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Geometrization of Mass in General Relativity

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Abstract In this paper we will extend the notion of tangent bundle to a $\mathbf{Z}_2$ graded tangent bundle. This graded bundle has a Lie algebroid structure and we can develop notions semi-Riemannian metrics, Levi-Civita connection, and curvature, on it. In case of space-times manifolds, even part of the tangent bundle is related to space and time structures (gravity) and odd part is related to mass distribution in space-time. In this structure, mass becomes part of the geometry, and Einstein field equation can be reconstructed in a new simpler form. The new field equation is purely geometric.

Keywords Graded tangent bundle · Algebroid · Mass · Gravitation · Field equation · Semi-Riemannian metric · Connection · Curvature

1 Graded Tangent Bundles

In general relativity, gravity can be formulated by Lorentzian metrics on the ordinary tangent bundle of a space-time [5]. Since we can extend ordinary tangent bundle of a manifold to a larger bundle, so we obtain additional degree of freedom to describe more concepts.

Tangent vectors to a manifold M, can be identified by point-derivations of the algebra $C^\infty(M)$ [4]. If we replace the algebra $C^\infty(M)$ by some other related algebra, we may find some new tangent vectors. We can consider the $\mathbf{Z}_2$ graded two dimensional algebra $\mathbb{R}\oplus\mathbb{R}$ which is the Clifford algebra of $\mathbb{R}$ furnished with its canonical positive definite inner product. The unit of this algebra is $(1, 1)$ which is denoted by $\mathbf{1}$. Even part of the algebra $\mathbb{R}\oplus\mathbb{R}$ is generated by $\mathbf{1}$. Odd part of this algebra is generated by $\tau = (1, -1)$. So, every elements of $\mathbb{R}\oplus\mathbb{R}$ has the form $\lambda\mathbf{1} + \mu\tau$, in fact

$$(a, b) = \frac{a+b}{2}\mathbf{1} + \frac{a-b}{2}\tau.$$

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For the sake of simplicity, denote $\lambda\mathbf{1}$ by λ. Denote the set of all $\mathbb{R}\oplus\mathbb{R}$ valued smooth function on M, by $\hat{C}^\infty(M)$. This set, by pointwise addition and multiplication is a $\mathbf{Z}_2$ graded algebra, and its elements have a form of $f+g\tau$ in which $f,g\in C^\infty(M)$. Even subalgebra of $\hat{C}^\infty(M)$ is $C^\infty(M)$, and its odd functions have a form of $g\tau$ for some $g\in C^\infty(M)$. Even and odd elements of a $\mathbf{Z}_2$ graded algebra are called homogeneous and their parity is defined as follows

$$|a|=\begin{cases}0 & a \text{ is even,}\\ 1 & a \text{ is odd.}\end{cases}$$

Derivations of $C^\infty(M)$ identify ordinary vector fields on M. Now, we are going to find graded derivations of $\hat{C}^\infty(M)$.

Definition 1.1 A linear map $D:\hat{C}^\infty(M)\longrightarrow\hat{C}^\infty(M)$ is called an even derivation iff even and odd subspaces of $\hat{C}^\infty(M)$ are invariant under D and for $\hat{f},\hat{g}\in\hat{C}^\infty(M)$, we have

$$D(\hat{f}\hat{g})=D(\hat{f})\hat{g}+\hat{f}D(\hat{g}).$$

Definition 1.2 A linear map $D:\hat{C}^\infty(M)\longrightarrow\hat{C}^\infty(M)$ is called an odd derivation iff D changes parity of homogeneous elements and for homogeneous elements $\hat{f},\hat{g}\in\hat{C}^\infty(M)$, we have

$$D(\hat{f}\hat{g})=D(\hat{f})\hat{g}+(-1)^{|\hat{f}|}\hat{f}D(\hat{g}).$$

Theorem 1.3 *Any vector field $X\in\mathfrak{X}M$ determine an even derivation on $\hat{C}^\infty(M)$ by $X(f+g\tau)=X(f)+X(g)\tau$ and every even derivation on $\hat{C}^\infty(M)$ can be obtained by this way.*

Proof Clearly the operation of X on $\hat{C}^\infty(M)$ defined as above, is an even derivation. Conversely, let D be an even derivation. Restriction of D to $C^\infty(M)$ is an ordinary derivation on $C^\infty(M)$ and determines some vector field $X\in\mathfrak{X}M$ such that for $f\in C^\infty(M)$ we have $D(f)=X(f)$. We can deduce $D(\tau)=0$.

$$0=D(1)=D(\tau\tau)=2\tau D(\tau)\Rightarrow D(\tau)=0.$$

It is easy to see that $D(f+g\tau)=X(f)+X(g)\tau$. □

Theorem 1.4 *Any function $h\in C^\infty(M)$ determine an odd derivation on $\hat{C}^\infty(M)$ by $D_h(f+g\tau)=gh$ and every odd derivation on $\hat{C}^\infty(M)$ can be obtained by this way.*

Proof Clearly D_h is an odd derivation. Conversely, suppose that D is an odd derivation. By commutativity of $\hat{C}^\infty(M)$, we can infer that for any function $f\in C^\infty(M)$ we have $D(f)=0$.

$$D(f)\tau+fD(\tau)=D(f\tau)=D(\tau f)=D(\tau)f-\tau D(f)\Rightarrow 2D(f)\tau=0\Rightarrow D(f)=0.$$

$D(\tau)$ is an even element of $\hat{C}^\infty(M)$, so for some $h\in C^\infty(M)$ we have $D(\tau)=h$. It is easy to show that $D=D_h$. □

These theorems show that the space of graded derivations of the graded algebra $\hat{C}^\infty(M)$ is $\mathfrak{X}(M)\oplus C^\infty(M)$. By pointwise addition and multiplication, this space is a $\mathbf{Z}_2$ graded module on $C^\infty(M)$, and we denote it by $\hat{\mathfrak{X}}(M)$. Even part of $\hat{\mathfrak{X}}(M)$ is $\mathfrak{X}(M)$ and its odd part is $C^\infty(M)$. For constant function $\mathbf{1}$, denote the odd derivation $D_\mathbf{1}$ by ξ, so $\xi(f+g\tau)=g$. Every odd derivation on $\hat{C}^\infty(M)$ is of the form $h\xi$ for some $h\in C^\infty(M)$, and we call them odd vector fields on M. Even vector fields, are ordinary vector fields on M. We can construct the bundle $\hat{T}M$ as follows

$$\hat{T}M=\bigcup_{p\in M} T_pM\oplus\mathbb{R}.$$

$\hat{\mathfrak{X}}(M)$ is the space of sections of $\hat{T}M$, so we call $\hat{T}M$ as the graded tangent bundle of M. $\hat{T}M$ is a $\mathbf{Z}_2$ graded bundle and is a natural graded extension of the ordinary tangent bundle. Henceforth, we use symbols $\hat{X},\hat{Y},\hat{Z},\dots$ for arbitrary sections of $\hat{T}M$ and symbols $X,Y,Z,\dots$ for ordinary (or even) vector fields on M. Lie bracket of two graded derivation D and D', is a graded derivation, defined as follows:

$$\left[D,\ D'\right]=D\circ D'-(-1)^{|D||D'|}D'\circ D.$$

This definition implies that Lie bracket of even vector fields is the ordinary Lie bracket of vector fields, and Lie bracket of even and odd vector fields is as follows:

$$[X,\ h\xi]=X(h)\xi,$$
$$[f\xi,\ g\xi]=0.$$

This Lie bracket, make $\hat{C}^\infty(M)$ into a super Lie algebra. Since, Lie bracket of odd vector fields are zero, $\hat{C}^\infty(M)$ is an ordinary Lie algebra too. This Lie algebra structure of $\hat{C}^\infty(M)$ and the anchor map $\rho:\hat{T}M\longrightarrow TM,\ \rho(X+h\xi)=X$, convert $\hat{T}M$ into a Lie algebroid. So, we can use properties of algebroid structures for $\hat{T}M$. In [3] we have used a similar structure for unification of electromagnetism and gravity, but $\hat{T}M$ play a different role here.

2 Graded Connections on Graded Tangent Bundles

Due to the theory of connections on Lie algebroids, a connection on $\hat{T}M$ is a bilinear operator $\hat{\nabla}:\hat{\mathfrak{X}}(M)\times\hat{\mathfrak{X}}(M)\longrightarrow\hat{\mathfrak{X}}(M)$ that satisfies the following relations.

For $\hat{X},\hat{Y}\in\hat{\mathfrak{X}}(M)$ and $f\in C^\infty(M)$:

$$\hat{\nabla}_{f\hat{X}}\hat{Y}=f\hat{\nabla}_{\hat{X}}\hat{Y},$$
$$\hat{\nabla}_{\hat{X}}f\hat{Y}=\rho(\hat{X})(f)\hat{Y}+f\hat{\nabla}_{\hat{X}}\hat{Y}.$$

Note that here, we have $\rho(\hat{X})(f)=\hat{X}(f)$, and we can rewrite the second equation in a more simple and natural form. If a connection $\hat{\nabla}$ on $\hat{T}M$, respects parity of vector fields such that for homogeneous vector fields $\hat{X},\hat{Y},\ \hat{\nabla}_{\hat{X}}\hat{Y}$ be homogeneous and $|\hat{\nabla}_{\hat{X}}\hat{Y}|=|\hat{X}|+|\hat{Y}|$, then we call it a graded connection.

Theorem 2.1 *For any graded connection $\hat{\nabla}$ on $\hat{T}M$, there exist a unique connection ∇ on M and two 1-form α,α' and a vector field X_0 on M such that for $X,Y\in\mathfrak{X}(M)$ and $h,k\in C^\infty(M)$:*

$$\hat{\nabla}_XY=\nabla_XY,\tag{1}$$

$$\hat{\nabla}_{h\xi}X=h\alpha(X)\xi,\tag{2}$$

$$\hat{\nabla}_Xh\xi=h\alpha'(X)\xi+X(h)\xi,\tag{3}$$

$$\hat{\nabla}_{h\xi} k\xi = hkX_0. \tag{4}$$

Proof Restriction of $\hat{\nabla}$ to even vector fields is an ordinary connection ∇ on M. $\hat{\nabla}_\xi X$ and $\hat{\nabla}_X \xi$ are odd vector fields, so we can define α and α' by $\alpha(X)\xi = \hat{\nabla}_\xi X$, $\alpha'(X)\xi = \hat{\nabla}_X \xi$. Properties of connections and odd vector fields imply that α and α' are 1-forms. Set $X_0 = \hat{\nabla}_\xi \xi$. Straight computations yield Eqs. (1)–(4). □

Conversely, any graded connection on $\hat{T}M$ is obtained by Eqs. (1)–(4) for some connection ∇ and 1-forms α and α' and some vector field X_0 on M. Torsion of $\hat{\nabla}$ is defined as follows.

$$\hat{T}(\hat{X}, \hat{Y}) = \hat{\nabla}_{\hat{X}} \hat{Y} - \hat{\nabla}_{\hat{Y}} \hat{X} - [\hat{X}, \hat{Y}].$$

$\hat{\nabla}$ is torsion free iff ∇ is torsion free and $\alpha = \alpha'$. So, torsion free graded connections on $\hat{T}M$ are obtained by triplet (∇, α, X_0) in which ∇ is a torsion free connection and α is a 1-form and X_0 is a vector field on M.

3 Semi-Riemannian Metrics on Graded Tangent Bundles

Definition 3.1 Any semi-Riemannian metric on the vector bundle $\hat{T}M$ is called a graded metric on $\hat{T}M$ iff even vectors are orthogonal to odd vectors.

Any semi-Riemannian metrics on $\hat{T}M$ determines a compatible and torsion free connection $\hat{\nabla}$ on $\hat{T}M$ by Koszul formula [2], and is called the Levi-Civita connection of the metric.

$$2\langle \hat{\nabla}_{\hat{X}} \hat{Y}, \hat{Z} \rangle = \hat{X}\langle \hat{Y}, \hat{Z} \rangle + \hat{Y}\langle \hat{Z}, \hat{X} \rangle - \hat{Z}\langle \hat{X}, \hat{Y} \rangle$$
$$\big\langle [\hat{X}, \hat{Y}], \hat{Z} \big\rangle - \big\langle [\hat{Y}, \hat{Z}], \hat{X} \big\rangle + \big\langle [\hat{Z}, \hat{X}], \hat{Y} \big\rangle.$$

Theorem 3.2 *Levi-Civita connection of a graded metric on $\hat{T}M$ is a graded connection.*

Proof If sum of the parity of homogeneous vector fields $\hat{X}$, $\hat{Y}$, $\hat{Z}$ be odd, then Koszul formula shows that $\hat{\nabla}_{\hat{X}} \hat{Y}$ is orthogonal to $\hat{Z}$. So, we can deduce $\hat{\nabla}_{\hat{X}} \hat{Y}$ is homogeneous and its parity is sum of the parity of $\hat{X}$ and $\hat{Y}$. □

Let $\hat{g}$ be a graded metric on $\hat{T}M$, then its restriction to TM is a semi-Riemannian metric on M and $h = \hat{g}(\xi, \xi)$ is a smooth nonzero function on M. Conversely, every semi-Riemannian metric g on M and smooth nonzero function h on M, determine a graded metric on $\hat{T}M$ as follows:

$$\langle X + f\xi, Y + k\xi \rangle = g(X, Y) + fkh.$$

Without lose of generality, we consider the case $\hat{g}(\xi, \xi)$ is positive and is of the form $\hat{g}(\xi, \xi) = e^{2\theta}$. So, graded metrics on M are determined by pairs (g, θ) in which g is a semi-Riemannian metric on M and $\theta \in C^\infty(M)$ such that $\langle X, Y \rangle = g(X, Y)$ and $\langle \xi, \xi \rangle = e^{2\theta}$.

The gradient of a smooth function f on a semi-Riemannian manifold (M, g) is denoted by $\vec{\nabla} f$ and defined by $g(\vec{\nabla} f, X) = df(X) = X(f)$. As we have already shown, any torsion free graded connection $\hat{\nabla}$ on $\hat{T}M$ is determined by a triplet (∇, α, X_0), in which ∇ is a torsion free connection and α is a 1-form and X_0 is a vector field on M.

Theorem 3.3 *If $\hat{g} = (g, \theta)$ is a graded semi-Riemannian metric on $\hat{T}M$, then its Levi-Civita connection is determined by the triplet $(\nabla, d\theta, -e^{2\theta}\vec{\nabla}\theta)$ in which ∇ is the Levi-Civita connection of g.*

Proof Let Levi-Civita connection of $\hat{g}$ be determined by (∇, α, X_0). Applying Koszul formula to even vector fields implies that ∇ is the Levi-Civita connection of g. Now,

$$\begin{aligned} X(e^{2\theta}) &= X\langle\xi, \xi\rangle = 2\langle\hat{\nabla}_X\xi, \xi\rangle = 2\big\langle\alpha(X)\xi, \xi\big\rangle \\ &= 2\alpha(X)\langle\xi, \xi\rangle = 2\alpha(X)e^{2\theta}. \end{aligned}$$

But $X(e^{2\theta}) = 2e^{2\theta}X(\theta)$, so $\alpha(X) = X(\theta)$. Consequently, $\alpha = d\theta$. Following computations show that $X_0 = -e^{2\theta}\vec{\nabla}\theta$.

$$\begin{aligned} 0 &= \xi\langle Y, \xi\rangle = \langle\hat{\nabla}_\xi Y, \xi\rangle + \langle Y, \hat{\nabla}_\xi\xi\rangle \\ &= \big\langle\alpha(Y)\xi, \xi\big\rangle + \langle Y, X_0\rangle = \alpha(Y)\langle\xi, \xi\rangle + \langle Y, X_0\rangle \\ &= e^{2\theta}d\theta(Y) + \langle Y, X_0\rangle \\ \Rightarrow \quad & \langle X_0, Y\rangle = \big\langle -e^{2\theta}\vec{\nabla}\theta, Y\big\rangle \;\Rightarrow\; X_0 = -e^{2\theta}\vec{\nabla}\theta. \end{aligned}$$

□

Explicitly, $\hat{\nabla}$ is determined as follows.

$$\hat{\nabla}_X Y = \nabla_X Y, \tag{5}$$

$$\hat{\nabla}_\xi X = \hat{\nabla}_X\xi = d\theta(X)\xi, \tag{6}$$

$$\hat{\nabla}_\xi\xi = -e^{2\theta}\vec{\nabla}\theta. \tag{7}$$

4 Curvature Tensors of Graded Metrics

In this section $\hat{g} = (g, \theta)$ is a graded metric on $\hat{T}M$ and $\hat{\nabla} = (\nabla, \alpha, X_0)$ is its Levi-Civita connection. Curvature tensors of $\hat{\nabla}$ as a connection in algebroid structures are defined. Curvature and Ricci curvature tensors of $\hat{g}$ and g are denoted respectively by $\hat{\mathrm{R}}$, $\widehat{\mathrm{Ric}}$, R, Ric.

Theorem 4.1 *Curvature tensor $\hat{R}$ respect parity of homogeneous vector fields and satisfies the following relations.*

$$\hat{\mathrm{R}}(X, Y)(Z) = \mathrm{R}(X, Y)(Z), \tag{8}$$

$$\hat{\mathrm{R}}(X, Y)(\xi) = 0, \tag{9}$$

$$\hat{\mathrm{R}}(X, \xi)(Y) = (\nabla_X\alpha)(Y)\xi + \alpha(X)\alpha(Y)\xi, \tag{10}$$

$$\hat{\mathrm{R}}(X, \xi)(\xi) = \nabla_X X_0 - \alpha(X)X_0. \tag{11}$$

Proof Formula of the curvature computation shows that the curvature tensor respect parity of vector fields. By straight forward computations we find Eqs. (8)–(11) hold. For example to prove (9), assume $[X, Y] = 0$, so

$$\hat{\mathrm{R}}(X, Y)(\xi) = \hat{\nabla}_X\hat{\nabla}_Y\xi - \hat{\nabla}_Y\hat{\nabla}_X\xi = \hat{\nabla}_X\alpha(Y)\xi - \hat{\nabla}_Y\alpha(X)\xi$$

$$= X\big(\alpha(Y)\big)\xi + \alpha(Y)\hat{\nabla}_X\xi - Y\big(\alpha(X)\big)\xi - \alpha(X)\hat{\nabla}_Y\xi$$
$$= d\alpha(X, Y)\xi + \alpha(Y)\alpha(X)\xi - \alpha(X)\alpha(Y)\xi = 0.$$

Note that $d\alpha = 0$ because $\alpha = d\theta$. □

Scalar part of the $\hat{\mathrm{R}}(X, \xi)(Y)$ is a tensor with respect to X and Y and it is convenient to have some name for it. Set

$$\tilde{T}(X, Y) = (\nabla_X\alpha)(Y) + \alpha(X)\alpha(Y).$$

Since α is closed and ∇ is torsion free, they imply that $(\nabla_X\alpha)(Y)$ is symmetric with respect to X and Y.

$$(\nabla_X\alpha)(Y) = (\nabla_X\alpha)(Y) - (\nabla_Y\alpha)(X) + (\nabla_Y\alpha)(X)$$
$$= d\alpha(X, Y) + (\nabla_Y\alpha)(X) = (\nabla_Y\alpha)(X).$$

Consequently, $\tilde{T}$ is a symmetric tensor. For a smooth function f on a semi-Riemannian manifold (M, g), its Hessian is a 2-covariant symmetric tensor denoted by $\mathrm{Hes}(f)$ and is defied as follows.

$$\mathrm{Hes}(f)(X, Y) = (\nabla_X df)(Y).$$

So, the symmetric tensor $\tilde{T}$ can be written as follows.

$$\tilde{T} = \mathrm{Hes}(\theta) + d\theta \otimes d\theta. \tag{12}$$

Laplacian of a smooth function f is also a smooth function denoted by $\triangle(f)$ and is defined by $\triangle(f) = \mathrm{div}(df) = \mathrm{tr}(\mathrm{Hes}(f))$.

Since, $\mathrm{tr}(d\theta \otimes d\theta) = g(\vec{\nabla}\theta, \vec{\nabla}\theta) = |\vec{\nabla}\theta|^2$, we find

$$\mathrm{tr}(\tilde{T}) = \triangle(\theta) + |\vec{\nabla}\theta|^2.$$

Theorem 4.2 *Ricci curvature of $\hat{g}$ satisfies the following relations.*

$$\widehat{\mathrm{Ric}}(X, Y) = \mathrm{Ric}(X, Y) - \tilde{T}(X, Y), \tag{13}$$
$$\widehat{\mathrm{Ric}}(X, \xi) = 0, \tag{14}$$
$$\widehat{\mathrm{Ric}}(\xi, \xi) = -e^{2\theta}\mathrm{tr}(\tilde{T}). \tag{15}$$

Proof Let $\{E_1, \dots, E_n\}$ be an orthonormal local base of (M, g) and $\hat{i} = \langle E_i, E_i\rangle = \pm 1$. Therefore, $\{E_1, \dots, E_n, e^{-\theta}\xi\}$ is an orthonormal local base for $\hat{T}M$. Following computations show that (13) hold.

$$\widehat{\mathrm{Ric}}(X, Y) = \sum_{i=1}^{n} \hat{i}\big\langle \hat{R}(X, E_i)(E_i), Y\big\rangle + \big\langle \hat{R}(X, e^{-\theta}\xi)(e^{-\theta}\xi), Y\big\rangle$$
$$= \sum_{i=1}^{n} \hat{i}\big\langle R(X, E_i)(E_i), Y\big\rangle + e^{-2\theta}\big\langle \hat{R}(X, \xi)(\xi), Y\big\rangle$$
$$= \mathrm{Ric}(X, Y) - e^{-2\theta}\big\langle \hat{R}(X, \xi)(Y), \xi)\big\rangle$$

$$= \mathrm{Ric}(X,Y) - e^{-2\theta}\langle \tilde{T}(X,Y)\xi,\xi\rangle = \mathrm{Ric}(X,Y) - e^{-2\theta}\tilde{T}(X,Y)\langle \xi,\xi\rangle$$
$$= \mathrm{Ric}(X,Y) - \tilde{T}(X,Y).$$

Proof of (14):

$$\widehat{\mathrm{Ric}}(X,\xi) = \sum_{i=1}^{n} \hat{i}\langle \hat{R}(X,E_i)(E_i),\xi\rangle + \langle \hat{R}(X,e^{-\theta}\xi)(e^{-\theta}\xi),\xi\rangle = 0.$$

Proof of (15):

$$\begin{aligned}\widehat{\mathrm{Ric}}(\xi,\xi) &= \sum_{i=1}^{n} \hat{i}\langle \hat{R}(\xi,E_i)(E_i),\xi\rangle + \langle \hat{R}(\xi,e^{-\theta}\xi)(e^{-\theta}\xi),\xi\rangle \\ &= -\sum_{i=1}^{n} \hat{i}\langle \hat{R}(E_i,\xi)(E_i),\xi\rangle = -\sum_{i=1}^{n} \hat{i}\langle \tilde{T}(E_i,E_i)\xi,\xi\rangle \\ &= -\sum_{i=1}^{n} \hat{i}\tilde{T}(E_i,E_i)\langle \xi,\xi\rangle = -e^{2\theta}\mathrm{tr}(\tilde{T}).\end{aligned}$$

□

Theorem 4.3 *If scalar curvature of $\hat{g}$ and g are denoted by $\hat{R}$ and R respectively, then*

$$\hat{R} = R - 2\mathrm{tr}(\tilde{T}). \tag{16}$$

Proof

$$\begin{aligned}\hat{R} &= \sum_{i=1}^{n} \hat{i}\widehat{\mathrm{Ric}}(E_i,E_i) + \widehat{\mathrm{Ric}}(e^{-\theta}\xi,e^{-\theta}\xi) \\ &= \sum_{i=1}^{n} \hat{i}(\mathrm{Ric}(E_i,E_i) - \tilde{T}(E_i,E_i)) + e^{-2\theta}\widehat{\mathrm{Ric}}(\xi,\xi) \\ &= R - \mathrm{tr}(\tilde{T}) - \mathrm{tr}(\tilde{T}) = R - 2\mathrm{tr}(\tilde{T}).\end{aligned}$$

□

5 Application to General Relativity

In this section we consider M as a space-time manifold whose dimension is n and $2 \leq n$. Assume $\hat{g} = (g,\theta)$ is a graded metric on $\hat{T}M$. g is an arbitrary semi-Riemannian metric in M and play the role of potential for gravity. We will find a field equation in which θ play the role of potential for mass distribution in space-time. In this structure, even part of $\hat{T}M$ is related to the structure of space and time (gravity) and its odd part relate to the structure of mass distribution. To find a proper field equation, we use Hilbert action in the context of graded metrics on $\hat{T}M$. Denote canonical volume form of a metric g on oriented manifold M by Ω_g. Hilbert action $\mathcal{L}$ on graded metrics is defined as follows.

$$\mathcal{L}(\hat{g}) = \mathcal{L}(g,\theta) = \int_M \hat{R}\Omega_g.$$

To be more precise, we must assume M is compact or we must integrate on open subset U of M such that $\overline{U}$ is compact [1]. To find a critical metric $\hat{g}$ for Hilbert action we must do some lengthy and tedious computations.

A variation for a metric $\hat{g}$ is obtained by a pair (s,h) in which s is a symmetric 2-covariant tensor on M and $h \in C^\infty(M)$. Set $\tilde{g}(t) = g + ts$ and $\tilde{\theta}(t) = \theta + th$. For small t, $\hat{g}(t) = (\tilde{g}(t), \tilde{\theta}(t))$ is a graded metric and is a variation of $\hat{g}$. $\hat{g}$ is a critical metric for Hilbert action iff for every pair (s,h):

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{L}\big(\tilde{g}(t), \tilde{\theta}(t)\big) = \frac{d}{dt}\bigg|_{t=0} \int_M \hat{R}(t)\varOmega_{g+ts} = 0. \tag{17}$$

$\hat{R}(t)$ is the scalar curvature of $\hat{g}(t)$ and

$$\hat{R}(t) = \tilde{R}(t) - 2\big(\triangle^t\big(\tilde{\theta}(t)\big) + \big|\vec{\nabla}^t\tilde{\theta}(t)\big|^2\big).$$

$\tilde{R}(t)$ is the scalar curvature of $\tilde{g}(t)$. Note that in above, all Levi-Civita connection, gradient, divergence, Laplacian, and volume form, depend on t. To find derivation in (17), we must compute derivations of $\tilde{R}(t)$, $\varOmega_{g+ts}$, $\triangle^t(\tilde{\theta}(t))$, $|\vec{\nabla}^t\tilde{\theta}(t)|^2$ at $t = 0$.

Note that an inner product in a vector space, extend to an inner product in tensor spaces. For example, if T and S are two 2-covariant tensors on an inner product space V, and g_{ij} are components of the inner product in some bases of V, then

$$\langle T, K\rangle = g^{im}g^{jn}T_{ij}K_{mn} = T^{mn}K_{mn}.$$

In particular, $\langle T, g\rangle = g^{ij}T_{ij} = \operatorname{tr}(T)$. Fix some chart (x,U) on M. For the sake of simplicity, denote $\frac{\partial}{\partial x^i}$ by ∂_i. Local components of the meter $\tilde{g}(t)$ are denoted by $\tilde{g}_{ij}(t)$, so $\tilde{g}_{ij}(t) = g_{ij} + ts_{ij}$ and $\tilde{g}'_{ij}(0) = s_{ij}$. By $\tilde{g}^{ik}(t)\tilde{g}_{kj}(t) = \delta^i_j$ and derivation with respect to t, it is deduced that $(\tilde{g}^{ij})'(0) = -s^{ij}$.

Components of Levi-Civita connection of $\tilde{g}(t)$ are denoted by $\tilde{\varGamma}^k_{ij}(t)$. It is well known that

$$\tilde{\varGamma}^k_{ij}(t) = \frac{1}{2}\tilde{g}^{kl}(t)\left(\frac{\partial\tilde{g}_{il}(t)}{\partial x^j} + \frac{\partial\tilde{g}_{jl}(t)}{\partial x^i} - \frac{\partial\tilde{g}_{ij}(t)}{\partial x^l}\right).$$

Derivation with respect to t, implies

$$\big(\tilde{\varGamma}^k_{ij}\big)'(0) = -\frac{1}{2}s^{kl}(t)\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right) + \frac{1}{2}g^{kl}\left(\frac{\partial s_{il}}{\partial x^j} + \frac{\partial s_{jl}}{\partial x^i} - \frac{\partial s_{ij}}{\partial x^l}\right).$$

$(\tilde{\varGamma}^k_{ij})'(0)$ is a tensor of type $(1,2)$, and we denote it by A, so $A^k_{ij} = (\tilde{\varGamma}^k_{ij})'(0)$. Tensor A is related to the 3-covariant tensor ∇s whose components are $s_{ij,k} = (\nabla_{\partial_k}s)(\partial_i, \partial_j)$. Direct computations show that

$$A^k_{ij} = \frac{1}{2}g^{kl}(s_{lj,i} + s_{li,j} - s_{ij,l}). \tag{18}$$

Denote components of the curvature tensor of $\tilde{g}(t)$ by $(\tilde{\mathrm{R}}^l_{ijk})(t)$. Due to the formula of computation of these components with respect to $\tilde{\varGamma}^k_{ij}(t)$ and derivation with respect to t, we find that [1]

$$\big(\tilde{\mathrm{R}}^l_{ijk}\big)'(0) = A^l_{jk,i} - A^l_{ik,j}. \tag{19}$$

Denote Ricci curvature of $\tilde{g}(t)$ by $\widetilde{\mathrm{Ric}}(t)$. Derivation of $\widetilde{\mathrm{Ric}}(t)$ for $t=0$ is as follows.

$$\widetilde{\mathrm{Ric}}'_{jk}(0) = \big(\tilde{R}^l_{ljk}\big)'(0) = A^l_{jk,l} - A^l_{lk,j}. \tag{20}$$

Now we can find derivation of $\tilde{R}(t)$ at $t=0$.

$$\begin{aligned}\tilde{R}'(0) &= \big(\tilde{g}^{ij}(t)\widetilde{\mathrm{Ric}}_{ij}(t)\big)'(0) = \big(\tilde{g}^{ij}\big)'(0)\mathrm{Ric}_{ij} + g^{ij}\widetilde{\mathrm{Ric}}'_{ij}(0)\\ &= -s^{ij}\mathrm{Ric}_{ij} + g^{ij}\widetilde{\mathrm{Ric}}'_{ij}(0) = -\langle s,\mathrm{Ric}\rangle + g^{ij}\big(A^l_{ij,l} - A^l_{lj,i}\big).\end{aligned}$$

Define the vector field W by components $W^l = g^{ij}A^l_{ij} - g^{il}A^j_{ij}$, so $\mathrm{div}(W) = g^{ij}(A^l_{ij,l} - A^l_{lj,i})$ [1]. Therefore,

$$\tilde{R}'(0) = -\langle s,\mathrm{Ric}\rangle + \mathrm{div}(W) \tag{21}$$

It is shown that the derivation of Ω_{g+ts} at $t=0$ is as follows [1].

$$\Omega'_{g+ts}(0) = \frac{1}{2}\langle g,s\rangle\Omega_g. \tag{22}$$

By local computations, we find derivation of $\triangle^t(\tilde{\theta}(t))$ at $t=0$. Local computation of Laplacian of a smooth function f is as follows.

$$\triangle(f) = g^{ij}\mathrm{Hes}(f)_{ij} = g^{ij}\left(\frac{\partial^2 f}{\partial x^i\partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k}\right). \tag{23}$$

So,

$$\begin{aligned}\triangle^t\big(\tilde{\theta}(t)\big)'(0) &= \triangle^t(\theta+th)'(0) = \left(\tilde{g}^{ij}(t)\left(\frac{\partial^2(\theta+th)}{\partial x^i\partial x^j} - \tilde{\Gamma}^k_{ij}(t)\frac{\partial(\theta+th)}{\partial x^k}\right)\right)'(0)\\ &= -s^{ij}\left(\frac{\partial^2\theta}{\partial x^i\partial x^j} - \Gamma^k_{ij}\frac{\partial\theta}{\partial x^k}\right) + g^{ij}\left(\frac{\partial^2 h}{\partial x^i\partial x^j} - A^k_{ij}\frac{\partial\theta}{\partial x^k} - \Gamma^k_{ij}\frac{\partial h}{\partial x^k}\right)\\ &= -\big\langle s,\mathrm{Hes}(\theta)\big\rangle + \triangle(h) - g^{ij}A^k_{ij}\frac{\partial\theta}{\partial x^k}.\end{aligned} \tag{24}$$

Define the vector field Y by components $Y^k = g^{ij}A^k_{ij}$. So,

$$\begin{aligned}Y^k &= g^{ij}A^k_{ij} = \frac{1}{2}g^{ij}g^{kl}(s_{lj,i} + s_{li,j} - s_{ij,l}) = \frac{1}{2}g^{kl}\big(g^{ij}s_{lj,i} + g^{ij}s_{li,j} - g^{ij}s_{ij,l}\big)\\ &= \frac{1}{2}g^{kl}\big(\mathrm{div}(s)_l + \mathrm{div}(s)_l - \mathrm{tr}(s)_{,l}\big) = \mathrm{div}(s)^k - \frac{1}{2}\big(\vec{\nabla}\mathrm{tr}(s)\big)^k.\end{aligned}$$

Consequently,

$$g^{ij}A^k_{ij}\frac{\partial\theta}{\partial x^k} = \mathrm{div}(s)^k\frac{\partial\theta}{\partial x^k} - \frac{1}{2}\big(\vec{\nabla}\mathrm{tr}(s)\big)^k\frac{\partial\theta}{\partial x^k} = \mathrm{div}(s)(\vec{\nabla}\theta) - \frac{1}{2}\big\langle\vec{\nabla}\mathrm{tr}(s),\vec{\nabla}\theta\big\rangle.$$

In the following, whenever it is convenient, we consider s as a (1, 1) symmetric tensor. For arbitrary vector field Z we have

$$\mathrm{div}(s)(Z) = g^{ij}\big\langle(\nabla_{\partial_i}s)(Z),\partial_j\big\rangle = g^{ij}\big\langle\nabla_{\partial_i}s(Z) - s(\nabla_{\partial_i}Z),\partial_j\big\rangle$$

$$= g^{ij}\big\langle \nabla_{\partial_i} s(Z), \partial_j\big\rangle - g^{ij}\big\langle s(\nabla_{\partial_i} Z), \partial_j\big\rangle$$
$$= \operatorname{div}\big(s(Z)\big) - g^{ij}\big\langle s(\partial_j), \nabla_{\partial_i} Z\big\rangle = \operatorname{div}\big(s(Z)\big) - \langle s, \nabla Z\rangle.$$

For $Z = \vec{\nabla}\theta$, the $(1,1)$ tensor $\nabla\vec{\nabla}\theta$ as a 2-covariant tensor is $\nabla d\theta$ and is equal to $\operatorname{Hes}(\theta)$. So,

$$g^{ij} A_{ij}^k \frac{\partial\theta}{\partial x^k} = \operatorname{div}\big(s(\vec{\nabla}\theta)\big) - \big\langle s, \operatorname{Hes}(\theta)\big\rangle - \frac{1}{2}\big\langle \vec{\nabla}\operatorname{tr}(s), \vec{\nabla}\theta\big\rangle.$$

Now, we go back to computations in (24). By above computations, we have

$$\begin{aligned}
\triangle^t\big(\tilde{\theta}(t)\big)'(0) &= -\big\langle s, \operatorname{Hes}(\theta)\big\rangle + \triangle(h) - g^{ij} A_{ij}^k \frac{\partial\theta}{\partial x^k} \\
&= -\big\langle s, \operatorname{Hes}(\theta)\big\rangle + \triangle(h) \\
&\quad - \left(\operatorname{div}\big(s(\vec{\nabla}\theta)\big) - \big\langle s, \operatorname{Hes}(\theta)\big\rangle - \frac{1}{2}\big\langle \vec{\nabla}\operatorname{tr}(s), \vec{\nabla}\theta\big\rangle\right) \\
&= \triangle(h) - \operatorname{div}\big(s(\vec{\nabla}\theta)\big) + \frac{1}{2}\big\langle \vec{\nabla}\operatorname{tr}(s), \vec{\nabla}\theta\big\rangle. \qquad (25)
\end{aligned}$$

Now, we compute derivation of $|\vec{\nabla}^t(\tilde{\theta}(t))|^2$ at $t = 0$.

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} \big|\vec{\nabla}^t\big(\tilde{\theta}(t)\big)\big|^2 &= \left(\tilde{g}^{ij}(t)\frac{\partial\tilde{\theta}(t)}{\partial x^i}\frac{\partial\tilde{\theta}(t)}{\partial x^j}\right)'(0) = -s^{ij}\frac{\partial\theta}{\partial x^i}\frac{\partial\theta}{\partial x^j} + 2g^{ij}\frac{\partial h}{\partial x^i}\frac{\partial\theta}{\partial x^j} \\
&= -\langle s, d\theta\otimes d\theta\rangle + 2\langle\vec{\nabla}h, \vec{\nabla}\theta\rangle. \qquad (26)
\end{aligned}$$

Remind that the integral of divergence of every vector fields on M is zero, consequently integral of Laplacian of any smooth function on M is zero. Also, for any two smooth function f and h we have

$$\int_M \langle\vec{\nabla}h, \vec{\nabla}f\rangle\Omega_g = -\int_M h\triangle(f)\Omega_g.$$

Now, we are ready to find critical metric $\hat{g}$ for Hilbert action.

$$\begin{aligned}
\mathcal{L}\big(\tilde{g}(t), \tilde{\theta}(t)\big)'(0) &= \left(\int_M \hat{R}(t)\Omega_{g+ts}\right)'(0) \\
&= \int_M \big(\tilde{R}(t) - 2\big(\triangle^t\big(\tilde{\theta}(t)\big) + \big|\vec{\nabla}^t\tilde{\theta}(t)\big|^2\big)\Omega_{g+th}\big)'(0) \\
&= \int_M \big(R - 2\triangle(\theta) - 2|\vec{\nabla}\theta|^2\big)\Omega'_{g+ts}(0) \\
&\quad + \big(\tilde{R}'(0) - 2\triangle^t\big(\tilde{\theta}(t)\big)'(0) - 2\big(\big|\vec{\nabla}^t\big(\tilde{\theta}(t)\big)\big|^2\big)'(0)\big)\Omega_g \\
&= \int_M \big(R - 2\triangle(\theta) - 2|\vec{\nabla}\theta|^2\big) \times \frac{1}{2}\langle s, g\rangle\Omega_g + \Big(-\langle s, \operatorname{Ric}\rangle + \operatorname{div}(W) \\
&\quad - 2\big(\triangle(h) - \operatorname{div}\big(s(\vec{\nabla}\theta)\big) + \frac{1}{2}\big\langle\vec{\nabla}\operatorname{tr}(s), \vec{\nabla}\theta\big\rangle\big) \\
&\quad - 2\big(-\langle s, d\theta\otimes d\theta\rangle + 2\langle\vec{\nabla}h, \vec{\nabla}\theta\rangle\big)\Big)\Omega_g
\end{aligned}$$

$$= \int_M \left(\left\langle s, \left(\frac{1}{2} R - \triangle\theta - |\vec{\nabla}\theta|^2 \right) g \right\rangle - \langle s, \mathrm{Ric} \rangle \right.$$
$$\left. + \operatorname{tr}(s) \triangle(\theta) + 2\langle s, d\theta \otimes d\theta \rangle + 4h \triangle(\theta) \right) \Omega_g$$
$$= \int_M \left(\left\langle s, -\mathrm{Ric} + \left(\frac{1}{2} R - |\vec{\nabla}\theta|^2 \right) g + 2d\theta \otimes d\theta \right\rangle + 4h \triangle(\theta) \right) \Omega_g.$$

The above expression is zero for all pair (s, h) iff

$$\mathrm{Ric} - \frac{1}{2} R g = 2d\theta \otimes d\theta - |\vec{\nabla}\theta|^2 g, \tag{27}$$
$$\triangle(\theta) = 0. \tag{28}$$

These are field equations for $\hat{g} = (g, \theta)$ and determine critical metrics for Hilbert action. Equation (28) express that $\operatorname{div}(\vec{\nabla}\theta) = 0$, so $\vec{\nabla}\theta$ can be interpreted as current of mass that satisfies conservation law. Absolute value of $2|\vec{\nabla}\theta|^2$ shows density of matter. Right hand side of Eq. (27) can be interpreted as energy-momentum tensor of matter and the following theorem shows that it satisfies conservation law.

Theorem 5.1 *If θ be a smooth function on a semi-Riemannian manifold (M, g) such that $\triangle(\theta) = 0$, then the divergence of symmetric tensor $2d\theta \otimes d\theta - |\vec{\nabla}\theta|^2 g$ is zero.*

Proof Let $\{E_1, \ldots, E_n\}$ be an orthonormal local base on M and $\hat{i} = \langle E_i, E_i \rangle = \pm 1$. So,

$$\begin{aligned} \operatorname{div}(d\theta \otimes d\theta)(X) &= \sum_{i=1}^{n} \hat{i} (\nabla_{E_i} d\theta \otimes d\theta)(E_i, X) \\ &= \sum_{i=1}^{n} \hat{i} \big((\nabla_{E_i} d\theta) \otimes d\theta + d\theta \otimes (\nabla_{E_i} d\theta) \big)(E_i, X) \\ &= \sum_{i=1}^{n} \hat{i} \big((\nabla_{E_i} d\theta)(E_i) d\theta(X) + d\theta(E_i)(\nabla_{E_i} d\theta)(X) \big) \\ &= \triangle(\theta) d\theta(X) + \sum_{i=1}^{n} \hat{i} d\theta(E_i) \mathrm{Hes}(\theta)(E_i, X) = \mathrm{Hes}(\theta)(\vec{\nabla}\theta, X). \end{aligned}$$

Remind that for any smooth function h on M: $\operatorname{div}(hg) = dh$. So,

$$\begin{aligned} \operatorname{div}\big(|\vec{\nabla}\theta|^2 g\big)(X) &= d\big(|\vec{\nabla}\theta|^2\big)(X) = X \langle \vec{\nabla}\theta, \vec{\nabla}\theta \rangle = 2\big\langle \nabla_X(\vec{\nabla}\theta), \vec{\nabla}\theta \big\rangle \\ &= 2(\nabla_X d\theta)(\vec{\nabla}\theta) = 2\mathrm{Hes}(\vec{\nabla}\theta, X). \end{aligned}$$

The above computations show that $\operatorname{div}(2d\theta \otimes d\theta - |\vec{\nabla}\theta|^2 g) = 0$. □

In case $3 \leq \dim(M)$, Eq. (27) can be written in a simple form.

Theorem 5.2 *Suppose $3 \leq n = \dim(M)$. The metric $\hat{g} = (g, \theta)$ on $\hat{T}M$, is a critical metric for Hilbert action iff*

$$\mathrm{Ric} = 2d\theta \otimes d\theta, \tag{29}$$

$$\triangle(\theta) = 0. \tag{30}$$

Proof Equation (30) is the same as (28). First suppose Eq. (27) holds. Compute traces of tow sides of Eq. (27).

$$R - \frac{n}{2}R = 2|\vec{\nabla}\theta|^2 - n|\vec{\nabla}\theta|^2 \Rightarrow \frac{2-n}{2}R = (2-n)|\vec{\nabla}\theta|^2 \Rightarrow R = 2|\vec{\nabla}\theta|^2.$$

By substitution into (27) we obtain (29). Now, assume (29) holds. By computation traces of tow sides of Eq. (29), we find $R = 2|\vec{\nabla}\theta|^2$, so by addition suitable expression to each side of Eq. (29) we obtain (27). □

Example Suppose $(N, \bar{g})$ be a Riemannian manifold of dimension $n \geq 2$. Set $M = N \times (0, \infty)$. Tangent vectors to M at (p, t) is of the form (v, λ) in which $v \in T_pN$ and $\lambda \in \mathbb{R}$. Denote the vector field $(0, 1)$ on M by ∂_t. As a derivation, for a smooth function $f(p, t)$ on M we have $\partial_t(f) = \frac{\partial f}{\partial t}$. Denote vector fields on N by $X, Y, Z, \dots$ and consider them as special vector fields on M. Denote second projection map $(p, t) \mapsto t$ on M by t. We can interpret t as time. For the 1-form dt we have $dt(v, \lambda) = \lambda$, so $dt(\partial_t) = 1$. For some smooth function $a : (0, \infty) \longrightarrow \mathbb{R}$ define a metric g on M as follows.

$$g = e^{2a(t)}\bar{g} - dt \otimes dt.$$

So, inner product of special vector fields on M and ∂_t are as follows.

$$\langle X, Y\rangle = e^{2a}\bar{g}(X, Y), \qquad \langle X, \partial_t\rangle = 0, \qquad \langle \partial_t, \partial_t\rangle = -1.$$

Consider $\theta : N \times (0, \infty) \longrightarrow \mathbb{R}$ such that its level sets be $N \times \{t\}$. Therefore, $\theta(p, t)$ must depend only on t, and we denote it by $\theta(t)$. Consequently, $d\theta = \theta'(t)dt$ and $\vec{\nabla}\theta = -\theta'(t)\partial_t$ and $|\vec{\nabla}\theta|^2 = \langle \vec{\nabla}\theta, \vec{\nabla}\theta\rangle = -|\theta'(t)|^2$.

Denote the Levi-Civita connection of N and M by $\bar{\nabla}$, ∇ respectively. Straightforward computations show that:

$$\nabla_X Y = \bar{\nabla}_X Y + a'\langle X, Y\rangle\partial_t, \tag{31}$$

$$\nabla_{\partial_t} Y = a'Y, \tag{32}$$

$$\nabla_{\partial_t}\partial_t = 0. \tag{33}$$

Suppose $\{E_1, \dots, E_n\}$ is an orthonormal local base on N, then, $\{e^{-a}E_1, \dots, e^{-a}E_n, \partial_t\}$ is an orthonormal local base on M. Laplacian of θ can be computed as follows.

$$\begin{aligned}
\Delta\theta &= \sum_{i=1}^{n} e^{-2a}\langle \nabla_{E_i}\vec{\nabla}\theta, E_i\rangle - \langle \nabla_{\partial_t}\vec{\nabla}\theta, \partial_t\rangle \\
&= \sum_{i=1}^{n} e^{-2a}\big\langle \nabla_{E_i}\big(-\theta'(t)\partial_t\big), E_i\big\rangle - \big\langle \nabla_{\partial_t}\big(-\theta'(t)\partial_t\big), \partial_t\big\rangle \\
&= \sum_{i=1}^{n} -\theta'(t)e^{-2a}\big\langle a'E_i, E_i\big\rangle - \theta''(t) \\
&= \left(\sum_{i=1}^{n} -a'\theta'(t)\right) - \theta''(t) = -na'\theta'(t) - \theta''(t).
\end{aligned}$$

Equation $\triangle\theta = 0$, implies that for some constant c we have $|\theta'(t)| = ce^{-na(t)}$.

Denote curvature tensors and Ricci curvature tensors of N and M respectively by $\bar{\mathrm{R}}$, R, $\overline{\mathrm{Ric}}$, Ric. Straight forward computations show that:

$$\mathrm{R}(X,Y)(Z) = \bar{\mathrm{R}}(X,Y)(Z) + a'^2\big(\langle Y,Z\rangle X - \langle X,Z\rangle Y\big), \tag{34}$$

$$\mathrm{R}(X,Y)(\partial_t) = 0, \tag{35}$$

$$\mathrm{R}(\partial_t,Y)(Z) = \big(a'' + a'^2\big)\langle Y,Z\rangle\partial_t, \tag{36}$$

$$\mathrm{R}(\partial_t,Y)(\partial_t) = \big(a'' + a'^2\big)Y, \tag{37}$$

$$\mathrm{Ric}(X,Y) = \overline{\mathrm{Ric}}(X,Y) + \big(a'' + na'^2\big)\langle X,Y\rangle, \tag{38}$$

$$\mathrm{Ric}(X,\partial_t) = 0, \tag{39}$$

$$\mathrm{Ric}(\partial_t,\partial_t) = -n\big(a'' + a'^2\big). \tag{40}$$

In this example, Eq. (29) becomes $\mathrm{Ric} = 2d\theta\otimes\theta = 2|\theta'(t)|^2 dt\otimes dt$, and it holds iff

$$\overline{\mathrm{Ric}}(X,Y) = -\big(a'' + na'^2\big)\langle X,Y\rangle, \tag{41}$$

$$-n\big(a'' + a'^2\big) = 2\big|\theta'(t)\big|^2 = 2c^2e^{-2na(t)}. \tag{42}$$

Left side of (41) dose not depend on t, so $a'' + na'^2$ must be constant and N is an Einstein manifold. In the case $a'' + na'^2 = 0$ we find solution $a(t) = \frac{1}{n}\ln(t)$ and for this solution, Eq. (42) also holds for $c = \sqrt{\frac{n-1}{2n}}$.

So, for an n dimensional Ricci flat manifold $(N,\bar{g})$ the meter $g = t^{\frac{2}{n}}\bar{g} - dt\otimes dt$ on M and the function $\theta(t) = \sqrt{\frac{n-1}{2n}}\ln(t)$ satisfies field equations (29) and (30). In this model, as t approaches zero, universe become smaller and density of matter increases to infinity. Time $t = 0$ is not in M and this time is the instant of Big-Bang. This example is an Einstein-de Sitter model in general relativity [6].

6 Field Equation on the Graded Tangent Bundle

We have found two Eqs. (29) and (30) as field equations on M. But, we can join these equations and make one equation on $\hat{T}M$.

Exterior derivation in algebroid structures is defined. Specially, for a smooth function f, its exterior derivation, denoted by $\hat{d}f$, is a 1-form on $\hat{T}M$ defined by $\hat{d}f(\hat{X}) = \hat{X}(f)$. Restriction of $\hat{d}f$ to TM is df, and its restriction to odd vectors is zero. Hessian of f which is denoted by $\widehat{\mathrm{Hes}}(f)$, is a 2-covariant symmetric tensor on $\hat{T}M$ and defined as follows.

$$\widehat{\mathrm{Hes}}(f)(\hat{X},\hat{Y}) = (\hat{\nabla}_{\hat{X}}\hat{d}f)(\hat{Y}) \quad \hat{X},\hat{Y} \in \hat{\mathfrak{X}}(M).$$

Straight computations show that $\widehat{\mathrm{Hes}}(f)$ satisfies the following relations.

$$\widehat{\mathrm{Hes}}(f)(X,Y) = \mathrm{Hes}(f)(X,Y),$$

$$\widehat{\mathrm{Hes}}(f)(\xi,X) = 0,$$

$$\widehat{\mathrm{Hes}}(f)(\xi,\xi) = e^{2\theta}\langle\vec{\nabla}f,\vec{\nabla}\theta\rangle.$$

Consequently, $\mathrm{tr}(\widehat{\mathrm{Hes}}(f)) = \triangle(f) + \langle \vec{\nabla} f, \vec{\nabla}\theta \rangle$. In particular,

$$\mathrm{tr}\big(\widehat{\mathrm{Hes}}(\theta)\big) = \triangle(\theta) + |\vec{\nabla}\theta|^2 = \mathrm{tr}(\tilde{T}). \tag{43}$$

Theorem 6.1 *Let M be a manifold, then a pair (g, θ) satisfies in field Eqs. (29) and (30) iff for the metric $\hat{g} = (g, \theta)$:*

$$\widehat{\mathrm{Ric}} = \hat{d}\theta \otimes \hat{d}\theta - \widehat{\mathrm{Hes}}(\theta). \tag{44}$$

$\widehat{\mathrm{Ric}}$ *is the Ricci curvature of the metric $\hat{g}$.*

Proof First, suppose (44) holds. So,

$$\begin{aligned}
&\widehat{\mathrm{Ric}}(\xi,\xi) = \big(\hat{d}\theta \otimes \hat{d}\theta - \widehat{\mathrm{Hes}}(\theta)\big)(\xi,\xi) = -\widehat{\mathrm{Hes}}(\theta)(\xi,\xi) \\
&\Rightarrow -e^{2\theta}\mathrm{tr}(\tilde{T}) = -e^{2\theta}|\vec{\nabla}\theta|^2 \ \Rightarrow\ \mathrm{tr}(\tilde{T}) = |\vec{\nabla}\theta|^2 \\
&\Rightarrow \triangle(\theta) + |\vec{\nabla}\theta|^2 = |\vec{\nabla}\theta|^2 \quad \Rightarrow \quad \triangle(\theta) = 0.
\end{aligned}$$

Therefore, (30) holds. Moreover,

$$\begin{aligned}
&\widehat{\mathrm{Ric}}(X,Y) = \big(\hat{d}\theta \otimes \hat{d}\theta - \widehat{\mathrm{Hes}}(\theta)\big)(X,Y) \\
&\Rightarrow (\mathrm{Ric} - \tilde{T})(X,Y) = \big(d\theta \otimes d\theta - \mathrm{Hes}(\theta)\big)(X,Y) \\
&\Rightarrow \mathrm{Ric} - \mathrm{Hes}(\theta) - d\theta \otimes d\theta = d\theta \otimes d\theta - \mathrm{Hes}(\theta) \\
&\Rightarrow \mathrm{Ric} = 2d\theta \otimes d\theta.
\end{aligned}$$

Therefore, (29) holds. Conversely, suppose (29) and (30) hold. All above computations are reversible, and show that each side of (44) are equal on even and odd vectors. Also, we can see directly, that each side of (44) on an even and an odd vectors are zero. So, (44) holds. □

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