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Dense Coding with Cluster State Via Local Measurements

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Abstract Two schemes, introducing generalized measurement and entanglement concentration respectively, for dense coding are investigated by using a one-dimensional four-particle cluster state, where the supervisors (Cliff and David) can control the average amount of information transmitted from the sender (Alice) to the receiver (Bob) by adjusting the local measurement angles θ_3 and θ_4. It is shown that the results for the average amounts of information are unique from the different two schemes.

Keywords Dense coding · Cluster state · Generalized measurement · Local measurement · Average amount of information

1 Introduction

Quantum entanglement [1] is a quintessential property of quantum mechanics that sets it apart from any classical physical theory. An important feature of entanglement is that it gives rise to correlations that cannot be explained by any local realistic description of quantum mechanics. In recent years, quantum entanglement has become an important physical resource for quantum teleportation [2–5], dense coding [6–8], quantum state sharing [9–12], and quantum computation [13, 14] and so on.

Since the original idea of quantum dense coding considered in 1992 by Bennett and Wiesner [15], dense coding has been generalized in various directions. For example, it is possible to generalize the dense coding for continuous variables [16, 17], multipartite communication [18–23]. The original controlled dense coding protocol was proposed in 2001 [24]. In this protocol, one party (Alice) can transmit information to the second party (Bob) whereas the local measurements of the third party (Cliff and David) serves as quantum erasure. Cliff and David can control the quantum channel between Alice and Bob and the average amount of information transmitted from Alice to Bob via a local measurements. Since that, Huang *et al.* studied controlled dense coding scheme between multi-parties with

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multi-qubit GHZ state [7]. Jiang *et al.* proposed a scheme to realize controlled dense coding with three-particle symmetric state [25].

In this paper, two methods are shown to realize dense coding with one-dimensional cluster state via local measurements [26]. One of the strategies is Alice directly applies one of the four unitary operators $\{I, \sigma_X, i\sigma_Y, \sigma_Z\}$ on her qubit and then sends it to Bob, then Bob can obtain 2 bits of information with a certain probability via a generalized measurement described by positive-operator-valued measure (POVM) elements on his two qubit states. The second one is Alice first concentrates the entanglement of the channel between she and Bob, and then performs dense coding. It is shown that the successful probability only depends on the local measurement angles θ_3 and θ_4 performed by Cliff and David, respectively. This implies that Cliff and David can control the average amount of information transmitted from Alice to Bob by adjusting local measurement angles. It is also shown that the results for the average amounts of information are unique from the different two schemes.

2 Dense Coding with Generalized Measurement

Let us consider that the quantum channel is a four-particle cluster state

$$|C\rangle_{1234} = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)_{1234}, \tag{1}$$

where qubits 1, 2, 3 and 4 are hold by Alice, Bob, Cliff and David, respectively. In order to control the quantum channel between Alice and Bob and the amount of information transmitted from Alice to Bob, David performs a von Neumann measurement on his qubit 4 under the basis,

$$|+\rangle_4 = \sin\theta_4|0\rangle_4 + \cos\theta_4|1\rangle_4, \qquad |-\rangle_4 = \cos\theta_4|0\rangle_4 - \sin\theta_4|1\rangle_4, \tag{2}$$

where θ_4 is a measured angle with the region $[0, \pi/4]$, After the measurement, David informs his measurement result to Alice and Bob. It is noted that the von Neuman measurement of qubit 4 gives the outcome $|+\rangle_4$ or $|-\rangle_4$ with equal probability. Thus, the cluster state (1) in the new basis $\{|+\rangle_4, |-\rangle_4\}$ can be rewritten as

$$|C\rangle_{1234} = \frac{1}{\sqrt{2}}\big(|\psi\rangle_{123} \otimes |+\rangle_4 + |\varphi\rangle_{123} \otimes |-\rangle_4\big), \tag{3}$$

with

$$\begin{aligned} |\psi\rangle_{123} &= \frac{1}{\sqrt{2}}\big[\sin\theta_4\big(|000\rangle_{123} + |110\rangle_{123}\big) + \cos\theta_4\big(|001\rangle_{123} - |111\rangle_{123}\big)\big], \\ |\varphi\rangle_{123} &= \frac{1}{\sqrt{2}}\big[\cos\theta_4\big(|000\rangle_{123} + |110\rangle_{123}\big) - \sin\theta_4\big(|001\rangle_{123} - |111\rangle_{123}\big)\big]. \end{aligned} \tag{4}$$

Corresponding to Cliff's measurement result $|+\rangle_4$ or $|-\rangle_4$, it is obvious that the state of qubits 1, 2 and 3 collapses to $|\psi\rangle_{123}$ or $|\varphi\rangle_{123}$, respectively. In this paper, we consider the case that David's measurement outcome gives $|+\rangle_4$ and the state of qubits 1, 2 and 3 collapses to $|\psi\rangle_{123}$, the other one can be treated in a similar way.

Then Cliff measures his qubit under the following basis

$$|+\rangle_3 = \sin\theta_3|0\rangle_3 + \cos\theta|1\rangle_3, \qquad |-\rangle_3 = \cos\theta_3|0\rangle_3 - \sin\theta_3|1\rangle_3, \tag{5}$$

and informs Alice and Bob his measurement result via a classical channel. The quantum state $|\psi\rangle_{123}$ can be rewritten in terms of the new basis $\{|+\rangle_3, |-\rangle_3\}$ as

$$\begin{aligned}
|\psi\rangle_{123} \\
&= \frac{1}{\sqrt{C_1}}\big[(\sin\theta_3\sin\theta_4+\cos\theta_3\cos\theta_4)|00\rangle_{12}-(\cos\theta_3\cos\theta_4-\sin\theta_3\sin\theta_4)|11\rangle_{12}\big]\otimes|+\rangle_3 \\
&\quad+\frac{1}{\sqrt{C_2}}\big[(\cos\theta_3\sin\theta_4-\sin\theta_3\cos\theta_4)|00\rangle_{12}+(\cos\theta_3\sin\theta_4+\sin\theta_3\cos\theta_4)|11\rangle_{12}\big] \\
&\quad\otimes|-\rangle_3 \\
&= \frac{1}{\sqrt{C_1}}\big(\alpha|00\rangle_{12}-\beta|11\rangle_{12}\big)\otimes|+\rangle_3+\frac{1}{\sqrt{C_2}}\big(\gamma|00\rangle_{12}+\delta|11\rangle_{12}\big)\otimes|-\rangle_3 \\
&= |\psi\rangle_{12}\otimes|+\rangle_3+|\varphi\rangle_{12}\otimes|-\rangle_3
\end{aligned} \tag{6}$$

with

$$\begin{aligned}
C_1 &= \sin^2\theta_3\sin^2\theta_4+\cos^2\theta_3\cos^2\theta_4, \\
C_2 &= \cos^2\theta_3\sin^2\theta_4+\sin^2\theta_3\cos^2\theta_4.
\end{aligned} \tag{7}$$

Generally, the states $|\psi\rangle_{12}$ and $|\varphi\rangle_{12}$ are not maximally entangled states, so the success probability of dense coding with them is less than 1. In the following, we discuss two schemes of dense coding with them. The first one is base on generalized measurement, and the second one is base on entanglement concentration.

At first, we consider the case in which Cliff's measurement result is $|+\rangle_3$ and the state of qubits 1 and 2 collapses to $|\psi\rangle_{12}$. After receiving the measurement result, Alice uses directly any one of the four unitary operators $\{I, \sigma_X, i\sigma_Y, \sigma_Z\}$ to operate the shared state $|\psi\rangle_{12}$. Such as

$$\begin{aligned}
(I\otimes I)|\psi\rangle_{12} &= \frac{1}{\sqrt{C_1}}\alpha|00\rangle_{12}-\frac{1}{\sqrt{C_1}}\beta|11\rangle_{12}=|\phi_1\rangle_{12}, \\
(\sigma_X\otimes I)|\psi\rangle_{12} &= \frac{1}{\sqrt{C_1}}\alpha|10\rangle_{12}-\frac{1}{\sqrt{C_1}}\beta|01\rangle_{12}=|\phi_2\rangle_{12}, \\
(i\sigma_Y\otimes I)|\psi\rangle_{12} &= \frac{1}{\sqrt{C_1}}\alpha|10\rangle_{12}+\frac{1}{\sqrt{C_1}}\beta|01\rangle_{12}=|\phi_3\rangle_{12}, \\
(\sigma_Z\otimes I)|\psi\rangle_{12} &= \frac{1}{\sqrt{C_1}}\alpha|00\rangle_{12}+\frac{1}{\sqrt{C_1}}\beta|11\rangle_{12}=|\phi_4\rangle_{12}.
\end{aligned} \tag{8}$$

Then Alice sends qubit 1 to Bob, and now Bob has at his disposal two qubits which could be in any one of the four possible states $\{|\phi_1\rangle_{12}, |\phi_2\rangle_{12}, |\phi_3\rangle_{12}, |\phi_4\rangle_{12}\}$. If Bob is able to distinguish all the four nonorthogonal states conclusively, he can extract two classical bits of information. It is noted that, however, the above four states are not mutually orthogonal. According to quantum theory, they cannot be distinguished with certainty. Fortunately, it is easy to find that the four possible states are actually linearly independent. Therefore Bob can conclusively distinguish these states with some probability of success.

To distinguish the above set $\{|\phi_1\rangle_{12}, |\phi_2\rangle_{12}, |\phi_3\rangle_{12}, |\phi_4\rangle_{12}\}$, first Bob performs a projection onto the subspaces spanned by the basis states $\{|00\rangle, |11\rangle\}$ and $\{|01\rangle, |10\rangle\}$ with corresponding projective operators are $P_1 = |00\rangle\langle 00| + |11\rangle\langle 11|$ and $P_2 = |01\rangle\langle 01| + |10\rangle\langle 10|$

respectively. Obviously, P_1 and P_2 are mutually orthogonal, so Bob can discriminate the two subsets of Alice's operators: $\{I, \sigma_Z\}$ and $\{\sigma_X, i\sigma_Y\}$. If Bob obtains P_1, then he knows that the state will be either $|\phi_1\rangle_{12}$ or $|\phi_4\rangle_{12}$. Similarly, if he obtains P_2, the state will be either $|\phi_2\rangle_{12}$ or $|\phi_3\rangle_{12}$. After this projective measurement he gets 1 bit of information [24]. Suppose Bob obtains P_1, then he performs a generalized measurement on his two qubit states. In the case, the positive operator valued measure (POVM) elements in the subspace $\{|00\rangle, |11\rangle\}$ are [27]

$$\begin{aligned} M_1 &= \frac{1}{2}\begin{pmatrix} \beta^2/\alpha^2 & \beta/\alpha \\ \beta/\alpha & 1 \end{pmatrix}, \qquad M_2 = \frac{1}{2}\begin{pmatrix} \beta^2/\alpha^2 & -\beta/\alpha \\ -\beta/\alpha & 1 \end{pmatrix}, \\ M_3 &= \begin{pmatrix} (\alpha^2-\beta^2)/\alpha^2 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{9}$$

It is easy to check that the condition $M_1 + M_2 + M_3 = 1$ is satisfied. The POVM has three outcomes, which is independent of the state of the measured system. Therefore, POVM provides the most general physically realized measurement in quantum mechanics.

If Bob gets M_1 then the state is $|\phi_1\rangle_{12}$, if he gets M_2 then the state is $|\phi_4\rangle_{12}$, and if he gets M_3 the state is completely indecisive. The absolutely success probability of distinguishing $|\phi_1\rangle_{12}$ and $|\phi_4\rangle_{12}$ is $2\beta^2/C_1$, which is also the probability that Bob obtains another 1 bit of information. Similar procedure can be applied for the case of P_2, one can easily to find that the relevant POVM elements and the success probability are the same. So, in this case, Alice can transmit

$$I_+^1 = \frac{1}{2} \times \frac{1}{2} C_1 \times \left(1 + \frac{2\beta^2}{C_1}\right) = \frac{1}{4}\left(\alpha^2 + 3\beta^2\right) \tag{10}$$

bits of information to Bob.

If Charlie's measurement result is $|-\rangle_3$, then the state of qubits 1 and 2 collapses to $|\varphi\rangle_{12}$. After Alice's encoding with one of the four operators $\{I, \sigma_X, i\sigma_Y, \sigma_Z\}$, the state $|\varphi\rangle_{12}$ undergoes one of the following transformations,

$$\begin{aligned} (I \otimes I)|\varphi\rangle_{12} &= \frac{1}{\sqrt{C_2}}\gamma|00\rangle_{12} + \frac{1}{\sqrt{C_2}}\delta|11\rangle_{12} = |\phi_1\rangle_{12}, \\ (\sigma_X \otimes I)|\varphi\rangle_{12} &= \frac{1}{\sqrt{C_2}}\gamma|10\rangle_{12} + \frac{1}{\sqrt{C_2}}\delta|01\rangle_{12} = |\phi_2\rangle_{12}, \\ (i\sigma_Y \otimes I)|\varphi\rangle_{12} &= -\frac{1}{\sqrt{C_2}}\gamma|10\rangle_{12} + \frac{1}{\sqrt{C_2}}\delta|01\rangle_{12} = |\phi_3\rangle_{12}, \\ (\sigma_Z \otimes I)|\varphi\rangle_{12} &= \frac{1}{\sqrt{C_2}}\gamma|00\rangle_{12} - \frac{1}{\sqrt{C_2}}\delta|11\rangle_{12} = |\phi_4\rangle_{12}. \end{aligned} \tag{11}$$

In this case, the POVM set may be expressed as

$$\begin{aligned} M_1' &= \frac{1}{2}\begin{pmatrix} \gamma^2/\delta^2 & \gamma/\delta \\ \gamma/\delta & 1 \end{pmatrix}, \qquad M_2' = \frac{1}{2}\begin{pmatrix} \gamma^2/\delta^2 & -\gamma/\delta \\ -\gamma/\delta & 1 \end{pmatrix}, \\ M_3' &= \begin{pmatrix} (\delta^2-\gamma^2)/\delta^2 & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{12}$$

and Bob can discriminate $|\phi_1\rangle_{12}$ from $|\phi_4\rangle_{12}$ (or $|\phi_2\rangle_{12}$ from $|\phi_3\rangle_{12}$) with absolute success probability $2\gamma^2/C_2$ and

$$I_-^1 = \frac{1}{2} \times \frac{1}{2} C_2 \times \left(1 + \frac{2\gamma^2}{C_2}\right) = \frac{1}{4}\left(\delta^2 + 3\gamma^2\right) \tag{13}$$

bits of information are transmitted.

Synthesizing all measurement cases, the average amount of information transmitted from Alice to Bob is a summation of the absolute success probability in the two POVM measurement and can be expressed as

$$I^1 = 2 \times \left(I_+^1 + I_-^1\right) = 2(1 - 2\sin\theta_3 \sin\theta_4 \cos\theta_3 \cos\theta_4). \tag{14}$$

From (14), we see that the average amount of information is only dependent on the measured angles θ_3 and θ_4. Therefore, it is helpful for David and Cliff to control the average amount of information transmitted from Alice to Bob by adjusting the measurement angles θ_3 and θ_4.

3 Probabilistic Dense Coding Via Entanglement Concentration

Now we discuss the second scheme. Like the above scheme, we first suppose that David's measurement result is $|+\rangle_4$ and Cliff's measurement gives $|+\rangle_3$. In this case, the state of qubits 1 and 2 collapses to $|\psi\rangle_{12}$. After receiving the measurement results, Alice shares the general entangled state $|\psi\rangle_{12}$ with Bob who need not know it. Now Alice takes a new way to realize dense coding with the non-maximally entangled state by introducing an auxiliary qubit with original state $|0\rangle_{aux}$. She first performs a unitary transformation

$$U_1 = \begin{pmatrix} \beta/\alpha & 0 & \sqrt{1-\beta^2/\alpha^2} & 0 \\ 0 & -1 & 0 & 0 \\ \sqrt{1-\beta^2/\alpha^2} & 0 & -\beta/\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{15}$$

on the auxiliary qubit and qubit 1 under the basis $\{|0\rangle_{aux}|0\rangle_1, |0\rangle_{aux}|1\rangle_1, |1\rangle_{aux}|0\rangle_1, |1\rangle_{aux}|1\rangle_1\}$. The two-qubit unitary matrix U_1 and single-qubit identity operation I_2, that is $U_1 \otimes I_2$, transforms the state $|0\rangle_{aux} \otimes |\psi\rangle_{12}$ to

$$|\psi\rangle_{aux12} = \sqrt{\frac{2}{C_1}}\beta|0\rangle_{aux}\left[\frac{1}{\sqrt{2}}(|00\rangle_{12} + |11\rangle_{12})\right] + \sqrt{\left(\alpha^2 - \beta^2\right)/C_1}|1\rangle_{aux}|11\rangle_{12}. \tag{16}$$

Then Alice performs a von Neumann measurement on the auxiliary qubit under the basis $\{|0\rangle_{aux}, |1\rangle_{aux}\}$. If she obtains $|0\rangle_{aux}$, qubits 1 and 2 are maximally entangled. Alice now performs one of the four unitary transformations $\{I, \sigma_X, i\sigma_Y, \sigma_Z\}$ on qubit 1 and sends it to Bob. By performing a Bell-basis measurement, Bob knows he has two qubits in one of the four Bell states resulted from Alice's transformation, so 2 bits of information are transmitted. If Alice obtains $|1\rangle_{aux}$, qubits 1 and 2 are unentangled. Bob can extract only 1 bit of information. Thus, an average number of

$$I_+^2 = \frac{1}{2} \times \frac{1}{2} C_1 \times \left[\frac{2}{C_1}\beta^2 \times 2 + \frac{\alpha^2 - \beta^2}{C_1} \times 1\right] = \frac{1}{4}\left(\alpha^2 + 3\beta^2\right), \tag{17}$$

bits of information is transmitted from Alice to Bob.

If Cliff's measurement result is $|-\rangle_3$, Alice's unitary transformation on the auxiliary qubit and qubit 1 should be changed as

$$U_2' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma/\delta & \sqrt{1-\gamma^2/\delta^2} & 0 \\ 0 & \sqrt{1-\gamma^2/\delta^2} & -\gamma/\delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{18}$$

under the basis $\{|0\rangle_{aux}|0\rangle_1, |0\rangle_{aux}|1\rangle_1, |1\rangle_{aux}|0\rangle_1, |1\rangle_{aux}|1\rangle_1\}$. Under the operation of $U_2' \otimes I_2$, the state $|0\rangle_{aux} \otimes |\varphi\rangle_{12}$ is transformed as

$$|\varphi\rangle_{aux12} = \sqrt{\frac{2}{C_2}}\gamma|0\rangle_{aux}\left[\frac{1}{\sqrt{2}}(|00\rangle_{12} + |11\rangle_{12})\right] + \sqrt{(\delta^2-\gamma^2)/C_2}|1\rangle_{aux}|11\rangle_{12}. \tag{19}$$

Then Alice performs a von Neumann measurement on the auxiliary qubit under the basis $\{|0\rangle_{aux}, |1\rangle_{aux}\}$. If Alice gets the result $|1\rangle_{aux}$, the state of qubits 1 and 2 is unentangled, and 1 bit information can be transmitted. If the measured result $|0\rangle_{aux}$, the state of qubits 1 and 2 is maximally entangled, and 2 bits information can be transmitted. In the case, therefore, Alice can transmit

$$I_-^2 = \frac{1}{2} \times \frac{1}{2}C_1 \times \left[\frac{2}{C_1}\gamma^2 \times 2 + \frac{\delta^2-\gamma^2}{C_1} \times 1\right] = \frac{1}{4}(\delta^2 + 3\gamma^2), \tag{20}$$

bits of information on average. The average amount of information transmitted from Alice to Bob adds up to

$$I^2 = 2 \times (I_+^2 + I_-^2) = 2(1 - 2\sin\theta_3 \sin\theta_4 \cos\theta_3 \cos\theta_4), \tag{21}$$

which is also a function of the measurement angles θ_3 and θ_4.

Comparing (14) with (21), we find that the results are same. Therefore, the two schemes are equivalent for dense coding via local measurements. That is to say, the results are unique.

4 Summary

In summary, two schemes, introducing generalized measurement and entanglement concentration respectively, of realizing dense coding are investigated by using a cluster state via local measurements, where Alice sends the information to Bob, while Cliff and David serves as quantum erasure by the local measurements. It is shown that the results for the average amounts of information are unique from the different two schemes.

Under the different measurement angles, the average amount of information transmitted from Alice to Bob is different. Therefore, one can obtain more information about the transmission. This is helpful for Cliff and David to choose some useful measurements for higher success probability.

It is shown that the success probability depends on the measurement angles θ_3 and θ_4. This implies that Cliff and David can control the average amount of information transmitted from Alice to Bob by adjusting measurement angles.

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