{Y}, \mathcal{M})} \mathcal{K}_{\mathbf{a}_t}(\alpha \wedge \beta)$.
 Knowledge is not generally closed under conjunction.
- (8) Let \mathcal{M} be a model and let \mathbf{a}_t be an agent of \mathcal{M} .

$$\not\models_{(\mathcal{M}, \mathfrak{T}_{\mathbf{a}_t})} \mathcal{K}_{\mathbf{a}_t}(\alpha \wedge \neg\alpha).$$

Contradictions are never known.

In order to prove (8), suppose, by contradiction, that there exists a model \mathcal{M} and an agent \mathbf{a}_t such that: $\models_{(\mathcal{M}, \mathfrak{T}_{\mathbf{a}_t})} \mathcal{K}_{\mathbf{a}_t}(\alpha \wedge \neg\alpha)$.

Then, $\mathfrak{p}_{\mathfrak{T}_{\mathbf{a}_t}}(\mathcal{M}_{\text{Hol}} \mathfrak{T}_{\mathbf{a}_t}(\mathcal{K}_{\mathbf{a}_t}(\alpha \wedge \neg\alpha))) = 1$.

By definition of model and by Theorem 2.1 we have:

$$\mathcal{M}_{\text{Hol}} \mathfrak{T}_{\mathbf{a}_t}(\mathcal{K}_{\mathbf{a}_t}(\alpha \wedge \neg\alpha)) = \mathbf{K}_{\mathbf{a}_t}(\mathcal{M}_{\text{Hol}} \mathfrak{T}_{\mathbf{a}_t}(\alpha \wedge \neg\alpha)).$$

Consequently, by hypothesis, $\mathfrak{p}_{\mathfrak{T}_{\mathbf{a}_t}}(\mathbf{K}_{\mathbf{a}_t}(\mathcal{M}_{\text{Hol}} \mathfrak{T}_{\mathbf{a}_t}(\alpha \wedge \neg\alpha))) = 1$. Thus, by the properties of knowledge operations: $\mathfrak{p}_{\mathfrak{T}_{\mathbf{a}_t}}(\mathcal{M}_{\text{Hol}} \mathfrak{T}_{\mathbf{a}_t}(\alpha \wedge \neg\alpha)) = 1$, which is impossible, owing to the following Lemma (of the holistic semantics).

Lemma 3.1 For any sentence α , for any truth-perspective \mathfrak{T} and for any holistic model \mathcal{M} ,

$$\mathfrak{p}_{\mathfrak{T}}(\mathcal{M}_{\text{Hol}} \mathfrak{T}(\alpha \wedge \neg\alpha)) \neq 1.$$

(9) In the non-harmonic semantics the following situation is possible:

$$\models_{(\mathcal{M}, \mathfrak{T}_{a_t})} \mathcal{K}a_t \mathcal{K}b_t \mathbf{f}.$$

In other words, according to the truth-perspective of Alice it is true that Alice (at time t) knows that Bob (at time t) knows the *Falsity* of Alice’s truth-perspective.

As an example, consider a (non-harmonic) model \mathcal{M} with two agents a_t and b_t satisfying the following conditions:

- (a) the epistemic distance between the truth-perspectives of a_t and of b_t is greater than or equal to $\frac{1}{2}$.⁷ In such a case we have:

$$\mathfrak{T}_{a_t} P_1^{(1)} \preceq_{\mathfrak{T}_{b_t}} \mathfrak{T}_{a_t} P_0^{(1)}$$

(according to Bob’s truth-perspective, Alice’s *Truth* precedes Alice’s *Falsity*);

- (b) $\mathbf{K}_{b_t} \mathfrak{T}_{a_t} P_0^{(1)} = \mathfrak{T}_{a_t} P_1^{(1)}$ (the information according to which Bob knows Alice’s *Falsity* is true with respect to Alice’s truth-perspective);

- (c) $\mathbf{K}_{a_t} \mathfrak{T}_{a_t} P_1^{(1)} = \mathfrak{T}_{a_t} P_1^{(1)}$
(Alice at time t has a sound epistemic capacity).

Consider the syntactical tree of $\mathcal{K}a_t \mathcal{K}b_t \mathbf{f}$:

$$Level_3(\mathcal{K}a_t \mathcal{K}b_t \mathbf{f}) = (\mathbf{f})$$

$$Level_2(\mathcal{K}a_t \mathcal{K}b_t \mathbf{f}) = (\mathcal{K}b_t \mathbf{f})$$

$$Level_1(\mathcal{K}a_t \mathcal{K}b_t \mathbf{f}) = (\mathcal{K}a_t \mathcal{K}b_t \mathbf{f}).$$

The qumixes assigned by $\mathcal{M}_{\text{HOL}} \perp_{\mathfrak{T}_{a_t}}$ to the levels of this tree are:

$$\mathcal{M}_{\text{HOL}} \perp_{\mathfrak{T}_{a_t}} (Level_3(\mathcal{K}a_t \mathcal{K}b_t \mathbf{f})) = \mathfrak{T}_{a_t} P_0^{(1)}$$

(by definition of model);

$$\mathcal{M}_{\text{HOL}} \perp_{\mathfrak{T}_{a_t}} (Level_2(\mathcal{K}a_t \mathcal{K}b_t \mathbf{f})) = \mathbf{K}_{b_t} (\mathcal{M}_{\text{HOL}} \perp_{\mathfrak{T}_{a_t}} (Level_3(\mathcal{K}a_t \mathcal{K}b_t \mathbf{f}))) = \mathfrak{T}_{a_t} P_1^{(1)}$$

(by definition of model and by (b));

$$\mathcal{M}_{\text{HOL}} \perp_{\mathfrak{T}_{a_t}} (Level_1(\mathcal{K}a_t \mathcal{K}b_t \mathbf{f})) = \mathbf{K}_{b_t} (\mathcal{M}_{\text{HOL}} \perp_{\mathfrak{T}_{a_t}} (Level_2(\mathcal{K}a_t \mathcal{K}b_t \mathbf{f}))) = \mathfrak{T}_{a_t} P_1^{(1)}$$

(by definition of model and by (c)).

Hence, $\mathfrak{p}_{\mathfrak{T}_{a_t}} (\mathcal{M}_{\text{HOL}} \perp_{\mathfrak{T}_{a_t}} (\mathcal{K}a_t \mathcal{K}b_t \mathbf{f})) = 1$ and $\models_{(\mathcal{M}, \mathfrak{T}_{a_t})} \mathcal{K}a_t \mathcal{K}b_t \mathbf{f}$. Notice that

$$\models_{(\mathcal{M}, \mathfrak{T}_{a_t})} \mathcal{K}a_t \mathcal{K}b_t \mathbf{f} \not\Rightarrow \models_{(\mathcal{M}, \mathfrak{T}_{b_t})} \mathcal{K}b_t \mathbf{f}.$$

In other words, the following situation is possible:

- According to Alice’s truth-perspective, it is true that Alice knows that Bob knows the *Falsity*.
- However, according to Bob’s truth-perspective it is not true that Bob knows the *Falsity*.

⁷The concept of *epistemic distance* has been defined in Sect. 2 of the first Part of this article.

Roughly, we might say: Alice knows that Bob is wrong. However, Bob is not aware of being wrong!

The epistemic situations illustrated above seem to reflect pretty well some characteristic limitations of the real processes of acquiring information and knowledge. Owing to the limits of epistemic domains, understanding and knowing are not generally closed under logical consequence. Hence, the unpleasant phenomenon of *logical omniscience* is here avoided. We have, in particular, that knowledge is not generally closed under logical conjunction, as in fact happens in the case of concrete memories both of human and of artificial intelligence. It is also admitted that an agent can understand (or know) a conjunction, without being able to understand (to know) its members. Such situation, which might appear *prima facie* somewhat “irrational”, seems to be instead deeply in agreement with our use of natural languages, where sometimes agents show to use correctly and to understand some *global* expressions without being able to understand their (meaningful) parts.

References

1. Beltrametti, E., Dalla Chiara, M.L., Giuntini, R., Leporini, R., Sergioli, G.: Epistemic quantum computational structures in a Hilbert-space environment. *Fundam. Inform.* **115**, 1–14 (2012). doi:[10.3233/FI-2012-637](https://doi.org/10.3233/FI-2012-637)
2. Beltrametti, E., Dalla Chiara, M.L., Giuntini, R., Sergioli, G.: Quantum teleportation and quantum epistemic semantics. *Math. Slovaca* **62**, 1–24 (2012)
3. Chiribella, G., D’Ariano, G.M., Perinotti, P.: Transforming quantum operations: quantum supermaps. *A Lett. J. Explor. Front. Phys.* **83**, 30004 (2008)
4. Dalla Chiara, M.L., Freytes, H., Giuntini, R., Ledda, A., Leporini, R., Sergioli, G.: Entanglement as a semantic resource. *Found. Phys.* **40**, 1494–1518 (2011)
5. Fan, H.-y., Hu, L.-y.: Infinite-dimensional Kraus operators for describing amplitude-damping channel and laser process. *Opt. Commun.* **282**, 932–935 (2009)
6. Kraus, K.: *States, Effects and Operations*. Springer, Berlin (1983)
7. Leporini, R., Sergioli, G.: Quantum channels and knowledge operations. Submitted