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Unified Field Theoretical Models from Generalized Affine Geometries III

Diego Julio Cirilo-Lombardo

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Abstract In this work the underlying structure of new type of Unified Field Theoretical model introduced in by the authors is elucidated and analyzed from the geometrical and group theoretical point of view. Our goal is to take advantage of the geometrical and topological properties of this theory in order to determine the minimal group structure of the resultant spacetime manifold able to support a fermionic structure. From this fact, the relation between antisymmetric torsion and Dirac structure of the spacetime is determined and important physical consequences enumerated. In the case of spacetime with torsion the real meaning of the spin-frame alignment is find and the question of the minimal coupling is discussed based in the important cases of tratorial, totally antisymmetric and general torsion fields.

Keywords Theories of gravitation · Affine geometries · Unified field theories

1 Motivation and Summary of the Results

As was pointed out by us in later works [1, 2], the cornerstone of the problem of the Unification is where to start conceptually to reformulate the theoretical arena where the Fundamental Theory will be placed, and where the geometry is the unifying essence. According to Mach spacetime doesn't exists without matter. Then, two basic ideas immediately arise to fulfill the observation given by Mach: the concept of dualistic or non-dualistic theories. In the first one the simplest and economical description can be formulated in terms of the gravitational field without torsion plus the energy momentum tensor that, however, is added "by hand" in order to cover the lack of knowledgement of a fundamental structure of the space

D.J. Cirilo-Lombardo (✉)
Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Moscow region, Russian Federation
e-mail: diego@thsun1.jinr.ru

D.J. Cirilo-Lombardo
Departamento de Fisica da Universidade Federal de Paraiba, 58051-970 João Pessoa, Paraiba, Brazil

time giving the matter plus energy distribution. In the second one there are not prescriptions for the interaction of gravity with the "matter" fields because they are arising from the same fundamental geometrical structure.

In previous works of the authors we present a new model of a non-dualistic Unified Theory. Although this nice model was studied in the most important cases of torsion (namely, the "trator" type and the totally antisymmetric case), the geometrical and mathematical framework introduced and several physical aspects discussed, many questions still remain without concrete answer. Basically the lack of knowledgement is because the basic underlying fermionic (Dirac) structure arising from the symmetries of the spacetime manifold was not analyzed in the previous references and (from the point of view of the exact solutions) the coexistence of the two types of torsion in the cosmological framework was not considered there.

Following the guidelines of our last works [1–3], here we complete the previous analysis considering the same fundamental model of UFT. The organization of the paper with the corresponding results is as follows: In Sect. 2 we show that, contrarily to the case of the Poincare theory of gravitation (see reference [4]), the possibility in our Theory of the co existence of both types of torsion in cosmological spacetimes certainly exists.

In Sect. 3 the fermionic structure of the spacetime is described and the possibility of geometrical unification realized: a unified theory of QED and GR can be derived from P(G,M), the Principal Fiber Bundle of frames over the 4D spacetime manifold with G as its structure group. Sections 4, 5 and 6 the action of the UFT is analyzed from the group-theoretical point of view, it is the G-symmetry of the model. In Sect. 7 the derivation of the Dirac equation from the G-manifold, the relation between the electromagnetic field/fermionic structure of the spacetime and the contribution of the torsion to the giromagnetic factor are explicitly shown. However, the physical consequences are explained. Finally, Sect. 8 is devoted to discuss the cohomological interplay between the fields involved in the spacetime structure and in Sect. 9 the concluding remarks are given.

2 Exact Solutions in UFT Theory: General Case

Let to assume the full form (71) of reference [2] for T^a

$$
\begin{aligned}
{}^*\mathbb{F}^a \equiv Mh\Big\{&h\delta^a_d\left(\partial_\tau \ln a\right)\varepsilon^{d0}{}_{ed}\sigma^e\wedge\sigma^d \\
&+\frac{h}{a}\left[\xi\left(\delta^a_i u_j-\delta^a_j u_i\right)+\varsigma h_\delta\varepsilon^{\delta a}{}_{ji}\right]\varepsilon^{ij}{}_{kl}\omega^k\wedge\omega^l \\
&+(-2h+h^2)\varepsilon^a{}_{bc}\varepsilon^{bc}{}_{0d}d\tau\wedge\sigma^a\Big\}
\end{aligned}
\tag{1}
$$

here, in order to avoid the cumbersome expression in the second term due the obvious splitting, $ij=0,a,b,c$ and the ω^k are the corresponding 1-forms $(d\tau,\sigma^a..)$ wherever the case. The YM type equation can be written as

$$
\begin{aligned}
&d^*\mathbb{F}^a+\frac{1}{2}\varepsilon^{abc}\left(A_b\wedge^*\mathbb{F}_c-{}^*\mathbb{F}_b\wedge A_c\right) \\
&\quad=Mh\left\{\left[h\delta^a_b\left(\partial_\tau\partial_\tau\ln a\right)+\partial_\tau\left(\frac{h}{a}\left(\xi\left(\delta^a_b u_0-\delta^a_0 u_b\right)+\varsigma h_c\varepsilon^{ca}{}_{b0}\right)\right)\right]\varepsilon^{b0}{}_{ed}d\tau\wedge\sigma^e\wedge\sigma^d\right.
\end{aligned}
$$

$$+\left[h\delta^a_b\left(\partial_\tau \ln a\right)+\frac{h}{a}\left(\xi\left(\delta^a_b u_0-\delta^a_0 u_b\right)+\varsigma h_c \varepsilon^{ca}{}_{b0}\right)\right]2d\left(\sigma^e\wedge\sigma^d\right)\Big\}$$

$$+M\left[\frac{h}{a}\left(\xi\left(\delta^a_b u_0-\delta^a_0 u_b\right)+\varsigma h_c \varepsilon^{ca}{}_{b0}\right)+\left(-2h+h^2\right)\right](h-1)d\tau\wedge\sigma^b\wedge\sigma^c=0 \quad (2)$$

from the above equation we obtain information about the determination of the non abelian f field, as in the previous cases, and on the determination of the torsion field: The first term

$$\left[h\delta^a_b\left(\partial_\tau\partial_\tau \ln a\right)+\partial_\tau\left(\frac{h}{a}\left(\xi\left(\delta^a_b u_0-\delta^a_0 u_b\right)+\varsigma h_c \varepsilon^{ca}{}_{b0}\right)\right)\right]=0 \quad (3)$$

leads immediately

$$\begin{aligned}
&\left[\eta_{ab}\partial_0 a+\left(\xi\left(\eta_{ab}u_0-\eta_{a0}u_b\right)+\varsigma h_c\varepsilon^{ca}{}_{b0}\right)\right]=\Xi^A_{ab0}+\Xi^s_{ab0}\\
&\Rightarrow\quad \varsigma h_c\varepsilon^c{}_{ab0}\equiv\Xi^A_{ab0}\\
&\Rightarrow\quad \eta_{ab}\partial_0 a+\xi\left(\eta_{ab}u_0-\eta_{a0}u_b\right)=\Xi^S_{ab0}
\end{aligned} \quad (4)$$

where the tensor $\Xi_{ab0}=\Xi^A_{ab0}+\Xi^S_{ab0}$ is independent of the time, and the superscripts A and S indicate the totally antisymmetric part of the another non-totally antisymmetric. Then, the second and third equalities above follows. Is not difficult to see, that contracting indices, tracing and considering the symmetries involved, we obtain explicitly

$$T^a_{b0}=\delta^a_{[b}\,\partial_{0]}a-a\,\widetilde{\Xi}^{Sa}{}_{b0}+\Xi^A_{ab0} \quad (5)$$

$$T^a_{bc}=-a\widetilde{\Xi}^{Sa}{}_{bc}+\varsigma h_0\varepsilon^{0a}{}_{bc} \quad (6)$$

$$T^0_{bc}=-a\widetilde{\Xi}^{S0}{}_{bc}+\varsigma h_c\varepsilon^{c0}{}_{bc} \quad (7)$$

where the integration tensor (independent on time) are related with u_i and $\widetilde{\Xi}^{Sj}{}_{kl}(ij..=0,a,b,c)$ as follows: $u_c=-\frac{a\Xi^S_c}{2\xi}, u_0=-\frac{1}{2\xi}(3\partial_0 a+a\Xi^S_c)$, $\Xi^S_c\equiv\Xi^{Sj}{}_{cj}$, $\Xi^S_0\equiv\Xi^{Sj}{}_{0j}$ and $\widetilde{\Xi}^{Sj}{}_{kl}\equiv\frac{-1}{2}(\delta^j_k\,\Xi^S_l-\delta^j_l\,\Xi^S_k)$. The last term, however, indicate us that there exist a simplest solution with $h=1$, as the previous case for the non abelian f. Then $f^a_{bc}=-\frac{\varepsilon^a_{bc}}{a^2}$, $f^a_{b0}=0$ again, and the second is identically zero due the symmetry of the torsion 2-form with respect to the tetrad defined by (2) of reference [2]. Now the question is if the system of equations is overdetermined or not: h^a and a are without determine. To this end, we carry the information into the expressions (5)–(7) to the second equation of the set of ref. [2], namely (19b). Again, the symmetry involved both: from the equations

$$\nabla_i T^i_{ab}+2T_i T^i_{ab}=-\lambda f^c_{ab}e_c \quad (8)$$

$$\nabla_i T^i_{ao}+2T_i T^i_{a0}=0 \quad (9)$$

fix the torsion tensor components as

$$T^a_{b0}=\delta^a_{[b}\,\partial_{0]}a \quad (10)$$

$$T^a_{bc}=-a\widetilde{\Xi}^{Sa}{}_{bc}+\varsigma h_0\varepsilon^{0a}{}_{bc} \quad (11)$$

$$T^0_{bc}=0. \quad (12)$$

Expression (9) turns a null identity, and from (8) only

$$
\begin{aligned}
4T_i T^i_{ab} &= 4a\widetilde{\Xi}^S_c \overbrace{\left(-a\widetilde{\Xi}^{Sa}{}_{bc} + \varsigma h_0 \varepsilon^{0a}{}_{bc}\right)}^{T^a_{bc}} = -\lambda f^c_{ab} e_c \\
&\Rightarrow \quad 4\left(a^2 \widetilde{\Xi}^S_c \widetilde{\Xi}^{Sc}{}_{ab} - a\widetilde{\Xi}^S_c \varsigma h_0 \varepsilon^{0c} \varepsilon_{ab}\right) = -\lambda f^c_{ab} e_c \\
& a\widetilde{\Xi}^S_c \varsigma h_0 \varepsilon^{0c} \varepsilon_{ab} = -\lambda f^c_{ab} e_c = \frac{\lambda \varepsilon^c_{ab}}{a^2} e_c \\
& \widetilde{\Xi}^S_c \varsigma h_0 \varepsilon^{0c} \varepsilon_{ab} = \frac{\lambda \varepsilon^c_{ab}}{a^2} \sigma_c
\end{aligned}
\tag{13}
$$

where in the last line with use the property $\widetilde{\Xi}^S_c \widetilde{\Xi}^{Sc}{}_{ab} = \Xi^S_c(\delta^c_a \Xi^S_b - \delta^c_b \Xi^S_a) \equiv 0$ (see definitions above). Is easily seen, that squaring both sides of (13) and from (12) we obtain

$$
h_0 = \frac{\lambda \sigma_0}{a^2 |\widetilde{\Xi}^S_c| 2\varsigma}, \qquad h_c = \frac{\lambda |\widetilde{\Xi}^S_c| a \sigma_c}{2\varsigma},
$$

and analogically to the previous cases, from (19a) of ref. [2] the equation to integrate takes the form

$$
\frac{da}{d\tau} = \pm \frac{4}{5}\left[1 + \frac{\lambda}{3}a^2 + \frac{2}{3}a^4 \left|\widetilde{\Xi}^S_c\right|^2 + \frac{3}{8}\left(\frac{\lambda}{|\widetilde{\Xi}^S_c| a}\right)^2\right]^{1/2}.
$$

One interesting case when the above equation can be integrated exactly is precisely when $d = 4$. This condition, besides improving the integrability condition of the equation, fix $|\widetilde{\Xi}^S_c|^2 > 3/2$. The scale factor $a(\tau)$ takes the following form

$$
a(\tau) = \sqrt{B + (A - B)\tanh^2\left[\frac{(\tau - \tau_0)\sqrt{(A - B)}}{2}\right]}
$$

where A and B are nonlinear functions of the norm square $|\widetilde{\Xi}^S_c|^2$. The explicit form of these functions are not crucial: only the bound for $|\widetilde{\Xi}^S_c|^2 > 3/2$ need to be preserved (also through the normalization of A and B into the graphic representation i.e. Fig. 1). Notice that the spacetime is asymptotically Minkowskian with a throat $a(\tau_0) = \sqrt{B}$ (however the values of the constants have been selected according the previous remarks). Other possibilities not enumerated here, lead spacetimes with cyclic singularities due transcendental functions into the denominator of the expression for the scale factor $a(\tau)$. This issue is a focus of a future discussion somewhere [5].

Is interesting to note that in reference [4] the field equations of vacuum quadratic Poincare gauge field theory (QPGFT) were solved for purely null tratorial torsion. The author there expressing the contortion tensor for such a case as

$$
K_{\lambda\mu\nu} = -2(g_{\lambda\mu} a_\nu - g_{\lambda\nu} a_\mu).
$$

However, the important thing is that the author have been discussed the relationship between this class (tratorial) and a similar class of solution with null axial vector torsion, arriving to the conclusion that cosmological solutions with different type of torsion are forbidden. The main reason of this situation can have 2 origins:

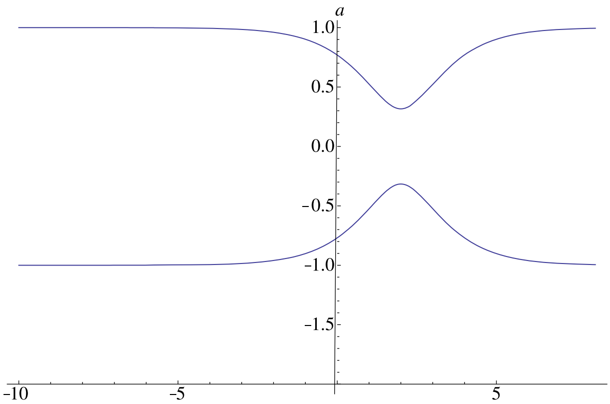

Fig. 1 Shape of the cosmological solution for values of $|\lambda| = 3$, $d = 4$

(i) the specific theory and action (QPGFT), or
(ii) the Newman-Penrose method used in the computations that works, as is well know, with null geometric quantities. Here we shown that this problem not arises in our theory.

3 The Underlying Dirac Structure of the Spacetime Manifold

The real structure of the Dirac equation in the form

$$(\gamma_0 p_0 - i\gamma \cdot \mathbf{p})\, \mathbf{u} = m\mathbf{v} \tag{14}$$

$$(\gamma_0 p_0 + i\gamma \cdot \mathbf{p})\, \mathbf{v} = m\mathbf{u} \tag{15}$$

with

$$\gamma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix} \tag{16}$$

where σ are the Pauli matrices and $p = (\widehat{p}_1, \widehat{p}_2, \widehat{p}_3)$, determines a 4D real vector space with G as its automorphism, such that $G \subset L(4)$. This real vector space can be make coincides with the tangent space to the spacetime manifold M, being this the idea. The principal fiber bundle (PFB) $P(G, M)$ with the structural group G determines the (Dirac) geometry of the spacetime. We suppose now G with the general form

$$G = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \qquad G^{+}G = I_4 \tag{17}$$

where A, B are 2×2 matrices. Also there exists a fundamental tensor $J_{\mu}{}^{\lambda} J_{\lambda}{}^{\nu} = \delta_{\mu}^{\nu}$ invariant under G with structure

$$J = \begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix} \tag{18}$$

where, however, the Lorentz metric $g_{\lambda\mu}$ is also invariant under G due its general form (17). Finally, a third fundamental tensor $\sigma_{\lambda\mu}$ is also invariant under G where the following relations between the fundamental tensors are

$$J_{\lambda}{}^{\nu} = \sigma_{\lambda\mu} g^{\lambda\nu}, \qquad g_{\mu\nu} = \sigma_{\lambda\mu} J_{\nu}{}^{\lambda}, \qquad \sigma_{\lambda\mu} = J_{\lambda}{}^{\nu} g_{\mu\nu} \tag{19}$$

where

$$g^{\lambda\nu}=\frac{\partial g}{\partial g_{\lambda\nu}}\qquad\left(g\equiv\det(g_{\mu\nu})\right).\tag{20}$$

Then, the necessary fundamental structure is given by

$$G=L\,(4)\cap Sp\,(4)\cap K\,(4)\tag{21}$$

which leaves concurrently invariant the three fundamental forms

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu\tag{22}$$

$$\sigma=\sigma_{\lambda\mu}dx^\lambda\wedge dx^\mu\tag{23}$$

$$\phi=J_\nu^{\ \lambda}w^\nu v_\lambda\tag{24}$$

where w^ν are components of a vector $w^\nu\in V^*$: the dual vector space. In expression (21) $L(4)$ is the Lorentz group in 4D, $Sp(4)$ is the Symplectic group in 4D real vector space and $K(4)$ denotes the almost complex group that leaves ϕ invariant [6].

For instance, G leaves the geometric product invariant [7]

$$\gamma_\mu\gamma_\nu=\frac{1}{2}\left(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu\right)+\frac{1}{2}\left(\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu\right)\tag{25}$$

$$=\gamma_\mu\cdot\gamma_\nu-\gamma_\mu\wedge\gamma_\nu=g_{\mu\nu}+\sigma_{\mu\nu}\tag{26}$$

where the are now regarded as a set of orthonormal basis vectors, of such a manner that any vector can be represented as $\mathbf{v}=v^\lambda\gamma_\lambda$ and

$$\varepsilon_{\alpha\beta\gamma\delta}\equiv\gamma_\alpha\wedge\gamma_\beta\wedge\gamma_\gamma\wedge\gamma_\delta.\tag{27}$$

In resume, the fundamental structure of the spacetime is then represented by $\mathrm{P}(G,M)$, where G is given by (21), which leaves invariant the fundamental forms (22)–(24), implying that

$$\nabla_\lambda g_{\mu\nu}=0\tag{28}$$

$$\nabla_\nu\sigma_{\lambda\mu}=0\tag{29}$$

$$\nabla_\lambda J_\nu^{\ \lambda}=0\tag{30}$$

where ∇_λ denotes the covariant derivative of the G connection. Is interesting to note that it is only necessary to consider two of above three equations: the third follows automatically. Then, we will consider (28), (29) because in some sense they represents the boson and fermion symmetry respectively.

4 Field Equations and Group Structure

Is necessary to introduce now other antisymmetric tensor $\sigma'_{\mu\nu}$ which is not helical, that means that is different of $\sigma_{\mu\nu}$ of (26) but also invariant with respect to the generalized connection G: $\nabla_\nu\sigma_{\lambda\mu}=0$. For instance, we can construct also the antisymmetric tensor $\vartheta_{\mu\nu}\equiv\sigma'_{\mu\nu}-\sigma_{\lambda\mu}\neq 0$, that obeys $\nabla_\nu\vartheta_{\mu\nu}=0$ and obviously $\frac{1}{6}(\partial_\mu\vartheta_{\nu\lambda}+\partial_\nu\vartheta_{\lambda\mu}+\partial_\lambda\vartheta_{\mu\nu})=T^{\rho}_{\nu\mu}\vartheta_{\rho\lambda}$ due the completely antisymmetric nature of T.

5 Antisymmetric Torsion and Fermionic Structure of the Spacetime

We know that [8]

$$\Gamma^{\rho}_{\mu\lambda} = \left\{{}^{\rho}_{\mu\lambda}\right\} + g^{\rho\nu}\left(T_{\mu\lambda\nu} + T_{\lambda\nu\mu} + T_{\nu\mu\lambda}\right) \tag{31}$$

where $\Gamma^{\rho}_{\mu\lambda}$ are he coefficients of the G-connection and $\{{}^{\rho}_{\mu\lambda}\}$ denotes the coefficients of the Levi-Civita connection whose covariant derivative is denoted by $\overset{\circ}{\nabla}_{\lambda}$. From (29) we make the link between the fermionic structure of the fundamental geometry of the manifold and the torsion tensor

$$\nabla_{[\nu}\sigma_{\lambda\mu]} = 0 \tag{32}$$

$$\Rightarrow \quad \frac{1}{2}\partial_{[\nu}\sigma_{\lambda\mu]} = T^{\rho}_{[\nu\mu}\sigma_{\rho\lambda]}. \tag{33}$$

A particular simplest solution for T arises when the torsion tensor is totally antisymmetric [9]

$$T_{\mu\lambda\nu} = T_{[\mu\lambda\nu]} \tag{34}$$

in order that the equivalence principle be obeyed [9, 10]. In this case, as we shown already in [1, 2, 9], we have

$$T_{\mu\lambda\nu} = \varepsilon_{\mu\lambda\nu\rho}h^{\rho} \tag{35}$$

where the axial vector h^{ρ} is still to be determined. As will be clear soon, is useful to put for d dimensions [9]

$$h^{\rho} = \frac{1}{\sqrt{w}}J^{\rho}_{\lambda}P^{\lambda} \tag{36}$$

where P^{λ} is the *generalized momentum vector*. If $d = 4$, $w = 6$.

Expression (35) can be simplified taking account on the symmetries of $T_{\mu\lambda\nu}$ and the contraction with the fundamental tensor J^{λ}_{τ}

$$T_{\lambda\mu\nu} = \frac{1}{w}J^{\rho}_{\lambda}\partial_{[\nu}\sigma_{\rho\mu]}. \tag{37}$$

6 The G-Invariance of the Action

As is well known, the Palatini principle has a twice role that is the determining of the connection required for the spacetime symmetry as the field equations. By means this principle, we were able to construct the action integral S. This action S necessarily need to yield the G-invariant conditions (28)–(30) without prior assumption; and, the Einstein, Dirac and Maxwel equations need to arise from S as a causally connected closed system. This equations will be generalized inevitably, so that causal connections between them can be established. Our action fulfill the above requirements, having account that the role of $f_{\mu\nu}$ that enters symmetrically with $g_{\mu\nu}$ in S, is linked with the fundamental tensor $\vartheta_{\mu\nu}$ of the previous section denoting the dual of $\vartheta_{\mu\nu}$ by

$$f_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\vartheta^{\rho\sigma} = *\vartheta_{\mu\nu}$$

(where $\vartheta^{\mu\nu}$ is the inverse tensor to $\vartheta_{\mu\nu}$).

The usual Euler-Lagrange equations from the action with the explicit computation of the determinant in ($d=4$) (see refs. [1, 2]) that will help us in order to compare the unitarian model introduced here (in the sense of Eddington (see [1, 2])) with the dualistic non abelian Born-Infeld model of [3], takes the familiar form [1–3]

$$S=\frac{b^2}{4\pi}\int\sqrt{-g}dx^4\left\{\overbrace{\sqrt{\gamma^4-\frac{\gamma^2}{2}\overline{G}^2-\frac{\gamma}{3}\overline{G}^3+\frac{1}{8}\left(\overline{G}^2\right)^2-\frac{1}{4}\overline{G}^4}}^{\equiv\mathbb{R}}\right\} \tag{38}$$

$$G_{\mu\nu}\equiv\left[\lambda^2\left(g_{\mu\nu}+f^a_{\ \mu}f_{a\nu}\right)+2\lambda R_{(\mu\nu)}+2\lambda f^a_{\ \mu}R_{[a\nu]}+R^a_{\ \mu}R_{a\nu}\right] \tag{39}$$

$$G^\nu_\nu\equiv\left[\lambda^2\left(d+f_{\mu\nu}f^{\mu\nu}\right)+2\lambda\left(R_S+R_A\right)+\left(R_S^2+R_A^2\right)\right] \tag{40}$$

with (the upper bar on the tensorial quantities indicates traceless condition)

$$\begin{aligned}
&R_S\equiv g^{\mu\nu}R_{(\mu\nu)}; \qquad R_A\equiv f^{\mu\nu}R_{[\mu\nu]};\\
&\gamma\equiv\frac{G^\nu_\nu}{d}; \qquad \overline{G}_{\mu\nu}\equiv G_{\mu\nu}-\frac{g_{\mu\nu}}{4}G^\nu_\nu; \qquad \overline{G}^\nu_\rho\overline{G}^\rho_\nu\equiv\overline{G}^2\\
&\overline{G}^\nu_\lambda\overline{G}^\lambda_\rho\overline{G}^\rho_\nu\equiv\overline{G}^3 \qquad \left(\overline{G}^\nu_\rho\overline{G}^\rho_\nu\right)^2\equiv\left(\overline{G}^2\right)^2 \qquad \overline{G}^\nu_\mu\overline{G}^\mu_\lambda\overline{G}^\lambda_\rho\overline{G}^\rho_\nu\equiv\overline{G}^4
\end{aligned} \tag{41}$$

where the variation was made with respect to the electromagnetic potential a_τ as follows

$$\frac{\delta\sqrt{G}}{\delta a_\tau}=\nabla_\rho\left(\frac{\partial\sqrt{G}}{\partial f_{\rho\tau}}\right)\equiv\nabla_\rho\mathbb{F}^{\rho\tau}=0. \tag{42}$$

Explicitly

$$\nabla_\rho\left[\frac{\lambda^2N^{\mu\nu}(\delta^\sigma_\mu\ f^\rho_\nu+\delta^\sigma_\nu\ f^\rho_\mu)}{2\mathbb{R}}\right]=0 \tag{43}$$

where $N^{\mu\nu}$ is given by

$$\begin{aligned}
N^{\mu\nu}=g\bigg[&-\gamma^2G^{\mu\nu}-\gamma(G^2)^{\mu\nu}+\frac{(G^2)^\mu_\mu G^{\mu\nu}}{2}-(G^3)^{\mu\nu}+\frac{4\gamma^3g^{\mu\nu}}{d}\\
&-\frac{\gamma(G^2)^\mu_\mu g^{\mu\nu}}{d}-\frac{(G^3)^\mu_\mu g^{\mu\nu}}{3d}\bigg]
\end{aligned} \tag{44}$$

(expression (32) of ref. [1]). The set of equations to solve for the geometrical action in this particular case is, explicitly from ref. [2]

$$R_{(\mu\nu)}=\overset{\circ}{R}_{\mu\nu}-T_{\mu\rho}^{\ \ \alpha}T_{\alpha\nu}^{\ \ \rho}=-\lambda g_{\mu\nu} \tag{19a}$$

$$R_{[\mu\nu]}=\nabla_\alpha T^\alpha_{\mu\nu}=-\lambda f_{\mu\nu} \tag{19b}$$

$$\nabla_\rho\left[\frac{\lambda^2N^{\mu\nu}(\delta^\sigma_\mu\ f^\rho_\nu+\delta^\sigma_\nu\ f^\rho_\mu)}{2\mathbb{R}}\right]=0 \tag{19c}$$

from this set, the link between T and f will be determined (f is not a priori potential for the torsion T).

The key point now is (36)

$$\overset{\circ}{R}_{\mu\nu} = -\lambda g_{\mu\nu} + T_{\mu\rho}{}^{\alpha} T_{\alpha\nu}{}^{\rho} \tag{45}$$

$$= -\lambda g_{\mu\nu} + w h_\mu h_\nu = -\lambda g_{\mu\nu} + P_\mu P_\nu \tag{46}$$

then we can obtain, as in mass shell condition

$$P^2 = m^2 \quad \Rightarrow \quad m = \pm\sqrt{\overset{\circ}{R} + \lambda d}. \tag{47}$$

Notice that there exists a link between the dimension of the spacetime and the scalar "Einsteinian" curvature $\overset{\circ}{R}$. Moreover, the curvature is constrained to take defined values $\in \mathbb{N}$ the natural number characteristic of the dimension. By the other hand, knowing that $|\lambda| = d - 1$ and accepting that the parameter $m \in \mathbb{R}$, the limiting condition on the physical values for the mass is $\overset{\circ}{R} \geqslant (1-d)d$.

Introducing the geometric product in above equation (e.g.: $\gamma^\mu\gamma^\nu P_\mu P_\nu = m^2$) plus the quantum condition: $P_\mu \to \widehat{P}_\mu - e\widehat{A}_\mu$, we have

$$\left[\gamma^\mu\gamma^\nu \left(\widehat{P}_\mu - e\widehat{A}_\mu\right)\left(\widehat{P}_\nu - e\widehat{A}_\nu\right) - m^2\right]\Psi = 0 \tag{48}$$

where $\Psi = \mathbf{u} + i\mathbf{v}$ given in (14), (15). That is

$$\left[\gamma^\mu \left(\widehat{P}_\mu - e\widehat{A}_\mu\right) + m\right]\left[\gamma^\nu \left(\widehat{P}_\nu - e\widehat{A}_\nu\right) - m\right] u^\lambda = 0 \tag{49}$$

which leads the Dirac equation

$$\left[\gamma^\mu \left(\widehat{P}_\mu - e\widehat{A}_\mu\right) + m\right] u^\lambda = 0 \tag{50}$$

with m given by (47). Notice that this condition, in the Dirac case, is not only to pass from classical variables to quantum operators, but in the case that the action does not contains explicitly $\widehat{A}_\mu$, h_μ remains without specification due the gauge freedom in the momentum. Applying the geometric product to (48) is not difficult to see that

$$\left[\left(\widehat{P}_\mu - e\widehat{A}_\mu\right)^2 - m^2 - \frac{1}{2}e\sigma^{\mu\nu}F_{\mu\nu}\right]u^\lambda + \frac{1}{2}\sigma^{\mu\nu}R^\lambda_{\rho[\mu\nu]}u^\rho - \frac{1}{2}e\sigma^{\mu\nu}\left(\widehat{A}_\mu\widehat{P}_\nu - \widehat{A}_\nu\widehat{P}_\mu\right)u^\lambda = 0. \tag{51}$$

Observations:

(i) the above formula is absolutely general for the type of geometrical Lagrangian's involved containing the generalized Ricci tensor inside,
(ii) for instance, the variation of the action will carry the symmetric contraction of components of the torsion tensor (i.e. (45)), then the arising of terms as $h_\mu h_\nu$,
(iii) the only thing that changes is the mass (47) and the explicit form of the tensors involved as $R^\lambda_{\rho[\mu\nu]}$, $F_{\mu\nu}$ etc., without variation of the Dirac general structure of the equation under consideration,
(iv) equation (51) differs from that obtained by Landau and Lifshitz by the appearance of the last two terms: the term involving the curvature tensor is due the spin interaction with the gravitational field (due torsion term in $R^\lambda_{\rho[\mu\nu]}$) and the last term is the spin interaction with the electromagnetic and mechanical momenta,

(v) expression (51) is valid for another vector v^λ, then is valid for a bispinor of the form $\Psi = \mathbf{u} + i\mathbf{v}$,
(vi) the meaning for a quantum measurement of the spacetime curvature is mainly due by the term in (51) involving explicitly the curvature tensor.

The important point here is that the spin-gravity interaction term is so easily derived as the spinors are represented as spacetime vectors whose covariant derivatives are defined in terms of the G-(affine) connection. In their original form the Dirac equations would have, in curved spacetime, their momentum operators replaced by covariant derivatives in terms of "spin-connection" whose relation is not immediately apparent.

7 Dirac Structure, Electromagnetic Field and Anomalous Giromagnetic Factor

The interesting point now is based in the observation that if we introduce expression (19b) in (51) then

$$\left[\left(\widehat{P}_\mu - e\widehat{A}_\mu\right)^2 - m^2 - \frac{1}{2}e\sigma^{\mu\nu}F_{\mu\nu}\right]u^\lambda - \frac{\lambda}{d}\frac{1}{2}\sigma^{\mu\nu}f_{[\mu\nu]}u^\lambda$$
$$- \frac{1}{2}e\sigma^{\mu\nu}\left(\widehat{A}_\mu\widehat{P}_\nu - \widehat{A}_\nu\widehat{P}_\mu\right)u^\lambda = 0 \tag{52}$$

$$\left[\left(\widehat{P}_\mu - e\widehat{A}_\mu\right)^2 - m^2 - \frac{1}{2}\sigma^{\mu\nu}\left(eF_{\mu\nu} + \frac{\lambda}{d}f_{\mu\nu}\right)\right]u^\lambda - \frac{e}{2}\sigma^{\mu\nu}\left(\widehat{A}_\mu\widehat{P}_\nu - \widehat{A}_\nu\widehat{P}_\mu\right)u^\lambda = 0 \tag{53}$$

we can see clearly that if $\widehat{A}_\mu = ja_\mu$ (with j arbitrary constant), $F_{\mu\nu} = jf_{\mu\nu}$ the last expression takes the suggestive form

$$\left[\left(\widehat{P}_\mu - e\widehat{A}_\mu\right)^2 - m^2 - \frac{1}{2}\left(ej + \frac{\lambda}{d}\right)\sigma^{\mu\nu}f_{\mu\nu}\right]u^\lambda - \frac{e}{2}\sigma^{\mu\nu}\left(\widehat{A}_\mu\widehat{P}_\nu - \widehat{A}_\nu\widehat{P}_\mu\right)u^\lambda = 0 \tag{54}$$

with the result that the giromagnetic factor have been modified to $2/(j + \frac{\lambda}{ed})$. Notice that in an Unified Theory with the characteristics introduced here is reasonable the identification introduced in the previous step ($F \leftrightarrows f$) in order that the fields arise from the same geometrical structure.

The concrete implications about this important contribution of the torsion to the giromagnetic factor will be given elsewhere with great detail on the dynamical property of the torsion field. Only we remark the following:

(i) there exists an important contribution of the torsion to the giromagnetic factor that can have implications to the trouble of the anomalous momentum of the fermionic particles,
(ii) this contribution appear (taking the second equality of expression (19b) ref. [2]), as a modification on the vertex of interaction, almost from the effective point of view;
(iii) is quite evident that this contribution will justify probably the little appearance of the torsion at great scale, because we can bounded the torsion due the other well know contributions to the anomalous momenta of the elementary particles (QED, weak, hadronic contribution, etc.),
(iv) the form of the coupling spin-geometric structure coming from the first principles, as the Dirac equation, not prescriptions,
(v) then, from (iii) how the covariant derivative works in presence of torsion is totally determined by the G structure of the spacetime,

(vi) the Dirac equation (52) (where was introduced the second part of the equivalence (19b) ref. [2] coming from the equation of motion), said us that the vertex was modified without a dynamical function of propagation. Then, other form to see the problem treated in this paragraph is to introduce the propagator for the torsion corresponding to the first part of the equivalence (19b). This important possibility will be studied elsewhere [5].

8 Space-Time and Structural Cohomologies

As is well know from the physical and mathematical point of view, the cohomological interplay between the fields involved in any well possessed geometrical and unified theory is crucial. This importance arises as a consequence of the logical (and causal) structure of the physical fields (sources, fields, conserved quantities) and not only as a mathematical play. In the theory presented here, there exist two cohomological structures: *Spacetime cohomology* and *structural cohomology*.

The difference between them is that in the *Spacetime cohomology* the Dirac (fermionic) structure of the space time is not involved directly in the relations between the fields involved. The main equations necessary for the construction are

$$\begin{aligned} \nabla_\alpha T^{\alpha}{}_{\mu\nu} &= -\lambda f_{\mu\nu} \\ d^* T &= -\lambda^* f = dh \end{aligned} \tag{55}$$

being the interplay schematically as

$$\begin{tikzcd} & \boxed{T} \arrow[dl, shift right, "A_-"'] \arrow[dr, shift left, "B_-"] & \\ \boxed{f} \arrow[ur, shift right, "A_+"'] \arrow[rr, shift right, "C_-"'] & & \boxed{h} \arrow[ul, shift left, "B_+"'] \arrow[ll, shift right, "C_+"'] \end{tikzcd} \tag{56}$$

where the operators are

$$\begin{aligned}
&A_- \equiv (-1)^{d+1}(-\lambda) * \int * \qquad A_+ \equiv (-\lambda)^{-1} * d* \\
&B_- \equiv (-1)^{d+1} * \qquad B_+ \equiv * \\
&C_- \equiv -\lambda \int^* \qquad C_+ \equiv \left[(-1)^{d+1}(-\lambda)\right]^{-1} * d \\
&D_- \equiv (-1)^{d+1} * d \qquad D_+ \equiv (-1)^{d+1} * \int \\
&E_- \equiv d \qquad E_+ \equiv \int \\
&G_- \equiv \left[(-1)^{d+1}(-\lambda)\right]^{-1} * \qquad G_+ \equiv -\lambda * .
\end{aligned} \tag{57}$$

The structural cohomology, in contrast, involve directly the fermionic structure of the space-time due that in the basic formulas $\vartheta_{\mu\nu}$ enters directly into the cohomological game, as is easily seen below

$$
\begin{array}{ccccc}
 & & \boxed{a} & & \\
 & {}_{E_-}\swarrow\nearrow_{E_+} & & {}_{D_+}\nwarrow\searrow^{D_-} & \\
\boxed{f} & & \overset{B_-}{\underset{B_+}{\rightleftarrows}} & & \boxed{\vartheta} \\
 & {}_{C_+}\nwarrow\searrow_{C_-} & & {}_{A_-}\swarrow\nearrow^{A_+} & \\
 & & \boxed{h} & & \\
 & & {}_{G_-}\uparrow\downarrow_{G_+} & & \\
 & & \boxed{a} & &
\end{array}
\tag{58}
$$

Notice the important thing that, in this case clearly the degree of the relations between the quantities involved are more fundamental that in the previous case (hierarchical sense).

9 Concluding Remarks

In this work we make an exhaustive analysis of the model based in the theory developed in early references of the authors, elucidate the simplest structure of the spacetime described by this new theory that make the link between the torsion and the spin, beside the connection between curvature and matter. As was well explained through all this paper, the model coming from the rupture of symmetry making that the geometrical Lagrangian can be written in a suggestive Eddington-Born-Infeld like form. Three cases were here treated from the point of view of the solutions, depending on the form of torsion used: totally antisymmetric with torsion potential, not totally antisymmetric ("tratorial" type), and with a torsion tensor with both characteristics. In all the cases they were compared from the point of view of the obtained solutions with the dualistic model of reference [3]. In all these cases the non-dualistic unified model proposed here have deep differences with the dualistic non-Abelian Born-Infeld model of our early reference [3].

The first obvious difference come from a conceptual framework: the geometrical action will provide, besides the spacetime structure, the matter-energy spin distribution. This fact is the same basis of the unification: all the (apparently disconnected) theories and interactions of the natural world appears naturally as a consequence of the intrinsic spacetime geometry.

For the general case, i.e. with torsion with totally antisymmetric and tratorial parts, the full analysis was given in a clear manner before. Here we point out that the Hosoya and Ogura anzats can be implemented as in the previous cases, and the most important, is the fact that wormhole solutions can be obtained for some particular cases. The solutions are asymptotically flat, where appear vector and tensor integration constants that are constrained in norm to bring physical consistency to the solution.

About the problem of the possibility of coexistence of the trace of the torsion due the tratorial part and the axial vector from the totally antisymmetric part of the torsion, we saw here that there are not problem in the new theory: there are tratorial and antisymmetric torsion fields without contradictions.

The fact that in reference [4] the field equations of vacuum quadratic Poincare gauge field theory (QPGFT) were solved for purely null tratorial torsion, if well permit to express the contortion tensor for such a case as (tratorial form, with notation of ref. [4])

$$K_{\lambda\mu\nu} = -2(g_{\lambda\mu}a_\nu - g_{\lambda\nu}a_\mu)$$

does not permit the coexistence with an axial torsion vector, as was clearly shown by Singh in the beautiful paper [4]. The two points that lead such discrepancy are:

(i) the different theories described, not only in foundations but also because one is unitarian and the other of [4] dualistic
(ii) and the fact that the Newman-Penrose formulation was used in [4], that as is well known such method works in a null tetrad.

9.1 On the Geometrical Structure

From the point of view of the concrete structure able to explain the content of the bosonic and fermionic matter of the universe, the present paper is left open-ended as many physical consequences need to be explored. Some words concerning to the realization and the choice of the correct group structure of the tangent space to M is that $G = L(4) \cap Sp(4) \cap K(4)$ preserves the boson and fermion symmetry simultaneously without imply supersymmetry of the model. As we like to show in a future work, the supergravitational extension of the model will be discussed joint with the problem of it quantization, where the key point will be precisely the group structure of the tangent space to the spacetime manifold M. Here we conclude enumerating the main results concerning to the basic structure of the manifold supporting an Unified Field Theoretical model:

(i) the simplest geometrical structure able to support the fermionic fields was constructed based in a tangent space with a group structure $G = L(4) \cap Sp(4) \cap K(4)$
(ii) then, the explicitly link of the fermionic structure with the torsion field was realized and the Dirac type equation was obtained from the same spacetime manifold
(iii) notice that the matter was not included on the Geometrical Lagrangian of the Unified theory presented here: only symmetry arguments (that will lead the correct dynamical equations for the material fields arising from the same manifold)

need to allow the appearance of matter and this fact is not the essence of the unification, of course (several references trying to include matter into the Eddington "type" theories by hand without physical and symmetry principles).

9.2 Discussion

(1) On the equation

$$\overset{\circ}{R}_{\mu\nu} = -\lambda g_{\mu\nu} + T_{\mu\rho}{}^{\alpha} T_{\alpha\nu}{}^{\rho}$$

notice that the concept here of the terms that arise as "energy-momentum" part coming from the symmetric contraction of the torsion components is different in essence to the concept coming from the inclusion of the energy-momentum tensor in the Einstein theory. The conceptual framework that "matter and energy curve the spacetime" implicitly carry the idea of some "embedding-like" situation where the matter and energy are putted on some Minkowskian flexible carpet and you see how it is curved under the "weight" of the "ball" (matter + energy). Here, in the theory presented, the situation is that the torsion terms (contributing as "energy momentum" in above equation) arise from the same geometry, then we have the picture as an unique entity: the interplay fields-spacetime. The idea is the same as the solitonic vortex in the water.

This fact can be also interpreted as that the concept of force is introduced due the torsion in the unified model, thing that is lost in the Einstein theory [10] where the concept is that there are not force, but curvature only.

(2) Some remarks on the general Hodge-de Rham decomposition of $h = h_\alpha dx^\alpha$.

Theorem 1 *If $h = h_\alpha dx^\alpha \notin F'(M)$ is a 1-form on M, then there exist a zero-form Ω, a 2-form $\alpha = A_{[\mu\nu]} dx^\mu \wedge dx^\nu$ and an harmonic 1-form $q = q_\alpha dx^\alpha$ on M that*

$$h = d\Omega + \delta\alpha + q \to h_\alpha = \nabla_\alpha \Omega + \varepsilon_\alpha^{\beta\gamma\delta} \nabla_\beta A_{\gamma\delta} + q_\alpha.$$

Notice that if even is not harmonic and assuming that q_α is a polar vector, an axial vector can be added such that above expression takes the form

$$h_\alpha = \nabla_\alpha \Omega + \varepsilon_\alpha^{\beta\gamma\delta} \nabla_\beta A_{\gamma\delta} + \varepsilon_\alpha^{\beta\gamma\delta} M_{\beta\gamma\delta} + q_\alpha$$

where $M_{\beta\gamma\delta}$ is a completely antisymmetric tensor.

(3) Notice the important fact that when the torsion is totally antisymmetric tensor field, $-2\lambda f_{\mu\nu}$ takes the role of "current" for the torsion field, as usually the terms proportional to the 1-form potential vector a_μ acts as current of the electromagnetic field $f_{\mu\nu}$ in the equation of motion for the electromagnetic field into the standard theory: $\nabla_\alpha f^\alpha{}_\mu = J_\mu$ (constants absorbed into the J_μ). The interpretation and implications of this question will be analyzed concretely in [5].

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