Int J Theor Phys (2013) 52:1768–1779
DOI 10.1007/s10773-012-1226-3

A Classification of Correlations of Tripartite Mixed States

Zhihua Guo · Huaixin Cao

Received: 28 May 2012 / Accepted: 1 June 2012 / Published online: 10 June 2012

Abstract To obtain a classification of correlations of tripartite mixed states, various correlated states are introduced by using measurement-induced disturbance, including CCC, QC, GQC, CCX, CXC, XCC, CXX, XCX and XXC-states. Standard forms of them are established and equivalent characterizations of them are obtained in terms of normality and commutativity of the associated component operators.

Keywords Correlation · Classification · Local projective measurement · Commutativity

1 Introduction

In quantum information, quantum and classical correlations of multipartite mixed states are important and complex quantum properties. Remarkably, quantum correlations may occur not only in entangled states, but also in separable ones and be still a potential resource in some quantum information processing tasks, such as deterministic quantum computation with one qubit [1], and quantum search algorithms without entanglement [2].

In this context, how to characterize and quantify quantum correlations has more attracted attentions. For the bipartite case, an important and usual measure is given by quantum discord (QD) [3], which arises as the difference between mutual information and classical correlations. In [4–6], some other methods for measure of quantum correlations were introduced. The problem of the separation of total correlations in a given quantum state into entanglement, dissonance, and classical correlations was discussed in [7] by using the concept of relative entropy as a distance measure of correlations. A global measure for quantum correlations in multipartite systems was introduced in [8] by suitably recasting the quantum discord in terms of relative entropy and local von Neumann measurements. A witness for nonclassical multipartite states was investigated in [9] based on their disturbance under local

Z. Guo (✉) · H. Cao
College of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710062, China
e-mail: zhguosx@gmail.com

H. Cao
e-mail: caohx@snnu.edu.cn

measurements, which provides a sufficient condition for nonclassicality without demanding an extremization procedure. Piani *et al.* in [10] proposed a different method to distinguish the classical correlated states from the set of all states and proved that a classical correlated state is equivalent to a local-broadcasted state.

The authors in [11] gave a new characterization of a bipartite classical correlated (CC) state, corresponding results for the left and right classical correlations (LCC and RCC) were also obtained, and a sufficient and necessary condition for a convex combination of two CC states to be CC was proved. Based on the characterization of CC states, a quantity $Q(\rho)$ was associated to a state ρ and it was shown that a state ρ is CC if and only if $Q(\rho) = 0$.

The aim of this paper is to classify the correlations of tripartite mixed states. Firstly, various correlated states will be introduced, including CCC, QC, GQC, CCX, CXC, XCC, CXX, XCX and XXC-states. Secondly, standard forms of them will be given by using measurement-induced disturbance and equivalent characterizations will be obtained in terms of normality and commutativity of the associated component operators. Lastly, some examples of each kind of correlated states will be list.

2 Correlations of Tripartite Mixed States

In what follows, we let $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_C$ be the state spaces (finite dimensional Hilbert spaces) of quantum mechanical systems A, B and C. We agree that, according to quantum mechanics, the inner product of a Hilbert space is right-linear and left conjugate-linear. By the postulates of quantum mechanics, the state space of the composite system of A, B and C is given by the tensor product $\mathcal{H}_{ABC} := \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ of spaces of dimensions d_A, d_B, d_C, respectively. We use $D(\mathcal{H}_X)$ to denote the set of all states (i.e., density operators) on $\mathcal{H}_X$, and I_X to stand for the identity on $\mathcal{H}_X$ for $X = A, B, C$. The adjoint operator of an operator T is denoted by $T^\dagger$. Also, we use $ONB(\mathcal{H}_X)$ to denote the set of all orthonormal bases for $\mathcal{H}_X$. Clearly, if $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_C$ have orthonormal bases

$$e := \{|e_i\rangle : 1 \le i \le d_A\}, \qquad f := \{|f_j\rangle : 1 \le j \le d_B\} \quad \text{and} \quad g := \{|g_k\rangle : 1 \le k \le d_C\}, \tag{2.1}$$

respectively, then $\mathcal{H}_{ABC}$ has an orthonormal basis

$$e \otimes f \otimes g := \{|e_i\rangle \otimes |f_j\rangle \otimes |g_k\rangle : 1 \le i \le d_A, 1 \le j \le d_B, 1 \le k \le d_C\}. \tag{2.2}$$

Recall that a quantum measurement of a quantum system with state space $\mathcal{H}$ is an operator family $\mathcal{M} := \{M_1, M_2, \dots, M_n\}$ on $\mathcal{H}$ such that $\sum_{i=1}^n M_i^\dagger M_i = I_H$. A quantum measurement Π on the tripartite system $\mathcal{H}_{ABC}$ is said to be a *local projective measurement* (LPM), if it is of the form

$$\Pi = \{\Pi_i^A \otimes \Pi_j^B \otimes \Pi_k^C : 1 \le i \le d_A, 1 \le j \le d_B, 1 \le k \le d_C\}, \tag{2.3}$$

where Π_n^X is a one-dimensional orthogonal projection on $\mathcal{H}_X$ for every n with sum I_X, for $X = A, B, C$.

Note that measurement operators of an LPM (2.3) have the property that $\sum_{n=1}^{d_X} \Pi_n^X = I_X$ and so $\Pi_n^X \Pi_m^X = 0 (m \ne n)$ for $X = A, B, C$.

Similar to [4], we introduce the following.

Definition 2.1 Let $\rho \in D(\mathcal{H}_{ABC})$. If there exists an LPM (2.3) such that $\Pi(\rho) = \rho$, then ρ is said to be *completely classical correlated* (shortly, CCC). Otherwise, ρ is said to be *quantum correlated* (QC).

After an LPM (2.3), a state $\rho \in D(\mathcal{H}_{ABC})$ is changed to

$$\Pi(\rho) := \sum_{i=1}^{d_A}\sum_{j=1}^{d_B}\sum_{k=1}^{d_C}\left(\Pi_i^A \otimes \Pi_j^B \otimes \Pi_k^C\right)\rho\left(\Pi_i^A \otimes \Pi_j^B \otimes \Pi_k^C\right). \tag{2.4}$$

Clearly, $\Pi(\Pi(\rho)) = \Pi(\rho)$ and so $\Pi(\rho)$ is always a CCC-state.

Different with the bipartite case, a tripartite state has some "partial correlations". For instance, there are another three different types of quantum correlations. To discuss further quantum correlated states, we introduce the following.

Definition 2.2 Let $\rho \in D(\mathcal{H}_{ABC})$. We say that ρ is *bi-classical correlated* (BCC) if there are one-rank projective measurements $\Pi_X = \{\Pi_n^X : n = 1, 2, \dots, d_X\}$ on $\mathcal{H}_X$ for $X = A, B, C$ such that one of the following holds:

(i) $\sum_{i=1}^{d_A}\sum_{j=1}^{d_B}(\Pi_i^A \otimes \Pi_j^B \otimes I_C)\rho(\Pi_i^A \otimes \Pi_j^B \otimes I_C) = \rho$ (ρ is called a CCX-state);
(ii) $\sum_{j=1}^{d_B}\sum_{k=1}^{d_C}(I_A \otimes \Pi_j^B \otimes \Pi_k^C)\rho(I_A \otimes \Pi_j^B \otimes \Pi_k^C) = \rho$ (ρ is called an XCC-state);
(iii) $\sum_{i=1}^{d_A}\sum_{k=1}^{d_C}(\Pi_i^A \otimes I_B \otimes \Pi_k^C)\rho(\Pi_i^A \otimes I_B \otimes \Pi_k^C) = \rho$ (ρ is called a CXC-state).

Definition 2.3 Let $\rho \in D(\mathcal{H}_{ABC})$. We say that ρ is *single-classical correlated* (SCC) if there are one-rank projective measurements $\Pi_X = \{\Pi_n^X : n = 1, 2, \dots, d_X\}$ on $\mathcal{H}_X$ for $X = A, B, C$ such that one of the following holds:

(i) $\sum_{i=1}^{d_A}(\Pi_i^A \otimes I_B \otimes I_C)\rho(\Pi_i^A \otimes I_B \otimes I_C) = \rho$ (ρ is called a CXX-state);
(ii) $\sum_{j=1}^{d_B}(I_A \otimes \Pi_j^B \otimes I_C)\rho(I_A \otimes \Pi_j^B \otimes I_C) = \rho$ (ρ is called an XCX-state);
(iii) $\sum_{k=1}^{d_C}(I_A \otimes I_B \otimes \Pi_k^C)\rho(I_A \otimes I_B \otimes \Pi_k^C) = \rho$ (ρ is called an XXC-state).

Put $\mathcal{S}(X) = \{\rho \in D(\mathcal{H}_{ABC}) : \rho \text{ is } X\}$. Obviously, $\mathcal{S}(CCC) \subset \mathcal{S}(BCC) \subset \mathcal{S}(SCC)$.

Definition 2.4 A state $\rho \in D(\mathcal{H}_{ABC})$ is said to be *genuine quantum correlated* (GQC) if it is not SCC.

By the method in [11], for any tripartite state $\rho \in D(\mathcal{H}_{ABC})$, and for any orthonormal bases (2.1), we have

$$\rho = \sum_{ijk\ell st} p_{ijk\ell st}|e_i\rangle\langle e_j| \otimes |f_k\rangle\langle f_\ell| \otimes |g_s\rangle\langle g_t|. \tag{2.5}$$

Let

$$A_{k\ell st}(\rho) = \sum_{ij} p_{ijk\ell st}|e_i\rangle\langle_j|, \tag{2.6}$$

$$B_{ijst}(\rho) = \sum_{k\ell} p_{ijk\ell st}|f_k\rangle\langle f_\ell|, \tag{2.7}$$

$$C_{ijk\ell}(\rho) = \sum_{st} p_{ijk\ell st}|g_s\rangle\langle g_t|. \tag{2.8}$$

With these notations, we can obtain the following.

Theorem 2.1 *Let $\rho \in D(\mathcal{H}_{ABC})$. Then*

(1) *ρ is CCC if and only if it can be represented as*

$$\rho = \sum_{mns} \delta_{mns} |e_m\rangle\langle e_m| \otimes |f_n\rangle\langle f_n| \otimes |g_s\rangle\langle g_s|, \tag{2.9}$$

for some bases $\{|e_m\rangle\} \in ONB(\mathcal{H}_A)$, $\{|f_n\rangle\} \in ONB(\mathcal{H}_B)$ and $\{|g_s\rangle\} \in ONB(\mathcal{H}_C)$, and a probability distribution $\{\delta_{mns}\}$.

(2) *ρ is a CCX-state if and only if there exist bases $\{|e_m\rangle\} \in ONB(\mathcal{H}_A)$, $\{|f_n\rangle\} \in ONB(\mathcal{H}_B)$ and operators $\gamma_{mn} \in B(\mathcal{H}_C)$ such that*

$$\rho = \sum_{mn} |e_m\rangle\langle e_m| \otimes |f_n\rangle\langle f_n| \otimes \gamma_{mn}. \tag{2.10}$$

(3) *ρ is a CXC-state if and only if there exist bases $\{|e_i\rangle\} \in ONB(\mathcal{H}_A)$, $\{|g_k\rangle\} \in ONB(\mathcal{H}_C)$ and operators $\beta_{ik} \in B(\mathcal{H}_C)$ such that*

$$\rho = \sum_{ik} |e_i\rangle\langle e_i| \otimes \beta_{ik} \otimes |g_k\rangle\langle g_k|. \tag{2.11}$$

(4) *ρ is an XCC-state if and only if there exist bases $\{|f_j\rangle\} \in ONB(\mathcal{H}_B)$, $\{|g_k\rangle\} \in ONB(\mathcal{H}_C)$ and operators $\alpha_{jk} \in B(\mathcal{H}_A)$ such that*

$$\rho = \sum_{jk} \alpha_{jk} \otimes |f_j\rangle\langle f_j| \otimes |g_k\rangle\langle g_k|. \tag{2.12}$$

(5) *ρ is a CXX-state if and only if there exist a basis $\{|e_m\rangle\} \in ONB(\mathcal{H}_A)$ and operators $\delta_m \in B(\mathcal{H}_{BC})$ such that*

$$\rho = \sum_{m} |e_m\rangle\langle e_m| \otimes \delta_m. \tag{2.13}$$

(6) *ρ is an XCX-state if and only if there exist a basis $\{|f_k\rangle\} \in ONB(\mathcal{H}_B)$ and operators $A_{k\ell} \in B(\mathcal{H}_A)$, $B_{k\ell} \in B(\mathcal{H}_C)$ such that*

$$\rho = \sum_{k} \sum_{\ell} A_{k\ell} \otimes |f_k\rangle\langle f_k| \otimes B_{k\ell}. \tag{2.14}$$

(7) *ρ is an XXC-state if and only if there exist a basis $\{|g_k\rangle\} \in ONB(\mathcal{H}_C)$ and operators $\varepsilon_k \in B(\mathcal{H}_{AB})$ such that*

$$\rho = \sum_{k} \varepsilon_k \otimes |g_k\rangle\langle g_k|. \tag{2.15}$$

Proof (1) Let ρ be CCC. Then there exists an LPM (2.3) such that $\Pi(\rho) = \rho$. Thus, there exist orthonormal bases (2.1) for $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_C$, respectively, such that $\Pi_m^A = |e_m\rangle\langle e_m|$ for all m, $\Pi_n^B = |f_n\rangle\langle f_n|$ for all n, $\Pi_s^C = |g_s\rangle\langle g_s|$ for all n. By using (2.5), we can get

$$\begin{aligned}\rho &= \sum_{mns} \left(\Pi_m^A \otimes \Pi_n^B \otimes \Pi_s^C\right) \rho \left(\Pi_m^A \otimes \Pi_n^B \otimes \Pi_s^C\right) \\ &= \sum_{mnsijk\ell uv} p_{ijk\ell uv} |e_m\rangle\langle e_m|e_i\rangle\langle e_j|e_m\rangle\langle e_m| \otimes |f_n\rangle\langle f_n|f_k\rangle\langle f_\ell|f_n\rangle\langle f_n|\end{aligned}$$

$$\otimes |g_s\rangle\langle g_s|g_u\rangle\langle g_v|g_s\rangle\langle g_s|$$
$$= \sum_{mns} \delta_{mns} |e_m\rangle\langle e_m| \otimes |f_n\rangle\langle f_n| \otimes |g_s\rangle\langle g_s|,$$

where $\delta_{mns} = p_{mmnnss}$, which gives a probability distribution $\{\delta_{mns}\}$.

Conversely, suppose that ρ is of the form (2.9). Then by taking $\Pi_m^A = |e_m\rangle\langle e_m|$, $\Pi_n^B = |f_n\rangle\langle f_n|$ and $\Pi_s^C = |g_s\rangle\langle g_s|$, we get an LPM (2.3) such that $\Pi(\rho) = \rho$.

(2) Let ρ be CCX. Then there are one-rank projective measurements $\Pi_X = \{\Pi_n^X : n = 1, 2, \dots, d_X\}$ on $\mathcal{H}_X$ for $X = A, B$ such that

$$\sum_{i=1}^{d_A} \sum_{j=1}^{d_B} \left(\Pi_i^A \otimes \Pi_j^B \otimes I_C\right)\rho\left(\Pi_i^A \otimes \Pi_j^B \otimes I_C\right) = \rho.$$

Thus, there exist an orthonormal basis $\{|e_i\rangle\}$ for $\mathcal{H}_A$ such that $\Pi_m^A = |e_m\rangle\langle e_m|$ for all $m = 1, 2, \dots, d_A$ and an orthonormal basis $\{|f_j\rangle\}$ for $\mathcal{H}_B$ such that $\Pi_n^B = |f_n\rangle\langle f_n|$ for all $n = 1, 2, \dots, d_B$. Taking any basis $\{|g_s\rangle\}$ for $\mathcal{H}_C$ and using by (2.5) imply that

$$\rho = \sum_{mn} \left(\Pi_m^A \otimes \Pi_n^B \otimes I_C\right)\rho\left(\Pi_m^A \otimes \Pi_n^B \otimes I_C\right)$$
$$= \sum_{mnijk\ell st} p_{ijk\ell st}\left(|e_m\rangle\langle e_m|e_i\rangle\langle e_j|e_m\rangle\langle e_m|\right) \otimes \left(|f_n\rangle\langle f_n|f_k\rangle\langle f_\ell|f_n\rangle\langle f_n|\right) \otimes |g_s\rangle\langle g_t|$$
$$= \sum_{mn} |e_m\rangle\langle e_m| \otimes |f_n\rangle\langle f_n| \otimes \gamma_{mn},$$

where $\gamma_{mn} = \sum_{st} p_{mmnnst}|g_s\rangle\langle g_t|$. This shows that ρ is of the form (2.10).

Conversely, let us assume (2.10) holds. Then by taking $\Pi_m^A = |e_m\rangle\langle e_m|$ and $\Pi_n^B = |f_n\rangle\langle f_n|$, we obtain one-rank projective measurements $\Pi_X = \{\Pi_n^X : n = 1, 2, \dots, d_X\}$ on $\mathcal{H}_X$ for $X = A, B$. From (2.10), we see that

$$\sum_{i=1}^{d_A} \sum_{j=1}^{d_B} \left(\Pi_i^A \otimes \Pi_j^B \otimes I_C\right)\rho\left(\Pi_i^A \otimes \Pi_j^B \otimes I_C\right) = \rho.$$

This shows that ρ is a CCX-state.

(3 and 4) Similar to (2).

(5) Let ρ be a CXX-state. Then there exists a rank-one projective measurement $\{\Pi_i^A : i = 1, 2, \dots, d_A\}$ on $\mathcal{H}_A$ such that

$$\sum_{i=1}^{d_A} \left(\Pi_i^A \otimes I_B \otimes I_C\right)\rho\left(\Pi_i^A \otimes I_B \otimes I_C\right) = \rho.$$

Thus, there exists an orthonormal basis $\{|e_i\rangle\}$ for $\mathcal{H}_A$ such that $\Pi_m^A = |e_m\rangle\langle e_m|$ for all $m = 1, 2, \dots, d_A$. Taking any bases $\{|f_j\rangle\}$ for $\mathcal{H}_B$ and $\{|g_s\rangle\}$ for $\mathcal{H}_C$ and using by (2.5) imply that

$$\rho = \sum_{m=1}^{d_A} \left(\Pi_m^A \otimes I_B \otimes I_C\right)\rho\left(\Pi_m^A \otimes I_B \otimes I_C\right)$$
$$= \sum_{mijk\ell st} p_{ijk\ell st}\left(|e_m\rangle\langle e_m|e_i\rangle\langle e_j|e_m\rangle\langle e_m|\right) \otimes |f_k\rangle\langle f_\ell| \otimes |g_s\rangle\langle g_t|$$

$$= \sum_{mk\ell st} p_{mmk\ell st} |e_m\rangle\langle e_m| \otimes |f_k\rangle\langle f_\ell| \otimes |g_s\rangle\langle g_t|$$

$$= \sum_{m=1}^{d_A} |e_m\rangle\langle e_m| \otimes \delta_m,$$

where $\delta_m = \sum_{k\ell st} p_{mmk\ell st} |f_k\rangle\langle f_\ell| \otimes |g_s\rangle\langle g_t| \in B(\mathcal{H}_{BC})$.

Conversely, we assume $\rho = \sum_{i=1}^{d_A} |e_i\rangle\langle e_i| \otimes \delta_i$, where $\{\delta_i\}_{i=1}^{d_A} \subset B(\mathcal{H}_{BC})$ and $\{|e_i\rangle\}$ is some orthonormal basis for $\mathcal{H}_A$. Put $\Pi_m^A = |e_m\rangle\langle e_m|$ for all m. Then we get a rank-one projective measurement $\{\Pi_m^A : m = 1, 2, \dots, d_A\}$ on $\mathcal{H}_A$ such that

$$\sum_{m=1}^{d_A} (\Pi_m^A \otimes I_B \otimes I_C) \rho (\Pi_m^A \otimes I_B \otimes I_C) = \sum_{m=1}^{d_A} |e_m\rangle\langle e_m| \otimes \delta_m = \rho.$$

Hence, ρ is a CXX-state.

(6) Let ρ be an XCX-state. Then there exists a rank-one projective measurement $\{\Pi_m^B : m = 1, 2, \dots, d_B\}$ on $\mathcal{H}_B$ such that

$$\sum_{m=1}^{d_B} (I_A \otimes \Pi_m^B \otimes I_C) \rho (I_A \otimes \Pi_m^B \otimes I_C) = \rho.$$

Thus, there exists an orthonormal basis $\{|f_j\rangle\}$ for $\mathcal{H}_B$ such that $\Pi_m^B = |f_m\rangle\langle f_m|$ for all $m = 1, 2, \dots, d_B$. Taking any bases $\{|e_i\rangle\}$ for $\mathcal{H}_A$ and $\{|g_s\rangle\}$ for $\mathcal{H}_C$ and using by (2.5) imply that

$$\begin{aligned}
\rho &= \sum_{m=1}^{d_B} (I_A \otimes \Pi_m^B \otimes I_C) \rho (I_A \otimes \Pi_m^B \otimes I_C) \\
&= \sum_{mijk\ell st} p_{ijk\ell st} |e_i\rangle\langle e_j| \otimes \big(|f_m\rangle\langle f_m| \cdot |f_k\rangle\langle f_\ell| \cdot |f_m\rangle\langle f_m|\big) \otimes |g_s\rangle\langle g_t| \\
&= \sum_{ijkst} p_{ijkkst} |e_i\rangle\langle e_j| \otimes |f_k\rangle\langle f_k| \otimes |g_s\rangle\langle g_t|.
\end{aligned}$$

Let

$$\alpha : \{1, 2, \dots, d_A^2\} \to \{(i, j) : i, j = 1, 2, \dots, d_A\} \quad \text{and}$$
$$\beta : \{1, 2, \dots, d_C^2\} \to \{(s, t) : s, t = 1, 2, \dots, d_C\}$$

be bijections. Define $P_m = |e_i\rangle\langle e_j|$, $Q_n = |g_s\rangle\langle g_t|$, $a_{mn}^{(k)} = p_{ijkkst}$ whenever $\alpha^{-1}((i, j)) = m$, $\beta^{-1}((s, t)) = n$. Then we get a $d_A^2 \times d_C^2$ matrix $A_k := [a_{mn}^{(k)}]$. For each k, by the singular value decomposition of A_k, we know that there exist a $d_A^2 \times d_A^2$ unitary matrix $U_k := [u_{mn}^{(k)}]$, a $d_C^2 \times d_C^2$ unitary matrix $V_k := [v_{mn}^{(k)}]$ and a $p \times p$ positive diagonal matrix $D_k := \mathrm{diag}[d_1^{(k)}, d_2^{(k)}, \dots, d_p^{(k)}]$ where $p = \min\{d_A^2, d_C^2\}$ such that

$$a_{mn}^{(k)} = \sum_{\ell=1}^{p} u_{m\ell}^{(k)} d_\ell^{(k)} v_{\ell n}^{(k)}.$$

Thus,

$$\begin{aligned}\rho &= \sum_k \sum_{mn} \sum_\ell u_{m\ell}^{(k)} d_\ell^{(k)} v_{\ell n}^{(k)} P_m \otimes |f_k\rangle\langle f_k| \otimes Q_n \\ &= \sum_k \sum_\ell \left(\sqrt{d_\ell^{(k)}} \sum_m u_{m\ell}^{(k)} P_m\right) \otimes |f_k\rangle\langle f_k| \otimes \left(\sqrt{d_\ell^{(k)}} \sum_n v_{\ell n}^{(k)} Q_n\right) \\ &= \sum_k \sum_\ell A_{k\ell} \otimes |f_k\rangle\langle f_k| \otimes B_{k\ell},\end{aligned}$$

where

$$A_{k\ell} = \sqrt{d_\ell^{(k)}} \sum_m u_{m\ell}^{(k)} P_m \in B(\mathcal{H}_A), \qquad B_{k\ell} = \sqrt{d_\ell^{(k)}} \sum_n v_{\ell n}^{(k)} Q_n \in B(\mathcal{H}_C).$$

Conversely, we assume that there exist a basis $\{|f_m\rangle\} \in ONB(\mathcal{H}_B)$ and operators $A_{k\ell} \in B(\mathcal{H}_A)$, $B_{k\ell} \in B(\mathcal{H}_C)$ such that

$$\rho = \sum_k \sum_\ell A_{k\ell} \otimes |f_k\rangle\langle f_k| \otimes B_{k\ell}.$$

Put $\Pi_j^B = |f_j\rangle\langle f_j|$ for all $j = 1, 2, \ldots, d_B$. Then we get a rank-one projective measurement $\{\Pi_j^B : j = 1, 2, \ldots, d_B\}$ on $\mathcal{H}_B$ such that

$$\sum_{j=1}^{d_B} \left(I_A \otimes \Pi_j^B \otimes I_C\right) \rho \left(I_A \otimes \Pi_j^B \otimes I_C\right) = \sum_j \sum_\ell A_{j\ell} \otimes |f_j\rangle\langle f_j| \otimes B_{j\ell} = \rho.$$

This shows that ρ is an XCX-state.

(7) Similar to (5). □

Corollary 2.1 *If $\rho \in D(\mathcal{H}_{ABC})$ is CCC, then for every positive integer k, $(\mathrm{tr}(\rho^k))^{-1}\rho^k$ is also a CCC state.*

Proof Since $\rho \in D(\mathcal{H}_{ABC})$ is CCC, Theorem 2.1(1) implies that it can be represented as

$$\rho = \sum_{mns} \delta_{mns} |e_m\rangle\langle e_m| \otimes |f_n\rangle\langle f_n| \otimes |g_s\rangle\langle g_s|,$$

for some bases $\{|e_m\rangle\} \in ONB(\mathcal{H}_A)$, $\{|f_n\rangle\} \in ONB(\mathcal{H}_B)$ and $\{|g_s\rangle\} \in ONB(\mathcal{H}_C)$, and a probability distribution $\{\delta_{mns}\}$. Hence,

$$\left(\mathrm{tr}\left(\rho^k\right)\right)^{-1} \rho^k = \sum_{mns} \left(\mathrm{tr}\left(\rho^k\right)\right)^{-1} (\delta_{mns})^k |e_m\rangle\langle e_m| \otimes |f_n\rangle\langle f_n| \otimes |g_s\rangle\langle g_s|.$$

Clearly, $\{(\mathrm{tr}(\rho^k))^{-1}(\delta_{mns})^k\}$ is also a probability distribution. It follows from Theorem 2.1(1) that the state $(\mathrm{tr}(\rho^k))^{-1}\rho^k$ is also a CCC state. □

Generally, by using Theorem 2.1 a proof similar to Corollary 2.1, we can obtain the following.

Corollary 2.2 *Let* $\rho \in D(\mathcal{H}_{ABC})$ *and* $P \in \{CCC, CCX, CXC, XCC, CXX, XCX, XXC\}$. *If* ρ *is a P-state, then for every nonzero polynomial* $Q(x) = \sum_{i=0}^{k} a_i x^i$ *with nonnegative real coefficients* a_i, *the state* $(\mathrm{tr}(Q(\rho))^{-1} Q(\rho))$ *is also a P-state.*

Corollary 2.3 *Let* $\rho \in D(\mathcal{H}_{ABC})$. *Then*

(1) ρ *is a CXX-state if and only if there exist a basis* $\{|e_m\rangle\} \in ONB(\mathcal{H}_A)$ *and states* $\rho_m^{BC} \in D(\mathcal{H}_{BC})$ *as well as a probability distribution* $\{p_m\}$ *such that*

$$\rho = \sum_m p_m |e_m\rangle\langle e_m| \otimes \rho_m^{BC}. \tag{2.16}$$

(2) ρ *is an XCX-state if and only if there exist a basis* $\{|f_m\rangle\} \in ONB(\mathcal{H}_B)$ *and states* $\rho_{k\ell}^A \in B(\mathcal{H}_A)$, $\rho_{k\ell}^C \in B(\mathcal{H}_C)$ *and a probability distribution* $\{q_{k\ell}\}$ *such that*

$$\rho = \sum_k \sum_\ell q_{k\ell} \rho_{k\ell}^A \otimes |f_k\rangle\langle f_k| \otimes \rho_{k\ell}^C. \tag{2.17}$$

(3) ρ *is an XXC-state if and only if there exist a basis* $\{|g_s\rangle\} \in ONB(\mathcal{H}_C)$ *and states* $\varepsilon_s^{AB} \in B(\mathcal{H}_{AB})$ *as well as a probability distribution* $\{p_s\}$ *such that*

$$\rho = \sum_s p_s \varepsilon_s^{AB} \otimes |g_s\rangle\langle g_s|.$$

(4) ρ *is a CCX-state if and only if there exist bases* $\{|e_m\rangle\} \in ONB(\mathcal{H}_A)$, $\{|f_n\rangle\} \in ONB(\mathcal{H}_B)$, *states* $\rho_{mn}^C \in B(\mathcal{H}_C)$ *and a probability distribution* $\{q_{mn}\}$ *such that*

$$\rho = \sum_{mn} q_{mn} |e_m\rangle\langle e_m| \otimes |f_n\rangle\langle f_n| \otimes \rho_{mn}^C.$$

(5) ρ *is a CXC-state if and only if there exist bases* $\{|e_m\rangle\} \in ONB(\mathcal{H}_A), \{|g_s\rangle\} \in ONB(\mathcal{H}_C)$ *and states* $\beta_{ms} \in B(\mathcal{H}_C)$ *and a probability distribution* $\{q_{ik}\}$ *such that*

$$\rho = \sum_{ik} q_{ik} |e_i\rangle\langle e_i| \otimes \beta_{ik} \otimes |g_k\rangle\langle g_k|.$$

(6) ρ *is an XCC-state if and only if there exist bases* $\{|f_n\rangle\} \in ONB(\mathcal{H}_B), \{|g_s\rangle\} \in ONB(\mathcal{H}_C)$ *and states* $\alpha_{ns} \in B(\mathcal{H}_A)$ *and a probability distribution* $\{q_{jk}\}$ *such that*

$$\rho = \sum_{jk} q_{jk} \alpha_{jk} \otimes |f_j\rangle\langle f_j| \otimes |g_k\rangle\langle g_k|.$$

Proof (1) Let ρ be a CXX-state. Then by Theorem 2.1(5) there exist a basis $\{|e_m\rangle\} \in ONB(\mathcal{H}_A)$ and operators $\delta_m \in B(\mathcal{H}_{BC})$ such that

$$\rho = \sum_m |e_m\rangle\langle e_m| \otimes \delta_m.$$

For every $|\psi\rangle \in \mathcal{H}_{BC}$, we compute that

$$0 \le \langle e_i, \psi|\rho|e_i, \psi\rangle = \langle \psi|\delta_i|\psi\rangle.$$

Thus, $\delta_i \geq 0$ for all i. Put

$$\rho_m^{BC} = \begin{cases} \frac{\delta_m}{\mathrm{tr}(\delta_m)}, & \delta_m \neq 0; \\ \frac{1}{d_B d_C} I_{BC}, & \delta_m = 0; \end{cases} \qquad p_m = \begin{cases} \mathrm{tr}(\delta_m), & \delta_m \neq 0; \\ 0, & \delta_m = 0. \end{cases}$$

Then $\rho_m^{BC} \in D(\mathcal{H}_{BC})$, $p_m \geq 0$ for all m and $\sum_{m=1}^{d_A} p_m = \mathrm{tr}(\mathrm{tr}_{BC}(\rho)) = 1$ as well as (2.16) holds.

Conversely, it is clear that ρ is a CXX-state.

(2) Let ρ be an XCX-state. Then Theorem 2.1(6) yields that there exist a basis $\{|f_m\rangle\} \in ONB(\mathcal{H}_B)$ and operators $A_{k\ell} \in B(\mathcal{H}_A)$, $B_{k\ell} \in B(\mathcal{H}_C)$ such that

$$\rho = \sum_k \sum_\ell A_{k\ell} \otimes |f_k\rangle\langle f_k| \otimes B_{k\ell}.$$

Put $\Delta = \{(k,\ell) : A_{k\ell} \neq 0, B_{k\ell} \neq 0\}$. Let $(k,\ell) \in \Delta$. Take $|\psi^A\rangle \in \mathcal{H}_A$ such that $\langle \psi^A | A_{k\ell} | \psi^A\rangle = 1$. For every $|\psi^C\rangle \in \mathcal{H}_C$, we have

$$0 \leq \langle \psi^A, f_k, \psi^C | \rho | \psi^A, f_k, \psi^C \rangle = \langle \psi^C | B_{k\ell} | \psi^C \rangle.$$

This shows that $B_{k\ell} \geq 0$. Similarly, $A_{k\ell} \geq 0$. Put

$$\rho_{k\ell}^A = \begin{cases} \frac{A_{k\ell}}{\mathrm{tr}(A_{k\ell})}, & (k,\ell) \in \Delta; \\ \frac{1}{d_A} I_A, & (k,\ell) \notin \Delta; \end{cases} \qquad \rho_{k\ell}^C = \begin{cases} \frac{B_{k\ell}}{\mathrm{tr}(B_{k\ell})}, & (k,\ell) \in \Delta; \\ \frac{1}{d_C} I_C, & (k,\ell) \notin \Delta; \end{cases}$$

$$q_{k\ell} = \begin{cases} \mathrm{tr}(A_{k\ell})\,\mathrm{tr}(B_{k\ell}), & (k,\ell) \in \Delta; \\ 0, & (k,\ell) \notin \Delta. \end{cases}$$

Then $\rho_{k\ell}^A \in D(\mathcal{H}_A)$, $\rho_{k\ell}^C \in D(\mathcal{H}_C)$, $q_{k\ell} \geq 0$ for all k,ℓ and $\sum_{k\ell} q_{k\ell} = \mathrm{tr}(\mathrm{tr}_{AC}(\rho)) = 1$ as well as (2.17) holds.

Conversely, it is clear that ρ is a XCX-state.

The proof of (3)–(6) is similar to that of (1), (2). □

An N-partite state ρ is called fully separable if $\rho = \sum_i p_i \rho_1^i \otimes \rho_2^i \otimes \cdots \otimes \rho_N^i$, where $\sum_i p_i = 1$ and $\rho_1^i, \rho_2^i, \ldots, \rho_N^i$ are all states in individual subsystems for every i. From Corollary 2.3, one can see that the states having certain classical correlations are all fully separable.

Corollary 2.4 *CCC-XCX-CCX-CXC- and XCC-states are all fully separable.*

Next theorem gives a characterization of each kind of correlated states in terms of normality and commutativity of the associated component operators.

Theorem 2.2 *If a state $\rho \in D(\mathcal{H}_{ABC})$ is CCC (resp. BCC, SCC), then for any orthonormal bases $\{|e_i\rangle\}$ for $\mathcal{H}_A$ and $\{|f_k\rangle\}$ for $\mathcal{H}_B$ and $\{|g_s\rangle\}$ for $\mathcal{H}_C$, $\{A_{k\ell st}(\rho)\}$, $\{B_{ijst}(\rho)\}$ and $\{C_{ijk\ell}(\rho)\}$ (resp. at least two of $\{A_{k\ell st}(\rho)\}$, $\{B_{ijst}(\rho)\}$, $\{C_{ijk\ell}(\rho)\}$, at least one of $\{A_{k\ell st}(\rho)\}$, $\{B_{ijst}(\rho)\}$, $\{C_{ijk\ell}(\rho)\}$) are commuting families of normal operators.*

Proof Let $\rho \in D(\mathcal{H}_{ABC})$ be CCC. Then there exist orthonormal bases $\{|\varepsilon_x\rangle\}$ for $\mathcal{H}_A$, $\{|\eta_y\rangle\}$ for $\mathcal{H}_B$ and $\{|\zeta_z\rangle\}$ for $\mathcal{H}_C$ such that

$$\rho = \sum_{x=1}^{d_A}\sum_{y=1}^{d_B}\sum_{z=1}^{d_C} c_{xyz}|\varepsilon_x\rangle\langle\varepsilon_x| \otimes |\eta_y\rangle\langle\eta_y| \otimes |\zeta_z\rangle\langle\zeta_z|.$$

For any orthonormal bases $\{|e_i\rangle\}$ for $\mathcal{H}_A$ and $\{|f_k\rangle\}$ for $\mathcal{H}_B$ and $\{|g_s\rangle\}$ for $\mathcal{H}_C$, we see that for all k, ℓ, s, t,

$$A_{k\ell st}(\rho) = \langle f_k, g_s|\rho|f_\ell, g_t\rangle = \sum_{x=1}^{d_A}\sum_{y=1}^{d_B}\sum_{z=1}^{d_C} c_{xyz}\langle f_k|\eta_y\rangle\langle\eta_y|f_\ell\rangle\langle g_s|\zeta_z\rangle\langle\zeta_z|g_t\rangle \cdot |\varepsilon_x\rangle\langle\varepsilon_x|.$$

This shows that $\{A_{k\ell st}(\rho)\}$ is a commuting family of normal operators. Similarly, $\{B_{ijst}(\rho)\}$ and $\{C_{ijk\ell}(\rho)\}$ are also commuting families of normal operators.

The proof is similar for the case for ρ being BCC or SCC. □

Corollary 2.5 *Let $e = \{|e_i\rangle\}$, $f = \{|f_k\rangle\}$ and $g = \{|g_s\rangle\}$ be any orthonormal bases for $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_C$, respectively. Then $\rho \in D(\mathcal{H}_{ABC})$ is CCC (resp. BCC, SCC) if and only if $\{A_{k\ell st}(\rho)\}$, $\{B_{ijst}(\rho)\}$ and $\{C_{ijk\ell}(\rho)\}$(resp. at least two of $\{A_{k\ell st}(\rho)\}$, $\{B_{ijst}(\rho)\}$, $\{C_{ijk\ell}(\rho)\}$, at least one of $\{A_{k\ell st}(\rho)\}$, $\{B_{ijst}(\rho)\}$, $\{C_{ijk\ell}(\rho)\}$) are normal and commutative.*

Proof *Necessity.* It is clear from Theorem 2.1.

Sufficiency. Suppose that $\{A_{k\ell st}(\rho)\}$, $\{B_{ijst}(\rho)\}$ and $\{C_{ijk\ell}(\rho)\}$ are commuting families of normal operators, then we can denote that

$$A_{k\ell st}(\rho) = \sum_x \langle e'_x|A_{k\ell st}(\rho)|e'_x\rangle|e'_x\rangle\langle e'_x|,$$

$$B_{ijst}(\rho) = \sum_y \langle f'_y|B_{ijst}(\rho)|f'_y\rangle|f'_y\rangle\langle f'_y|,$$

$$C_{ijk\ell}(\rho) = \sum_z \langle g'_z|C_{ijk\ell}(\rho)|g'_z\rangle|g'_z\rangle\langle g'_z|,$$

where $\{|e'_x\rangle\}$, $\{|f'_y\}$ and $\{|g'_z\}$ are some orthonormal bases for $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_C$, respectively. Then

$$\begin{aligned}
\rho &= \sum_{k\ell st} A_{k\ell st}(\rho) \otimes |f_k\rangle\langle f_\ell| \otimes |g_s\rangle\langle g_t| \\
&= \sum_{k\ell st}\left(\sum_x \langle e'_x|A_{k\ell st}(\rho)|e'_x\rangle|e'_x\rangle\langle e'_x|\right) \otimes |f_k\rangle\langle f_\ell| \otimes |g_s\rangle\langle g_t| \\
&= \sum_x |e'_x\rangle\langle e'_x| \otimes \left[\sum_{k\ell st}\sum_{ij} p_{ijk\ell st}\langle e'_x|e_i\rangle\langle e_j|e'_x\rangle|f_k\rangle\langle f_\ell| \otimes |g_s\rangle\langle g_t|\right] \\
&= \sum_x |e'_x\rangle\langle e'_x| \otimes \left[\sum_{st}\sum_{ij}\langle e'_x|e_i\rangle\langle e_j|e'_x\rangle\left(\sum_y \langle f'_y|\sum_{k\ell} p_{ijk\ell st}|f_k\rangle\langle f_\ell|f'_y\rangle|f'_y\rangle\langle f'_y|\right)\right. \\
&\quad \left.\otimes |g_s\rangle\langle g_t|\right]
\end{aligned}$$

$$= \sum_{xyz} \left(\sum_{ijk\ell st} p_{ijk\ell st} \langle e'_x | e_i \rangle \langle e_j | e'_x \rangle \langle f'_y | f_k \rangle \langle f_\ell | f'_y \rangle \langle g'_z | g_s \rangle \langle g_t | g'_z \rangle \right) | e'_x \rangle \langle e'_x | \otimes | f'_y \rangle \langle f'_y | \otimes | g'_z \rangle \langle g'_z |,$$

where

$$\Delta_{xyz} = \sum_{ijk\ell st} p_{ijk\ell st} \langle e'_x | e_i \rangle \langle e_j | e'_x \rangle \langle f'_y | f_k \rangle \langle f_\ell | f'_y \rangle \langle g'_z | g_s \rangle \langle g_t | g'_z \rangle$$

satisfies $\sum_{xyz} \Delta_{xyz} = 1$ and $\Delta_{xyz} \geq 0$ for all x, y, z, so ρ is CCC.

The proof is similar for the case for ρ being BCC or SCC. □

3 Examples

By using Corollary 2.5 and Theorem 2.2, we can give some examples of CCC, CCX, CXC, XCC, CXX, XCX, XXC and GQC-states of the system $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, respectively.

(1) CCC: arbitrary product state, e.g., $|\phi\rangle = |000\rangle$;

(2) GQC: the GHZ state $|\phi\rangle = |000\rangle + |111\rangle$;

Take

$$\rho_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \rho_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad 0 < \lambda < 1.$$

(3) CCX: $\rho = \lambda |0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes \rho_1 + (1-\lambda)|0\rangle\langle 0| \otimes |1\rangle\langle 1| \otimes \rho_2$, which is not CCC since the operators $C_{0000}(\rho) = \lambda \rho_1$ and $C_{0011}(\rho) = (1-\lambda)\rho_2$ are not commutative;

(4) CXC: $\rho = \lambda |0\rangle\langle 0| \otimes \rho_1 \otimes |0\rangle\langle 0| + (1-\lambda)|1\rangle\langle 1| \otimes \rho_2 \otimes |1\rangle\langle 1|$, which is not CCC since the operators $B_{0000}(\rho) = \lambda \rho_1$ and $B_{1111}(\rho) = (1-\lambda)\rho_2$ are not commutative;

(5) XCC: $\rho = \lambda \rho_1 \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0| + (1-\lambda)\rho_2 \otimes |1\rangle\langle 1| \otimes |1\rangle\langle 1|$, which is not CCC since the operators $A_{0000}(\rho) = \lambda \rho_1$ and $A_{1111}(\rho) = (1-\lambda)\rho_2$ are not commutative;

(6) CXX: $\rho = |0\rangle\langle 0| \otimes \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)$, which is not CCC since the operators $B_{0000}(\rho) = \frac{1}{2}|0\rangle\langle 0|$ and $B_{0001}(\rho) = \frac{1}{2}|0\rangle\langle 1|$ are not commutative, $C_{0000}(\rho) = \frac{1}{2}|0\rangle\langle 0|$ and $C_{0001}(\rho) = \frac{1}{2}|0\rangle\langle 1|$ are not commutative;

(7) XCX: $\rho = \lambda \rho_1 \otimes |0\rangle\langle 0| \otimes \rho_1 + (1-\lambda)\rho_2 \otimes |0\rangle\langle 0| \otimes \rho_2$ which is not CCC since $A_{0000}(\rho) = \frac{\lambda}{2}\rho_1 + (1-\lambda)\rho_2$ and $A_{0001}(\rho) = \frac{\lambda}{2}\rho_1$ are not commutative, $C_{0000}(\rho) = \frac{\lambda}{2}\rho_1 + (1-\lambda)\rho_2$ and $C_{0100}(\rho) = \frac{\lambda}{2}\rho_1$ are not commutative;

(8) XXC: $\rho = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|) \otimes |0\rangle\langle 0|$ which is not CCC since the operators $A_{0000}(\rho) = \frac{1}{2}|0\rangle\langle 0|$ and $A_{0100}(\rho) = \frac{1}{2}|0\rangle\langle 1|$ are not commutative, $B_{0000}(\rho) = \frac{1}{2}|0\rangle\langle 0|$ and $B_{0100}(\rho) = \frac{1}{2}|0\rangle\langle 1|$ are not commutative.

According to [12], a k-partite state ρ in $\mathbb{C}^N \otimes \mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N$ is said to be a Schmidt-correlated (SC) state if it can be expressed as

$$\rho = \sum_{m,n=0}^{N-1} a_{mn} |mm\ldots m\rangle\langle nn\ldots n|,$$

where $\sum_{m=0}^{N-1} a_{mm} = 1$. It was proved in [12] that for an SC state ρ, it is fully separable if and only if it has a positive partial transposition, and ρ is genuinely entangled if and only if it has no positive partial transpositions.

From the correlation point of view, for a tripartite SC state $\rho = \sum_{m,n=0}^{N-1} a_{mn} |mmm\rangle\langle nnn|$ in $\mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N$, we get easily that it is CCC if and only if $a_{mn} = 0$ for all $m \neq n$, and it is GQC if and only if $a_{mn} = 0$ for some $m \neq n$. In fact, under bases $e = f = g = \{|m\rangle\}_{m=0}^{N-1}$ for $\mathcal{H}_A = \mathcal{H}_B = \mathcal{H}_C = \mathbb{C}^N$, respectively, we have

$$A_{mnmn} = B_{mnmn} = C_{mnmn} = a_{mn}|m\rangle\langle n|.$$

Thus, it implies from Corollary 2.5 that ρ is CCC if and only if $\{a_{mn}|m\rangle\langle n|\}$ is a commuting family of normal operators if and only if $a_{mn} = 0 (m \neq n)$. Similarly, ρ is GQC if and only if $\{a_{mn}|m\rangle\langle n|\}$ is not a commuting family of normal operators if and only if there exists a pair $(m, n)(m \neq n)$ such that $a_{mn} \neq 0$.

Acknowledgements This subject was supported by the NNSF of China (No. 11171197), IFGP of Shaanxi Normal University (No. 2011CXB004), and the FRF for the Central Universities (No. GK201104010).

References

1. Gian, L.G., Bruno, B., Fernando, G., Roberta, Z.: Phys. Rev. Lett. **107**, 190501 (2011)
2. Knill, E., Laflamme, R.: Phys. Rev. Lett. **81**, 5672 (1998)
3. Harold, O., Wojciech, H.Z.: Phys. Rev. Lett. **88**, 017901 (2002)
4. Luo, S.: Phys. Rev. A **77**, 022301 (2008)
5. Aharon, B., Kavan, M.: arXiv:1108.3649v1 (2011)
6. Zhou, T., Cui, J., Long, G.: Phys. Rev. A **84**, 062105 (2011)
7. Modi, K., Paterek, T., Son, W., Vedral, V., Williamson, M.: Phys. Rev. Lett. **104**, 080501 (2010)
8. Rulli, C.C., Sarandy, M.S.: Phys. Rev. A **84**, 042109 (2011)
9. Saguia, A., Rulli, C.C., de Oliveira, T.R., Sarandy, M.S.: Phys. Rev. A **84**, 042123 (2011)
10. Piani, M., Horodecki, P., Horodecki, R.: Phys. Rev. Lett. **100**, 119903 (2008)
11. Guo, Z., Cao, H., Chen, Z.: J. Phys. A, Math. Theor. **45**, 145301 (2012)
12. Zhao, M., Fei, S., Wang, Z.: Phys. Lett. A **372**, 2552 (2008)