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Algebraic Structure and Poisson's Integral Theory of $f(R)$ Cosmology

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Abstract In this paper, the algebraic structure and the Poisson's integral theory of $f(R)$ cosmology are presented. Firstly, the Hamilton canonical equations are derived for the system. Secondly, the contravariant algebraic forms of $f(R)$ cosmology are obtained. Thirdly, the Lie algebraic structure admitted and Poisson's integral methods are investigated for $f(R)$ cosmology. Further, the first integrals and solution of $f(R)$ cosmology are given. Finally, an example is given to illustrate the results.

Keywords $f(R)$ cosmology · Algebraic structure · Poisson integral method · First integral

1 Introduction

Extended theories of gravity have become a sort of paradigm in modern physics since they seem to solve several problems of standard general relativity related to cosmology, astrophysics and quantum field theory. In recent years, modified theories of gravity constructed by adding correction terms in the usual Einstein–Hilbert action, have opened a new window to study the accelerated expansion of the universe. It has been shown that such correction terms could give rise to accelerating solutions of the field equations without having to invoke concepts such as dark energy [1]. In a more general setting, one can use a generic function $f(R)$, instead of the usual Ricci scalar R as the action of Hilbert–Einstein formulation. This method relaxing the hypothesis that gravitational Lagrangian has to be a linear function of the Ricci curvature scalar R in Hilbert–Einstein formulation, one can take into account, as a minimal extension, an effective action where the gravitational Lagrangian is

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a generic $f(R)$ function. Such $f(R)$ gravity theories have been extensively studied in the literature over the past few years [2] for a review. Vakili has studied a flat FRW space–time in the framework of the metric formalism of $f(R)$ gravity, and have constructed an effective Lagrangian in the minisuperspace $\{a, R\}$ where a and R being the scale factor and Ricci scalar, respectively [3].

It is well known that many methods, such as Ermakov technique [4, 5], Lutzky's approach [6, 7], group transformation method [8–10], dynamical algebraic method [11–13], and algebraic structure and Poisson's method for constrained mechanical systems [14–16], have been developed to seek invariants of mechanical and physical systems. Among these methods, the invariants of the mechanical systems studied by using Lie group of transformation seem to have an extra advantage of a straightforward extension to the corresponding quantum mechanics, cosmological models and $f(R)$ cosmology [17–22]. Capozziello, Stabile and Troisi searched for spherically symmetric solutions of $f(R)$ theories of gravity via the Noether symmetry approach. A general formalism in the metric framework is developed considering a point-like $f(R)$ Lagrangian [23]. As mentioned above, Vakili presented Noether symmetries of a generic $f(R)$ classical and quantum cosmological models by utilizing the behavior of the corresponding Lagrangian under the infinitesimal generators with respect to the scale factor a and Ricci scalar R [3, 22].

In fact the algebraic structure and the Poisson's theory have been used to study the relativistic Birkhoffian mechanics and electromechanical systems by Fu et al. [24, 25]. The algebraic structures and Poisson's theories of $f(R)$ cosmology which have not been explored so far. In this paper, we make an effort in this direction to obtain the algebraic structure and Poisson's theory of $f(R)$ cosmology, and further study the solutions with respect to this model. It would be of interest to employ such $f(R)$ cosmology model in this study.

2 The Phase Space of the Model

In this section we consider a spatially flat FRW cosmology within the framework of $f(R)$ gravity. Since our goal is to study models which exhibit algebra structure and Poisson's integral approach, we do not include any matter contribution in the action. Let us start from the $(n+1)$-dimensional action (we work in units where $c = 16\pi G = 1$)

$$S = \int d^{n+1}x\sqrt{-g}\,f(R), \tag{1}$$

where R is the scalar curvature and $f(R)$ is an arbitrary function of R. By varying the above action with respect to metric we obtain the equation of motion as

$$\frac{1}{2}g_{\mu\nu}f(R) - R_{\mu\nu}f'(R) + \nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R) = 0, \tag{2}$$

where a prime represents differentiation with respect to R. We assume that the geometry of space–time is described by the flat FRW metric which seems to be consistent with the present cosmological observations

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\sum_{i=1}^{n}(\mathrm{d}x^i)^2. \tag{3}$$

With this background geometry the field equations read [20]

$$(n-1)P + \frac{n(n-1)}{2}P^2 = -\frac{1}{f'}\left[f'''\dot{R}^2 + (n-1)P\dot{R}f'' + f''\ddot{R} + \frac{1}{2}(f - Rf')\right], \quad (4)$$

$$P^2 = \frac{1}{n(n-1)f'}[(f'R - f) - 2n\dot{R}Pf''], \quad (5)$$

where $P = \dot{a}/a$ is the Hubble parameter and a dot represents differentiation with respect to t. To study the algebra structure and Poisson's approach of the minisuperspace under consideration, we need an effective Lagrangian for the model whose variation with respect to its dynamical variables yields the correct equations of motion. Following [3], by considering the action described above as representing a dynamical system in which the scale factor a and scalar curvature R play the role of independent dynamical variables, we can rewrite action (1) as [3]

$$S = \int L(a, \dot{a}, R, \dot{R})dt = \int dt\left\{a^n f(R) - \lambda\left[R - n(n-1)\frac{\dot{a}^2}{a^2} - 2n\frac{\ddot{a}}{a}\right]\right\}, \quad (6)$$

where we introduce the definition of R in terms of a and its derivatives as a constraint. This procedure allows us to remove the second order derivatives from action (6). The Lagrange multiplier λ can be obtained by variation with respect to R, that is, $\lambda = a^n f(R)$. Thus, we obtain the following Lagrangian for the model [3]

$$L(a, \dot{a}, R, \dot{R}) = n(n-1)\dot{a}^2 a^{n-2} f' + 2n\dot{a}\dot{R}a^{n-1}f'' + a^n(f'R - f). \quad (7)$$

The generic momenta with respect to variables a and R are

$$p_a = \frac{\partial L}{\partial \dot{a}} = 2n(n-1)\dot{a}a^{n-2}f' + 2na^{n-1}\dot{R}f'', \quad (8)$$

$$p_R = \frac{\partial L}{\partial \dot{R}} = 2na^{n-1}\dot{a}f''(R). \quad (9)$$

Then we have

$$\dot{a} = \frac{p_R}{2na^{n-1}f''}, \quad (10)$$

$$\dot{R} = \frac{af''p_a - (n-1)p_R f'}{2na^n f''^2}. \quad (11)$$

The Hamiltonian corresponding to Lagrangian (7) can then be written in terms

$$H(a, p_a, R, p_R) = p_a\dot{a} + p_R\dot{R} - L = \frac{ap_R p_a}{2na^n f''} - \frac{(n-1)f'p_R^2}{4na^n f''^2} - a^n(f'R - f). \quad (12)$$

Then, the Hamilton canonical equations associated with Hamiltonian (12) are obtain

$$\dot{p}_a = -\frac{\partial H}{\partial a} = \frac{(n-1)p_R p_a}{2na^n f''} - \frac{(n-1)nf'p_R^2}{4na^{n+1}f''^2} + na^{n-1}(f'R - f), \quad (13)$$

$$\dot{p}_R = -\frac{\partial H}{\partial R} = \frac{ap_a p_R f'''}{2na^n f''^2} + \frac{n(n-1)p_R^2 f''^2 - 2f'f'''}{4na^n f''^3} + na^{n-1}f''R. \quad (14)$$

3 Contravariant Algebraic Form of $f(R)$ Cosmology

Introduce contravariant vectors for the $f(R)$ cosmology

$$a^{\mu} = \begin{cases} q^{\mu}, & (\mu = 1, 2), \\ p_{\mu-2}, & (\mu = 3, 4), \end{cases} \tag{15}$$

where $a_1 = a$, $a_2 = R$, $a_3 = p_a$, $a_4 = p_R$, then Hamiltonian of the $f(R)$ cosmology can be written in the form

$$H(\boldsymbol{a}, \boldsymbol{p}) = H(t, a^{\mu}), \tag{16}$$

where $\boldsymbol{a} = \{a, R\}$, $\boldsymbol{p} = \{p_a, p_R\}$. Using the contravariant tensor

$$(\omega^{\mu\nu}) = \begin{pmatrix} 0_{2\times 2} & I_{2\times 2} \\ -I_{2\times 2} & 0_{2\times 2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{pmatrix}. \tag{17}$$

We express the (10), (11), (13) and (14) the contravariant algebraic form as

$$\dot{a}^{\mu} - \omega^{\mu\nu} \frac{\partial H}{\partial a^{\nu}} = 0, \quad (\mu, \nu = 1, 2, 3, 4). \tag{18}$$

namely

$$\dot{a}^2 = \frac{a^1 f'' a^3 - (n-1) a^4 f'}{2n (a^1)^n f''^2}, \tag{19}$$

$$\dot{a}^2 = \frac{a^1 f'' a^3 - (n-1) a^4 f'}{2n (a^1)^n f''^2}, \tag{20}$$

$$\dot{a}^3 = -\frac{\partial H}{\partial a^1} = \frac{(n-1) a^3 a^4}{2n (a^1)^n f''} - \frac{(n-1) n f' (a^4)^2}{4n (a^1)^{n+1} f''^2} + n (a^1)^{n-1} (f' a^2 - f), \tag{21}$$

$$\dot{a}^4 = -\frac{\partial H}{\partial a^2} = \frac{a^1 a^3 a^4 f'''}{2n (a^1)^n f''^2} + \frac{n(n-1)(a^4)^2 f''^2 - 2 f' f'''}{4n (a^1)^n f''^3} + n (a^1)^{n-1} f'' a^2. \tag{22}$$

4 Algebraic Structure of $f(R)$ Cosmology

Firstly, we study the algebraic structure of $f(R)$ cosmology.

Performing full derivative of function $A(\boldsymbol{a})$ along (18), one has

$$\dot{A} = \frac{\partial A}{\partial a^{\mu}} \omega^{\mu\nu} \frac{\partial H}{\partial a^{\nu}} \quad (\mu, \nu = 1, 2, 3, 4), \tag{23}$$

the right-hand side of (23) is defined as a double-linear product $A \bullet H$, i.e.

$$\frac{\partial A}{\partial a^{\mu}} \omega^{\mu\nu} \frac{\partial H}{\partial a^{\nu}} = A \bullet H, \tag{24}$$

which satisfies the right-hand assignment law

$$A \bullet (B + C) = A \bullet B + A \bullet C, \tag{25}$$

left-hand assignment law

$$(A+B)\bullet C = A\bullet C + B\bullet C, \tag{26}$$

and scalar law

$$(\alpha A)\bullet B = A\bullet(\alpha B) = \alpha(A\bullet B). \tag{27}$$

so (18) possesses the compatible algebraic structure.

Expanding (18) yields

$$\frac{\partial A}{\partial a^{\mu}}\omega^{\mu\nu}\frac{\partial H}{\partial a^{\nu}} = \frac{\partial A}{\partial q^{k}}\frac{\partial H}{\partial p_{k}} - \frac{\partial A}{\partial p_{k}}\frac{\partial H}{\partial q^{k}}, \tag{28}$$

which is the classical Poisson's bracket (A,B), i.e. $(A,B) = A\bullet B$. It is well known that Poisson's bracket possesses the anti-symmetrical property

$$A\circ B + B\circ A = 0, \tag{29}$$

and satisfies Jacobi identical equation

$$A\circ(B\circ C) + B\circ(C\circ A) + C\circ(A\circ B) = 0. \tag{30}$$

Equations (29) and (30) are also called Lie algebra axiom, then one has

Theorem *Equations of motion of $f(R)$ cosmology possess the compatible algebraic structure as well as the Lie algebraic structure.*

5 Poisson's Theory of $f(R)$ Cosmology

We have known that the theoretical foundation of Poission's integral method being equations of motion of systems possess Lie algebraic structure [21]. The $f(R)$ cosmology (23) possesses Lie algebraic structure, then the Poisson's integral methods of conservative holonomic dynamical systems can all be used in the system. Then we have

Proposition 1 *The necessary and sufficient condition on which $I(a^{\mu},t)$ $(\mu = 1,\ldots,n+m)$ is first integral of the $f(R)$ cosmology model (23) is that the $I(a^{\mu},t)$ satisfies*

$$\frac{\partial I}{\partial t} + (I,H) = 0, \tag{31}$$

expanding (31) one has

$$\begin{aligned}
\frac{\partial I}{\partial t} + I\bullet H &= \frac{\partial I}{\partial t} + \frac{\partial I}{\partial a^{\mu}}\omega^{\mu\nu}\frac{\partial H}{\partial a^{\nu}} = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial a^{1}}\dot{a}^{1} + \frac{\partial I}{\partial a^{2}}\dot{a}^{2} + \frac{\partial I}{\partial a^{3}}\dot{a}^{3} + \frac{\partial I}{\partial a^{4}}\dot{a}^{4} \\
&= \frac{\partial I}{\partial t} + \frac{\partial I}{\partial a^{1}}\frac{a^{4}}{2n(a^{1})^{n-1}f''} + \frac{\partial I}{\partial a^{2}}\frac{a^{1}f''a^{3} - (n-1)a^{4}f'}{2n(a^{1})^{n}f''^{2}} \\
&\quad + \frac{\partial I}{\partial a^{3}}\left\{\frac{(n-1)a^{3}a^{4}}{2n(a^{1})^{n}f''} - \frac{(n-1)nf'(a^{4})^{2}}{4n(a^{1})^{n+1}f''^{2}} + n(a^{1})^{n-1}(f'a^{2} - f)\right\}
\end{aligned}$$

$$+\frac{\partial I}{\partial a^4}\left\{\frac{a^1a^3a^4f'''}{2n(a^1)^nf''^2}+\frac{n(n-1)(a^4)^2f''^2-2f'f'''}{4n(a^1)^nf''^3}+n(a^1)^{n-1}f''a^2\right\}$$

$$=0 \tag{32}$$

this is a one-order linear homogeneous partial differential equation which the characteristic equation is

$$\begin{aligned}\frac{\mathrm{d}t}{1}&=\frac{2n(a^1)^{n-1}f''\mathrm{d}a^1}{a^4}=\frac{2n(a^1)^nf''^2\mathrm{d}a^2}{a^1f''a^3-(n-1)a^4f'}\\&=\frac{\mathrm{d}a^3}{\frac{(n-1)a^3a^4}{2n(a^1)^nf''}-\frac{(n-1)nf'(a^4)^2}{4n(a^1)^{n+1}f''^2}+n(a^1)^{n-1}(f'a^2-f)}\\&=\frac{da^4}{\frac{a^1a^3a^4f'''}{2n(a^1)^nf''^2}+\frac{n(n-1)(a^4)^2f''^2-2f'f'''}{4n(a^1)^nf''^3}+n(a^1)^{n-1}f''a^2},\end{aligned} \tag{33}$$

where, *we taking $f(R)$ cosmology as* [3]

$$f(R)=R^{\frac{3}{2}}=(a^2)^{\frac{3}{2}}, \tag{34}$$

the characteristic equation is written in the form

$$\begin{aligned}\frac{\mathrm{d}t}{1}&=\frac{3(a^1)^{n-1}\mathrm{d}a^1}{2(a^2)^{\frac{1}{2}}a^4}=\frac{3n(a^1)^n\mathrm{d}a^2}{2a^1a^3(a^2)^{\frac{1}{2}}-4(n-1)a^4(a^2)^{\frac{3}{2}}}\\&=\frac{\mathrm{d}a^3}{\frac{2(n-1)a^3a^4(a^2)^{\frac{1}{2}}}{3n(a^1)^n}-\frac{2}{3}\frac{(n-1)(a^2)^{\frac{3}{2}}(a^4)^2}{(a^1)^{n+1}}+\frac{3}{2}n(a^1)^{n-1}(a^2)^{\frac{3}{2}}}\\&=\frac{\mathrm{d}a^4}{\frac{a^3a^4}{3n(a^1)^{n-1}(a^2)^{\frac{1}{2}}}+\frac{(n-1)(a^4)^2(a^2)^{\frac{1}{2}}}{(a^1)^n}+\frac{3}{4}n(a^1)^{n-1}(a^2)^{\frac{1}{2}}}.\end{aligned} \tag{35}$$

Equation (35) is called the Poisson's condition of the first integral for $f(R)$ cosmology. In (35),

$$\frac{3(a^1)^{n-1}\mathrm{d}a^1}{2(a^2)^{\frac{1}{2}}}=\frac{3n(a^1)^n\mathrm{d}a^2}{2a^1a^3(a^2)^{\frac{1}{2}}-4(n-1)a^4(a^2)^{\frac{3}{2}}}$$

then, we can obtain the first integral

$$I_1=a^2(a^1)^{\frac{4(n-1)a^4}{n}}-\frac{a^3}{4(n-1)a^4}(a^1)^{\frac{4(n-1)a^4+n}{n}}=C_1. \tag{36}$$

Equation (35) result in

$$\frac{3(a^1)^{n-1}\mathrm{d}a^1}{2(a^2)^{\frac{1}{2}}}=\frac{\mathrm{d}a^3}{\frac{2(n-1)a^3a^4(a^2)^{\frac{1}{2}}}{3n(a^1)^n}-\frac{2}{3}\frac{(n-1)(a^2)^{\frac{3}{2}}(a^4)^2}{(a^1)^{n+1}}+\frac{3}{2}n(a^1)^{n-1}(a^2)^{\frac{3}{2}}},$$

then one has

$$I_2 = a^3(a^1)^{-(n-1)a^4} + (n-1)a^2(a^4)^2\frac{1}{(n-1)a^4-1}(a^1)^{-(n-1)a^4-1} - \frac{9a^2}{4(n-1)a^4+2n-1}(a^1)^{-(n-1)a^4+2n-1} = C_2. \tag{37}$$

We eliminating a^3 by first integrals (36) and (37) lead to

$$\begin{aligned}4(a^1)^{1-(n-1)a^4}C_1 = (a^1)^{\frac{-4}{n}}\Bigg(&4(a^1)^{-(n-1)d+5}a^2 - \frac{9(a^1)^{1-(n-1)a^4+\frac{4}{n}-\frac{4(n-1)a^4}{n}+2n}}{(n-1)a^4}a^2 \\ &- \frac{36(a^1)^{1-(n-1)a^4+\frac{4}{n}-\frac{4(n-1)a^4}{n}+2n}}{1-4(n-1)a^4-2n}a^2 - \frac{18(a^1)^{1-(n-1)a^4+\frac{4}{n}-\frac{4(n-1)a^4}{n}+2n}}{(1-4(n-1)a^4-2n)a^4}a^2 \\ &- \frac{(a^1)^{1+(n-1)a^4+\frac{4}{n}-\frac{4(n-1)a^4}{n}+2n}}{1-(n-1)a^4}a^2a^4 - \frac{(a^1)^{2+\frac{4}{n}-\frac{4(n-1)a^4}{n}}}{(n-1)a^4}C_2\Bigg).\end{aligned} \tag{38}$$

In (35),

$$\frac{da^1}{1} = \frac{da^4}{\frac{a^3a^4}{na^2} + \frac{3(n-1)(a^4)^2}{2a^1} + \frac{9}{8}n(a^1)^{2n-2}},$$

namely

$$\frac{da^4}{da^1} - \frac{a^3a^4}{na^2} - \frac{3(n-1)(a^4)^2}{2a^1} = \frac{9}{8}n(a^1)^{2n-2}, \tag{39}$$

which can be expressed as a Riccati equation. It is very known that we to obtain generic solution is difficult from Riccati equation. Now, we given the following several special solutions of (39):
when $n=1$, the (38) can be written as

$$\frac{da^4}{da^1} - \frac{a^3}{a^2}a^4 = \frac{9}{8}, \tag{40}$$

then, one has

$$I_3 = \left(a^4 + \frac{9a^3}{8a^2}\right)e^{-\frac{a^3}{a^2}a^1} = C_3. \tag{41}$$

When $n=2$, the (38) can be expressed as

$$\frac{da^4}{da^1} - \frac{a^3a^4}{2a^2} - \frac{3(a^4)^2}{2a^1} = \frac{9}{4}(a^1)^2, \tag{42}$$

which has the series solution

$$I_4 = a^4 - \frac{3}{4}(a^1)^3 - \frac{3a^3}{16a^2}(a^1)^4 - \frac{(a^3)^2}{160(a^2)^2}(a^1)^5 - \left[\frac{(a^3)^3}{1920(a^2)^3} + \frac{9}{64}\right](a^1)^6 + \cdots = C_4. \tag{43}$$

Similar, when $n = 3, 4, 5, \ldots$, we can obtain the first integrals I_i $(i = 3, 4, 5, \ldots)$.

From (35), one has

$$\frac{\mathrm{d}t}{1} = \frac{3(a^1)^{n-1}\mathrm{d}a^1}{2(a^2)^{\frac{1}{2}}a^4}. \tag{44}$$

Using first integrals (38), the a^4 can be written in the form

$$a^4 = a^4(a^1, a^2, C_1, C_2). \tag{45}$$

On the other hand, we give the following propositions:

Proposition 2 *For the Hamiltonian of the $f(R)$ cosmology does not depend explicitly on time t, then the Hamiltonian of system is first integral, i.e.*

$$\begin{aligned} I_5 = H &= \frac{ap_R p_a}{2na^n f''} - \frac{(n-1)f' p_R^2}{4na^n f''^2} - a^n(f'R - f) \\ &= \frac{8a^1a^2a^3a^4}{9n(a^1)^n} - \frac{2(n-1)(a^2)^{\frac{3}{2}}(a^4)^2}{3n(a^1)^n} - \frac{3}{2}(a^1)^n\big((a^2)^{\frac{3}{2}} - (a^2)^{\frac{1}{2}}\big) = C_5, \end{aligned} \tag{46}$$

where C_5 is an arbitrary constant.

Now, substituting (45) and (46) into (44), we can obtain the first integral

$$I_5 = I_5(a^1, a^2, C_1, C_2, C_5, t), \tag{47}$$

where C_i $(i = 1, 2, 5.6)$ are arbitrary constants.

Using (36), (37), (46) and (47), we obtain the following solution of $f(R)$ cosmology:

$$a^1 = a = a(t, C_1, C_2, C_5, C_6), \tag{48}$$

$$a^2 = R = R(t, C_1, C_2, C_5, C_6), \tag{49}$$

$$a^3 = p_a = p_a(t, C_1, C_2, C_5, C_6), \tag{50}$$

$$a = p_R = p_R(t, C_1, C_2, C_5, C_6). \tag{51}$$

Proposition 3 *If the system of $f(R)$ cosmology (18) possesses two first integrals $I_1(a^\mu, t)$ and $I_2(a^\mu, t)$ having not involution, their Poisson's bracket (I_1, I_2) is also the first integral of the system.*

Proof Supposing that the system of $f(R)$ cosmology (18) possesses two first integrals having not involution

$$I_1(a^\mu, t) = c_1, \qquad I_2(a^\mu, t) = c_2 \quad (\mu = 1, 2, 3, 4), \tag{52}$$

which satisfy Poisson's conditions

$$\frac{\partial I_1}{\partial t} + (I_1, H) = 0, \qquad \frac{\partial I_2}{\partial t} + (I_2, H) = 0. \tag{53}$$

Performing the following operation

$$\frac{\partial}{\partial t}(I_1, I_2) = \left(\frac{\partial I_1}{\partial t}, I_2\right) + \left(I_1, \frac{\partial I_2}{\partial t}\right) + \frac{\partial I_1}{\partial a^\mu}\frac{\partial \omega^{\mu\nu}}{\partial t}\frac{\partial I_2}{\partial t}, \tag{54}$$

in respect that

$$\frac{\partial \omega^{\mu\nu}}{\partial t} = 0,$$

then (53) becomes

$$\frac{\partial}{\partial t}(I_1, I_2) = \left(\frac{\partial I_1}{\partial t}, I_2\right) + \left(\frac{\partial I_2}{\partial t}, I_1\right). \tag{55}$$

Using Lie algebraic axioms (29), (30) leads to

$$((I_1, I_2), H) = -((I_2, H), I_1) - ((H, I_1), I_2) = (I_1, (I_2, H)) + ((I_1, H), I_2). \tag{56}$$

Combining (54) and (56) and considering (53) yields

$$\frac{\partial}{\partial t}(I_1, I_2) + ((I_1, I_2), H) = \left(\left(\frac{\partial I_1}{\partial t}\right) + (I_1, H), I_2\right) + \left(I_1, \frac{\partial I_2}{\partial t} + (I_2, H)\right) = 0, \tag{57}$$

then (I_1, I_2) being also a first integral of the system of $f(R)$ cosmology (18). □

Proposition 4 *If the system of $f(R)$ cosmology (18), which possesses a first integral $I(a^\mu, t)$ containing t, and Hamiltonian does not depend explicitly on t, then $\frac{\partial I}{\partial t}, \frac{\partial^2 I}{\partial t^2}, \ldots$, are also first integrals of the system.*

Proof By partially differentiating (37) with respect to t, we obtain

$$\frac{\partial}{\partial t}\frac{\partial I}{\partial t} + \left(\frac{\partial I}{\partial t}, H\right) + \left(I, \frac{\partial H}{\partial t}\right) = 0,$$

where H does not depending explicitly on t, and

$$\frac{\partial}{\partial t}\frac{\partial I}{\partial t} + \left(\frac{\partial I}{\partial t}, H\right) = 0, \tag{58}$$

where $\frac{\partial I}{\partial t}$ is a first integral of (18). Similarly, one can prove that $\frac{\partial^2 I}{\partial t^2}, \frac{\partial^3 I}{\partial t^3}, \ldots$, are also the first integrals of the system.

For example, for the system of $f(R)$ cosmology (18), the first integral (47) obvious containing t and the Hamiltonian does not one, then $\frac{\partial I_5}{\partial t}, \frac{\partial^2 I_5}{\partial^2 t}, \ldots$ are also the first integrals of the system. □

Proposition 5 *If the system of $f(R)$ cosmology (18), which possesses a first integral $I(a^\mu, t)$ containing a^ρ, and Hamiltonian H does not depend explicitly on a^ρ, then $\frac{\partial I}{\partial a^\rho}, \frac{\partial^2 I}{\partial a^{\rho 2}}, \ldots$, are also first integrals of the system.*

Proof By partially differentiating (37) with respect to a^ρ, we have

$$\frac{\partial}{\partial a^\rho}\frac{\partial I}{\partial t} + \left(\frac{\partial I}{\partial a^\rho}, H\right) + \left(I, \frac{\partial H}{\partial a^\rho}\right) = 0,$$

where H does not depend explicitly on a^ρ, and

$$\frac{\partial}{\partial t}\frac{\partial I}{\partial a^\rho} + \left(\frac{\partial I}{\partial a^\rho}, H\right) = 0, \tag{59}$$

therefore $\frac{\partial I}{\partial a^\rho}$ is a first integral of the (18). Similarly we can prove that $\frac{\partial^2 I}{\partial a^{\rho^2}}, \frac{\partial^3 I}{\partial a^{\rho^3}}, \dots$, are also first integrals of the system.

In the Hamiltonian of (12) including all a^ρ, then the $f(R)$ cosmology (12) have not the first integrals of $\frac{\partial I}{\partial a^\rho}, \frac{\partial^2 I}{\partial a^{\rho^2}}, \dots$. □

6 An Example

In this section we consider a spatially flat FRW cosmology within the framework of $f(R)$ gravity. Since our aim is to the first integral which exhibit Poisson's method, one does not include any matter contribution in the action. Let us tart from a string-dilaton four-dimensional effective action, neglecting the torsion terms and other scalar fields except the dilation φ, is [26]

$$S = \int \mathrm{d}^4 x \sqrt{-g} e^{-\varphi} [(R + 4 g^{\mu\nu} \varphi_\mu \varphi_\nu - 2\Lambda(R))], \tag{60}$$

this action is nothing else than a particular case of the most general action, then we have

$$R = 2e^{-\varphi}, \quad f(R) = \frac{1}{8} R^2 = \frac{1}{2} e^{-2\varphi}, \quad V(R) = e^{-2\varphi} V(\varphi). \tag{61}$$

In a FRW flat metric, action (60) gives rise to a Lagrangian density,

$$L = e^{-2\varphi}(3\dot{a}^2 a - 6\dot{a}a^2\dot{\varphi} + 2a^3\dot{\varphi}^2 - a^3 V(\varphi)). \tag{62}$$

We find that a particular form of $V(\varphi)$ for the existence of a Noether symmetry is [26]

$$V(\varphi) = e^{(-5\pm\sqrt{3})\varphi}. \tag{63}$$

We introduce the generic momenta with respect to variables a and φ as

$$P_a = \frac{\partial L}{\partial \dot{a}} = e^{-2\varphi}[6a\dot{a} - 6a^2\dot{\varphi}], \quad P_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = e^{-2\varphi}[-6\dot{a}a^2 + 4a^3\dot{\varphi}] \tag{64}$$

then, one has

$$\dot{a} = -e^{2\varphi} \frac{2aP_a + 3P_\varphi}{6a^2}, \qquad \dot{\varphi} = -e^{2\varphi} \frac{aP_a + P_\varphi}{2a^3}. \tag{65}$$

The Hamiltonian corresponding to Lagrangian (62) can then be written in terms

$$\begin{aligned} H &= P_a\dot{a} + P_\varphi\dot{\varphi} - e^{-2\varphi}(3\dot{a}^2 a - 6\dot{a}a^2\dot{\varphi} + 2a^3\dot{\varphi}^2 - e^{(-1\pm\sqrt{3})\varphi} a^3) \\ &= -e^{2\varphi} \frac{2aP_a^2 + 6aP_a P_\varphi + 3P_\varphi^2}{12a^3} + a^3 e^{(-3\pm\sqrt{3})\varphi}. \end{aligned} \tag{66}$$

Let

$$a_1 = a, \qquad a_2 = \varphi, \qquad a_3 = p_a, \qquad a_4 = p_R, \tag{67}$$

then Hamiltonian of the $f(R)$ cosmology can be written in the form

$$H = -e^{2a_2} \frac{2a_1^2 a_3^2 + 6a_1 a_3 a_4 + 3a_4^2}{12a_1^3} + a_1^3 e^{(-3\pm\sqrt{3})a_2}. \tag{68}$$

We express the equations of motion for systems the contravariant algebraic form as

$$\begin{pmatrix} \dot a_1 \\ \dot a_2 \\ \dot a_3 \\ \dot a_4 \end{pmatrix} - \begin{pmatrix} -e^{2a_2}\dfrac{2a_1^2a_3+3a_1a_4}{6a_1^3} \\ -e^{2a_2}\dfrac{a_1a_3+a_4}{2a_1^3} \\ -e^{2a_2}\dfrac{a_1a_3a_4+a_4^2}{2a_1^4}-3a_1^2e^{(-3\pm\sqrt{3})a_2} \\ e^{2a_2}\dfrac{2a_1^2a_3^2+6a_1a_3a_4+3a_4^2}{12a_1^3}-(-3\pm\sqrt{3})a_1^3e^{(-3\pm\sqrt{3})a_2} \end{pmatrix} = 0, \tag{69}$$

i.e.

$$\dot a_1 = -e^{2a_2}\frac{2a_1^2a_3+3a_1a_4}{6a_1^3}, \tag{70}$$

$$\dot a_2 = -e^{2a_2}\frac{a_1a_3+a_4}{2a_1^3}, \tag{71}$$

$$\dot a_3 = -e^{2a_2}\frac{a_1a_3a_4+a_4^2}{2a_1^4}-3a_1^2e^{(-3\pm\sqrt{3})a_2}, \tag{72}$$

$$\dot a_4 = e^{2a_2}\frac{2a_1^2a_3^2+6a_1a_3a_4+3a_4^2}{12a_1^3}-(-3\pm\sqrt{3})a_1^3e^{(-3\pm\sqrt{3})a_2}. \tag{73}$$

We can prove that the Lagrangian (62) of $f(\varphi)$ cosmology possesses Lie algebraic structure, then the Poisson's integral methods of conservative holonomic dynamical systems can all be used in the system.

The necessary and sufficient condition on which $I(a^\mu,t)$ $(\mu=1,\dots,n+m)$ is first integral of the $f(R)$ cosmology model (23) is that the $I(a^\mu,t)$ satisfies

$$\begin{aligned} \frac{\partial I}{\partial t}+I\bullet H &= \frac{\partial I}{\partial t}+\frac{\partial I}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}=\frac{\partial I}{\partial t}+\frac{\partial I}{\partial a^1}\dot a^1+\frac{\partial I}{\partial a^2}\dot a^2+\frac{\partial I}{\partial a^3}\dot a^3+\frac{\partial I}{\partial a^4}\dot a^4 \\ &= \frac{\partial I}{\partial t}+\frac{\partial I}{\partial a^1}\left(-e^{2a_2}\frac{2a_1^2a_3+3a_1a_4}{6a_1^3}\right)+\frac{\partial I}{\partial a^2}\left(-e^{2a_2}\frac{a_1a_3+a_4}{2a_1^3}\right) \\ &\quad+\frac{\partial I}{\partial a^3}\left(-e^{2a_2}\frac{a_1a_3a_4+a_4^2}{2a_1^4}-3a_1^2e^{(-3\pm\sqrt{3})a_2}\right) \\ &\quad+\frac{\partial I}{\partial a^4}\left(e^{2a_2}\frac{2a_1^2a_3^2+6a_1a_3a_4+3a_4^2}{12a_1^3}-(-3\pm\sqrt{3})a_1^3e^{(-3\pm\sqrt{3})a_2}\right) \\ &= 0, \end{aligned} \tag{74}$$

the characteristic equation is written in the form

$$\begin{aligned} \frac{\mathrm{d}t}{1} &= -\frac{6a_1^3\mathrm{d}a_1}{e^{2a_2}(2a_1^2a_3+3a_1a_4)} = -\frac{2a_1^3\mathrm{d}a_2}{e^{2a_2}(a_1a_3+a_4)} \\ &= -\frac{2a_1^4da_3}{e^{2a_2}(a_1a_3a_4+a_4^2)+6a_1^6e^{(-3\pm\sqrt{3})a_2}} \end{aligned}$$

$$= \frac{12a_1^3 da_4}{e^{2a_2}(2a_1^2a_3^2 + 6a_1a_3a_4 + 3a_4^2) - 12(-3 \pm \sqrt{3})a_1^6 e^{(-3\pm\sqrt{3})a_2}}. \tag{75}$$

Equation (75) result in

$$\frac{3(a_1a_3 + a_4)\mathrm{d}a_1}{(2a_1^2a_3 + 3a_1a_4)} = \frac{\mathrm{d}a_2}{1} \tag{76}$$

then we obtain the first integral

$$I_1 = \frac{3}{4}\ln(2a_1^2a_3 + 3a_1a_4) + \frac{a_4}{4}\ln a^1 - \frac{a_4}{4}\ln(2a_1a_3 + 3a_4) - a_2$$
$$= \frac{1}{4}\ln(2a_1^2a_3 + 3a_1a_4)^3\left(\frac{a^1}{2a_1a_3 + 3a_4}\right)^{a_4} - a_2 = \mathrm{C}_1. \tag{77}$$

From (75) one has

$$-\frac{2a_1^3\mathrm{d}a_2}{e^{2a_2}(a_1a_3 + a_4)} = -\frac{2a_1^4 da_3}{e^{2a_2}(a_1a_3a_4 + a_4^2) + 6a_1^6 e^{(-3\pm\sqrt{3})a_2}}, \tag{78}$$

which is been expressed as

$$\frac{da_3}{da_2} = \frac{a_4}{a_1} + \frac{6a_1^5}{a_4(1 + \frac{a_1}{a_4}a_3)}e^{(-5\pm\sqrt{3})a_2}, \tag{79}$$

we let $a_1a_3 < a_4$, and expanding $\frac{1}{1+\frac{a_1}{a_4}a_3} \doteq 1 - \frac{a_1}{a_4}a_3$ in series, then (79) is written as

$$\frac{da_3}{da_2} + \frac{6a_1^6}{a_4^2}a^3 e^{(-5\pm\sqrt{3})a_2} = \frac{a_4^2 + 6a_1^6}{a_1a_4} \tag{80}$$

which has solution

$$a_3 = e^{-\frac{6a_1^6}{a_4^2(-5\pm\sqrt{3})}e^{(-5\pm\sqrt{3})a_2}}\left[\frac{a_4^2(a_4^2 + 6a_1^6)}{6a_1^7a_4}e^{\frac{6a_1^6}{a_4^2(-5\pm\sqrt{3})}e^{(-5\pm\sqrt{3})a_2}} + I_2\right], \tag{81}$$

i.e.

$$I_2 = a_3 e^{\frac{6a_1^6}{a_4^2(-5\pm\sqrt{3})}e^{(-5\pm\sqrt{3})a_2}} - \frac{a_4^2(a_4^2 + 6a_1^6)}{6a_1^7a_4} = a^3 e^{(\frac{6a_1^6}{a_4^2(-5\pm\sqrt{3})} + \frac{6a_1^6}{a_4^2}a_2)} - \frac{a_4^2(a_4^2 + 6a_1^6)}{6a_1^7a_4} = \mathrm{C}_2. \tag{82}$$

In (75)

$$-\frac{\mathrm{d}a_2}{(a_1a_3 + a_4)} = \frac{6da_4}{(2a_1^2a_3^2 + 6a_1a_3a_4 + 3a_4^2) - 12(-3 \pm \sqrt{3})a_1^6 e^{(-5\pm\sqrt{3})a_2}}, \tag{83}$$

then one has

$$\frac{da_4}{da_2} = -\frac{a_3}{3} - \frac{a_4}{2} - \frac{a_1a_3}{6(1 + \frac{a_1a_3}{a_4})} + \frac{a_1^6(-3 \pm \sqrt{3})e^{(-5\pm\sqrt{3})a_2}}{a_4(1 + \frac{a_1a_3}{a_4})} \tag{84}$$

which is a nonlinear differential equation. It is very known that we to obtain a first integral I_3 is difficult from this equation.

In (75)

$$\frac{\mathrm{d}t}{1} = -\frac{6a_1^2 \mathrm{d}a_1}{e^{2a_2}(2a_1a_3 + 3a_4)}. \tag{85}$$

On the other hand, for the Hamiltonian of the $f(\varphi)$ cosmology does not depend explicitly on time t, then the Hamiltonian of system is first integral, i.e.

$$I_4 = H = -e^{2a_2}\frac{2a_1^2a_3^2 + 6a_1a_3a_4 + 3a_4^2}{12a_1^3} + a_1^3 e^{(-3\pm\sqrt{3})a_2} = \mathrm{C}_4. \tag{86}$$

Using first integrals (77), (82) and (86), the a^4 and a^3 can be written in the forms

$$a^3 = a^3(a^1, C_1, C_2, C_4), \qquad a^4 = a^4(a^1, C_1, C_2, C_4), \tag{87}$$

substituting (87) into (85), we can obtain the first integral

$$I_5 = I_5(a^1, C_1, C_2, C_4, t). \tag{88}$$

Using first integrals (77), (82), (86) and (88), we obtain the following solution of $f(\varphi)$ cosmology

$$\begin{aligned} a^1 &= a = a(C_1, C_2, C_4, C_5, t), \\ a^2 &= \varphi = \varphi(C_1, C_2, C_4, C_5, t), \\ a^3 &= P_a = P_a(C_1, C_2, C_4, C_5, t), \\ a^4 &= P_\varphi = P_\varphi(C_1, C_2, C_4, C_5, t), \end{aligned} \tag{89}$$

where P_a and P_ϕ are given by (64).

7 Conclusion

The algebraic structure and the Poisson's theory to $f(R)$ cosmology are studied in this paper. The results here present significant approaches to seeking for first integrals and solution in $f(R)$ cosmology.

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