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Applications of Lie Symmetries to Higher Dimensional Gravitating Fluids

A.M. Msomi · K.S. Govinder · S.D. Maharaj

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Abstract We consider a radiating shear-free spherically symmetric metric in higher dimensions. Several new solutions to the Einstein's equations are found systematically using the method of Lie analysis of differential equations. Using the five Lie point symmetries of the fundamental field equation, we obtain either an implicit solution or we can reduce the governing equations to a Riccati equation. We show that known solutions of the Einstein equations can produce infinite families of new solutions. Earlier results in four dimensions are shown to be special cases of our generalised results.

Keywords Gravitating fluids · Symmetries · Higher dimensional physics

1 Introduction

The spherically symmetric radiating spacetimes with vanishing shear are important for applications in relativistic astrophysics, radiating stars and cosmology. In the literature, there exists a large number of studies of various models involving gravitational collapse with radiative processes. Studies modeling relativistic stars show that a necessary requirement for these models is that the interior radiating spacetime has to be matched at the boundary, with the radial pressure being nonzero, to the exterior Vaidya radiating spacetime. Krasinski [1] pointed out the significance of relativistic heat conducting fluids in modeling inhomogeneous processes. Some exact solutions in the presence of heat flow have been developed by Bergmann [2], Maiti [3] and Modak [4]. In considering spherical gravitational collapse, the appearance of singularities and the formation of horizons, Banerjee and Chatterjee [5] and Banerjee et al. [6] have investigated heat conducting fluids in higher dimensional cosmological models. Davidson and Gurwich [7] and Maartens and Koyama [8] highlighted the role

A.M. Msomi · K.S. Govinder · S.D. Maharaj (✉)
Astrophysics and Cosmology Research Unit, School of Mathematical Sciences, University of KwaZulu-Natal, Private Bag X54001, Durban 4000, South Africa
e-mail: maharaj@ukzn.ac.za

A.M. Msomi
Department of Mathematical Sciences, Mangosuthu University of Technology, P.O. Box 12363, Jacobs 4026, South Africa

of heat flow in gravitational dynamics and perturbations in the framework of brane world cosmological models.

A proper and complete model of a radiating relativistic star requires the presence of heat flux. The result given by Santos et al. [9] indicates that the interior spacetime must contain a nonzero heat flux so that the matching of the interior at the boundary to the exterior Vaidya spacetime is possible. Models of heat flow in astrophysics have been used in gravitational collapse, black hole physics, formation of singularities and particle production at the stellar surface in four and higher dimensions. Herrera et al. [10], Maharaj and Govender [11] and Misthry et al. [12] showed that heat conducting relativistic radiating stars are also useful in the investigation of the cosmic censorship hypothesis and in describing collapse with vanishing tidal forces. Solutions to the Einstein field equations for a shear-free spherically symmetric spacetime with a homothetic vector, together with radial heat flux, have been presented by Wagh et al. [13]. Analytical solutions to the field equations for radiating collapsing spheres in the diffusion approximation have been found by Herrera et al. [14]. Recent examples of radiating stars, with generalised energy momentum tensors, are given by Herrera et al. [15] and Pinheiro and Chan [16].

Shear-free fluids, in the presence of heat flux, are also important in modeling inhomogeneous cosmological processes. The need for radiating models in the formation of structure, evolution of voids, the study of singularities, and investigations of the cosmic censorship hypothesis, has been pointed out by Krasinski [1]. Banerjee et al. [17] generated a model of a heat conducting sphere which radiates energy during collapse without the appearance of a horizon at any stage. This result holds in four dimensions but may be extended to models in higher dimensions. Banerjee and Chatterjee [5] studied heat conducting fluids in cosmological models in higher dimensions, and determined that gravitational collapse is also possible without the appearance of an event horizon. The presence of heat flow in brane world models sometimes allows for more general behaviour than is the case in standard general relativity. Govender and Dadhich [18] proved that the analogue of the Oppenheimer-Snyder model of a collapsing dust permits a radiating brane.

In this paper we analyse the master equation for higher dimensional radiating fluids studied by Banerjee and Chatterjee [5] applicable to a $(n+2)$-dimensional spherically symmetric metric. In our analysis, we used the Lie theory of extended groups as a systematic approach to generalise known solutions and generate new solutions of the same equation. The higher dimensional radiating model is derived in Sect. 2. In Sect. 3, we give a brief outline of the Lie theory. In Sect. 4, we discuss the new solutions of the master equation that can be found via Lie symmetries by taking one potential to be a function of the remaining potential. Also in this section, we systematically study other group invariant solutions admitted by the fundamental equation by taking specific ratios of the potentials. In Sect. 5, we extend known solutions to new solutions of the fundamental equation utilising Lie theory. Regardless of the complexity of the generating function chosen it is possible to find new exact solutions; we demonstrate this in two cases. We conclude this paper with some brief observations about the nature of the new exact solutions in Sect. 6.

2 Radiating Model

We consider the shear-free, spherically symmetric line element with an exterior $(n+2)$-dimensional manifold given by

$$ds^2 = -A^2dt^2 + \frac{1}{F^2}\left[dr^2 + r^2dX_n^2\right] \tag{1}$$

where $A = A(t,r)$ and $F = F(t,r)$ and

$$X_n^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \cdots + \sin^2\theta_1 \sin^2\theta_2 \ldots \sin^2\theta_{n-1} d\theta_n^2 \tag{2}$$

The energy momentum tensor for a nonviscous heat conducting fluid is given by

$$T_{ij} = (\rho + p)v_i v_j + p g_{ij} + q_i v_j + q_j v_i \tag{3}$$

where ρ is the energy density of the fluid, p the isotropic fluid pressure, v_i is the $(n+2)$-velocity and q_i is the heat flow vector. Using (1) and (3) we find that the nontrivial Einstein field equations in comoving coordinates are

$$\rho = \frac{n(n+1)F_t^2}{2A^2F^2} - \frac{n(n+1)F_r^2}{2} + nFF_{rr} + \frac{n^2FF_r}{r}, \tag{4a}$$

$$p = -\frac{nA_rFF_r}{A} + \frac{nA_rF^2}{rA} + \frac{n(n-1)F_r^2}{2} - \frac{n(n-1)FF_r}{r} + \frac{nF_{tt}}{A^2F} - \frac{n(n+3)F_t^2}{2A^2F^2} - \frac{nA_tF_t}{A^3F}, \tag{4b}$$

$$p = \frac{F^2A_{rr}}{A} - (n-1)FF_{rr} + \frac{n(n-1)F_r^2}{2} + \frac{(n-1)F^2A_r}{rA} - \frac{(n-1)^2FF_r}{r} - \frac{(n-2)FF_rA_r}{A} + \frac{nF_{tt}}{A^2F} - \frac{n(n+3)F_t^2}{2A^2F^2} - \frac{nA_tF_t}{A^3F}, \tag{4c}$$

$$q = -\frac{nFF_{tr}}{A} + \frac{nF_tF_r}{A} + \frac{nFF_tA_r}{A^2} \tag{4d}$$

The isotropy of pressure is given by (4b) and (4c) together in the form

$$FA_{xx} + 2A_xF_x - (n-1)AF_{xx} = 0 \tag{5}$$

with $x = r^2$. Equation (5) is the master equation for the system of higher dimensional Einstein field equations with $n \geq 2$. In this paper, we reduce the order of (5) via Lie analysis in order to find general solutions in higher dimensions.

3 Lie Analysis

The symmetry analysis for a system of ordinary differential equations in two dependent variables requires the determination of the one-parameter (ε) Lie group of transformations

$$\begin{aligned}\bar{x} &= f(x, F, A, \varepsilon)\\ \bar{F} &= g(x, F, A, \varepsilon)\\ \bar{A} &= h(x, F, A, \varepsilon)\end{aligned} \tag{6}$$

that leaves the solution set of the system invariant. It is difficult to calculate these transformations directly, and as such, we must resort to approximations via

$$\begin{aligned} \bar{x} &= x + \varepsilon\xi(x, F, A) + O(\varepsilon^2) \\ \bar{F} &= F + \varepsilon\eta(x, F, A) + O(\varepsilon^2) \\ \bar{A} &= A + \varepsilon\zeta(x, F, A) + O(\varepsilon^2) \end{aligned} \tag{7}$$

The transformations (7) can be obtained once we find the (symmetry) operator

$$Z = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial F} + \zeta\frac{\partial}{\partial A} \tag{8}$$

which is a set of vector fields. Once these symmetries are determined, it is possible to regain the finite (global) form of the transformation, given by (6), on solving Lie's equations

$$\begin{aligned} \frac{\mathrm{d}\bar{x}}{\mathrm{d}\varepsilon} &= \xi(\bar{x}, \bar{F}, \bar{A}) \\ \frac{\mathrm{d}\bar{F}}{\mathrm{d}\varepsilon} &= \eta(\bar{x}, \bar{F}, \bar{A}) \\ \frac{\mathrm{d}\bar{A}}{\mathrm{d}\varepsilon} &= \zeta(\bar{x}, \bar{F}, \bar{A}) \end{aligned} \tag{9}$$

subject to initial conditions

$$\bar{x}|_{\varepsilon=0} = x, \qquad \bar{F}|_{\varepsilon=0} = F, \qquad \bar{A}|_{\varepsilon=0} = A \tag{10}$$

The full details on the symmetry approach to solving differential equations can be found in a number of excellent texts (Bluman and Kumei [19], Olver [20]).

The determination of the generators is a straight forward process and has been automated by computer algebra packages (Dimas and Tsoubelis [21], Cheviakov [22]). In practice, we have found the package `PROGRAM LIE` (Head [23]) to be the most useful. It is quite accomplished given its age—it often yields results when its modern counterparts fail.

Utilising `PROGRAM LIE`, we show that (5) admits the following Lie point symmetries/vector fields:

$$Z_1 = \frac{\partial}{\partial x}, \tag{11a}$$

$$Z_2 = x\frac{\partial}{\partial x}, \tag{11b}$$

$$Z_3 = A\frac{\partial}{\partial A}, \tag{11c}$$

$$Z_4 = \frac{F}{n-1}\frac{\partial}{\partial F}, \tag{11d}$$

$$Z_5 = x^2\frac{\partial}{\partial x} + \frac{xF}{n-1}\frac{\partial}{\partial F} \tag{11e}$$

where $n > 2$. It is normal practice to use the symmetries (11a)–(11e) to reduce the order of the equation in the hope of finding solutions of the master equation. We need to proceed with some caution due to the overdetermined nature of (5). Thereafter we indicate how known solutions can be extended using these symmetries.

4 New Solutions via Lie Symmetries

One of the main purposes of calculating symmetries is to use them for symmetry reductions and hopefully obtain group invariant solutions. The goal of this section is to apply the symmetries calculated in Sect. 3 to obtain symmetry reductions and exact solutions where possible. The application of symmetries (11a)–(11e) to the master equation results in either an implicit solution of (5) or we can reduce the governing equations to complicated Riccati equations that are difficult to solve. However there are two cases in which we can find new solutions regardless of the complexity of the function chosen.

4.1 The Choice $A = A(F)$

An obvious case to consider in this subsection, is when one dependent variable in (5) is a function of the other. Usually such an approach results in a more complicated equation to solve. In spite of this, we can make significant progress if we use the Lie symmetry Z_1 (which gives the same result as Z_2). For our purposes we use the partial set of invariants of

$$Z_1 = \frac{\partial}{\partial x} \tag{12}$$

given by

$$\begin{aligned} p &= F \\ q(p) &= F_x \\ r(p) &= A \end{aligned} \tag{13}$$

This transformation reduces equation (5) to

$$q'(p)\left[(n-1)r(p) - pr'(p)\right] = q(p)\left[pr''(p) + 2r'(p)\right] \tag{14}$$

which can be integrated to give

$$q = q_0 e^{\int \frac{2r' + pr''}{(n-1)r - r'p} dp} \tag{15}$$

Substituting for the metric functions via (13), we can integrate one more time to give the solution

$$\int \left[e^{-\int \frac{2A_F + FA_{FF}}{(n-1)A - FA_F} dF}\right] dF = q_0 x + x_0 \tag{16}$$

where q_0 and x_0 are arbitrary functions of time. Equation (16) suggests that, given any function A depending on arbitrary F, we can work out F explicitly from (16). Such an explicit relationship between F and A has not been found previously. Note that since (5) is linear, once we obtain F via (16) we can use it to obtain the general solution of (5) using standard techniques for solving linear equations.

We illustrate this method with simple examples. Using $A = 1$, we evaluate (16) to obtain

$$F = q_0(t)x + x_0(t) \tag{17}$$

We can easily generate the general solution to (5) as

$$A = \frac{-C_1}{q(x_0 + qx)} + C_2 \tag{18}$$

$$F = \frac{-1+n}{qC_1(1+n)}(x_0+qx)^{\frac{2}{1-n}} \times \left[\frac{1+n}{-1+n}q(x_0+qx)^{\frac{1+n}{-1+n}}C_1 + (-C_1+q(x_0+qx)C_2)^{\frac{1+n}{-1+n}}C_2\right] \tag{19}$$

If we take $A = F^2$, then (16) is reduced to

$$\int F^{\left(\frac{6}{3-n}\right)} dF = q_0 x + x_0 \tag{20}$$

and hence

$$\begin{aligned} F &= \left[\frac{9-n}{3-n}(\bar{q}_0 x + \bar{x}_0)\right]^{\frac{3-n}{9-n}} \\ A &= \left[\frac{9-n}{3-n}(\bar{q}_0 x + \bar{x}_0)\right]^{\frac{6-2n}{9-n}} \end{aligned} \tag{21}$$

The functional form of F can be easily extended to obtain

$$F = C_1 \frac{(\frac{9-n}{3-n})^{\frac{3-n}{9-n}}(9-10n+n^2)(q_0x+x_0)^{\frac{3-n}{9-n}+\frac{-9+2n-n^2}{9-10n+n^2}}}{q_0(-9+2n-n^2)} + C_2\left[\frac{9-n}{3-n}(q_0x+x_0)\right]^{\frac{3-n}{9-n}} \tag{22}$$

where C_1 and C_2 are arbitrary functions of time, which is the general solution to (5) when

$$A = \left[\frac{9-n}{3-n}(\bar{q}_0 x + \bar{x}_0)\right]^{\frac{6-2n}{9-n}} \tag{23}$$

4.2 The Choice $W = \frac{F}{A^{1/(n-1)}}$

The combination of symmetries given by

$$Z_3 + Z_4 = A\frac{\partial}{\partial A} + \frac{F}{n-1}\frac{\partial}{\partial F} \tag{24}$$

gives rise to the invariant

$$W = \frac{F}{A^{1/(n-1)}} \tag{25}$$

Then (5) is transformed by (25) to the form

$$-nWA_x^2 + (n-1)A^2 W_{xx} = 0 \tag{26}$$

with solution

$$A = C_1(t)\exp\left(\int \pm \frac{\sqrt{(n-1)^2 W_{xx}}}{\sqrt{nW}}dx\right) \tag{27}$$

which comes as a result of treating equation (26) as a nonlinear first order ordinary differential equation in A. Given any function W we can integrate the right hand side of (27) and find a form for A.

If we take $W = a(t)x + b(t)$, then (27) gives

$$A = \bar{C}_1(t) \tag{28}$$

and

$$F = \bar{C}_1(t)^{1/(n-1)}(a(t)x + b(t)) \tag{29}$$

which are new solutions of (5) for $n > 2$.

Alternatively, we could substitute the inverse of (25), *i.e.*

$$\widehat{W} = \frac{A^{1/(n-1)}}{F} \tag{30}$$

into (5) and obtain

$$nA_x{}^2\widehat{W}^2 - 2(n-1)^2A^2\widehat{W}_x{}^2 + (n-1)^2A^2\widehat{W}\widehat{W}_{xx} = 0 \tag{31}$$

with solution

$$A = C_2(t)\exp\left(\pm\int\frac{\sqrt{2(n-1)^2\widehat{W}_x{}^2 - (n-1)^2\widehat{W}\widehat{W}_{xx}}}{\sqrt{n}\widehat{W}}\mathrm{d}x\right) \tag{32}$$

Again, given any function W we can integrate the right hand side of (32) and find a form for A.

If we take $W = a(t)x + b(t)$ as before we find that

$$A_1 = C_2(t)\,[a(t)x + b(t)]^{\frac{\sqrt{2}(n-1)}{\sqrt{n}}} \tag{33}$$

and

$$F_1 = \frac{\left[C_2(t)(a(t)x + b(t))^{\frac{\sqrt{2}(n-1)}{\sqrt{n}}}\right]^{\frac{1}{n-1}}}{a(t)x + b(t)} \tag{34}$$

which is essentially a new solution of the master equation in higher dimensional space. We can also have

$$A_2 = \frac{\bar{C}_2(t)}{(a(t)x + b(t))^{\frac{\sqrt{2}(n-1)}{\sqrt{n}}}} \tag{35}$$

and

$$F_2 = \frac{\left[\bar{C}_2(t)(a(t)x + b(t))^{\frac{-\sqrt{2}(n-1)}{\sqrt{n}}}\right]^{1/n-1}}{a(t)x + b(t)} \tag{36}$$

thus obtaining two different solutions from the same seed function.

Observe that (27) and (32) will contain all solutions of the master equation (5) for appropriately chosen seed functions W or $\widehat{W}$ which are ratios of the metric functions. We are always able to reduce (5) to the quadratures (27) or (32) regardless of the complexity of the seed functions.

5 Extending Known Solutions

Another use of Lie point symmetries is the extension of known solutions of differential equations. This is possible due to the fact that the symmetries generate transformations that leave the equations invariant. As a result, applying those transformations to known solutions will (usually) result in new solutions.

We illustrate the approach by using the simple infinitesmal generator Z_1, where we observe that

$$\xi = 1, \qquad \eta = 0, \qquad \zeta = 0 \tag{37}$$

We solve the Lie equations (9), subject to initial conditions (10), to obtain

$$\begin{aligned} \bar{x} &= x + a_1 \\ \bar{F} &= F \\ \bar{A} &= A \end{aligned} \tag{38}$$

This means that using (38) we can map (5) to the form

$$\bar{F}\bar{A}_{\bar{x}\bar{x}} + 2\bar{A}_{\bar{x}}\bar{F}_{\bar{x}} - (n-1)\bar{A}\bar{F}_{\bar{x}\bar{x}} = 0 \tag{39}$$

As a result of this mapping, any existing solution to (5) can be transformed to a solution of (39) by (38). Usually, a_1 is an arbitrary constant. However, since F and A depend on x and t we take a_1 to be an arbitrary function of time, $a_1 = a_1(t)$.

If we now take each of the remaining symmetries successively, we obtain the general transformation

$$\begin{aligned} \bar{x} &= \frac{e^{a_2}(a_1 + x)}{1 - a_5 e^{a_2}(a_1 + x)} \\ \bar{F} &= \frac{e^{a_4} F}{1 - a_5 e^{a_2}(a_1 + x)} \\ \bar{A} &= e^{a_3} A \end{aligned} \tag{40}$$

where the a_i are all arbitrary functions of time and $n > 2$.

Thus any known solution of (5) can be transformed to a new solution of (5) via (40). For example, if we start with the solution

$$A = 1, \qquad F = \alpha(t)x + \beta(t) \tag{41}$$

the transformation (40) yields the new solution

$$\begin{aligned} \bar{x} &= \frac{e^{a_2}(a_1 + x)}{1 - a_5 e^{a_2}(a_1 + x)} \\ \bar{F} &= \frac{e^{a_4}(\alpha(t)x + \beta(t))}{1 - a_5 e^{a_2}(a_1 + x)} \\ \bar{A} &= e^{a_3} \end{aligned} \tag{42}$$

All the new results that we derived in the previous section can be similarly extended via (40).

6 Conclusion

In this work, we have provided symmetry reductions and exact solutions of the Einstein field equations governing shear-free heat conducting fluids in higher dimensions. Explicit relationships were provided between the gravitational potentials, obviating a need to start with "simple" forms for one to calculate the other. We were also able to provide a general transformation to extend our (and any other) known solution into new solutions. When $n = 2$ we regain the results of Deng [24] who developed a method to generate solutions when simple forms of A or F are chosen. The case $n = 2$ also contains the results of Msomi et al. [25] who adopted a more geometric and systematic approach using Lie theory to generalise known solutions and generate new solutions. The new solutions of this paper may be used to study the physics of radiating astrophysical and cosmological models in higher dimensions.

It is interesting to observe an important feature of the solutions admitted by (5). When $F_{xx} = 0$ we find that (5) becomes

$$FA_{xx} + 2A_x F_x = 0$$

This equation has the remarkable feature that it is independent of the dimension n. Thus any solution with $F_{xx} = 0$ presented by Deng [24] and Msomi et al. [25] in four dimensions will also be applicable in higher dimensions. This direct application of four dimensional solutions into higher dimensional spacetimes is rather unusual in general relativity.

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