Maximal Entanglement of Two-Qubit States Constructed by Linearly Independent Coherent States

G. Najarbashi · Y. Maleki

Abstract In this paper, we find the necessary and sufficient condition for the maximal entanglement of the state, $|\psi\rangle = \mu|\alpha\rangle|\beta\rangle + \lambda|\alpha\rangle|\delta\rangle + \rho|\gamma\rangle|\beta\rangle + \nu|\gamma\rangle|\delta\rangle$, constructed by linearly independent coherent states with *real parameters* when $\langle\alpha|\gamma\rangle = \langle\beta|\delta\rangle$. This is a further generalization of the classified nonorthogonal states discussed in Fu et al. (Phys. Lett. A 291:73–76, 2001).

Keywords Entanglement · Coherent state · Concurrence

1 Introduction

Quantum entanglement is one of the most profound features of quantum mechanics and has been considered to be a valuable physical resource in the rapidly developing field of quantum information science. In fact a fundamental difference between quantum and classical physics is the possible existence of quantum entanglement between distinct systems. Therefore, manipulation of entangled states is very important and a challenging problem. By definition, a pure quantum state of two or more subsystems is said to be entangled if it is not a product of the states of each component [1, 2]. Among various entangled states, entanglement of coherent states has attracted special attention, and due to its applications in quantum optics [3], quantum computation and information, a lot of work has been devoted to this problem in the last two decades [4–11].

The required conditions for the maximal entanglement in superposed bosonic coherent states of the form

$$|\varphi\rangle = \mu|\alpha\rangle|\beta\rangle + \nu|\gamma\rangle|\delta\rangle, \tag{1.1}$$

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have been studied, and the maximally entangled coherent states have been classified in Ref. [7]. This investigation has been done using the concurrence measure [12]. We note that in the framework of two-qubit states, constructed by linearly independent coherent states, the above state is a special case of the state

$$|\psi\rangle = \mu|\alpha\rangle|\beta\rangle + \lambda|\alpha\rangle|\delta\rangle + \rho|\gamma\rangle|\beta\rangle + \nu|\gamma\rangle|\delta\rangle. \tag{1.2}$$

Note that the above state can not be deduced from the state (1.1) by means of local unitary operations $SU(2) \otimes SU(2)$, and consequently the state (1.1) is not the Schmidt form of (1.2). This is due to the fact that the coefficients μ and ν are generally complex numbers while in the Schmidt form they must be real positive numbers. On the other hand, the basis $|\alpha\rangle$ and $|\gamma\rangle$ (also $|\beta\rangle$ and $|\delta\rangle$) are not orthonormal, while in Schmidt form they must be orthonormal. Here we apply the maximal condition of the concurrence measure on the previous state with *real parameters*, and show that in the case $\langle\alpha|\gamma\rangle = \langle\beta|\delta\rangle$, there would be only two disjoint classes of maximal entangled coherent states

(a) $\nu = 1$ and $\lambda + \rho = -2\langle\alpha|\gamma\rangle$,
(b) $\lambda = \rho$ and $\nu + 1 = -2\lambda\langle\alpha|\gamma\rangle$.

2 Entanglement

A bosonic coherent state is defined as the eigen-state of the annihilation operator as below

$$a|\alpha\rangle = \alpha|\alpha\rangle, \tag{2.3}$$

where α is a complex number, and a is the annihilation operator for the bosonic harmonic oscillator. Thus, considering the definition of the coherent state in the number states space, it can be written as follows

$$|\alpha\rangle = e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle = D(\alpha)|0\rangle, \tag{2.4}$$

where $D(\alpha)$ is the displacement operator, and we have

$$D(\alpha) := \exp(a^\dagger\alpha - \alpha^*a). \tag{2.5}$$

The overlap of two coherent states is

$$\langle\alpha|\beta\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2-2\alpha^*\beta)}. \tag{2.6}$$

One of the useful measures which can be utilized to quantify the amount of entanglement in two-qubit states is concurrence, which is defined as [12]

$$C = |\langle\zeta|\sigma_y \otimes \sigma_y|\zeta^*\rangle|, \tag{2.7}$$

where σ_y is the y component of the usual Pauli spin matrices. Now let us consider the general form of the two-qubit coherent state i.e.,

$$|\psi\rangle = \mu|\alpha\rangle|\beta\rangle + \lambda|\alpha\rangle|\delta\rangle + \rho|\gamma\rangle|\beta\rangle + \nu|\gamma\rangle|\delta\rangle, \tag{2.8}$$

where $|\alpha\rangle$ and $|\gamma\rangle$ are linearly independent normalized states, which span the space of the system 1, and $|\beta\rangle$ and $|\delta\rangle$ are linearly independent normalized states, which span the space of the system 2. The normalization of the state (2.8) is

$$
\begin{aligned}
N^2 = \langle\psi|\psi\rangle = {}& |\mu|^2 + \mu^*\lambda\langle\beta|\delta\rangle + \mu^*\rho\langle\alpha|\gamma\rangle + \mu^*\nu\langle\alpha|\gamma\rangle\langle\beta|\delta\rangle + \lambda^*\mu\langle\beta|\delta\rangle^* + |\lambda|^2 \\
& + \lambda^*\rho\langle\alpha|\gamma\rangle\langle\beta|\delta\rangle^* + \lambda^*\nu\langle\alpha|\gamma\rangle + \rho^*\mu\langle\alpha|\gamma\rangle^* + \rho^*\lambda\langle\alpha|\gamma\rangle\langle\beta|\delta\rangle + |\rho|^2 \\
& + \rho^*\nu\langle\beta|\delta\rangle + \nu^*\mu\langle\alpha|\gamma\rangle^*\langle\beta|\delta\rangle^* + \nu^*\lambda\langle\alpha|\gamma\rangle^* + \nu^*\rho\langle\beta|\delta\rangle^* + |\nu|^2. \quad (2.9)
\end{aligned}
$$

In the case $\rho = \lambda = 0$, $|\psi\rangle$ reduces to the state discussed in [7]. Here we are going to develop the discussion to some general form in *real field*. We assume, without loss of generality, that $\mu = 1$ and get

$$
|\psi\rangle = |\alpha\rangle|\beta\rangle + \lambda|\alpha\rangle|\delta\rangle + \rho|\gamma\rangle|\beta\rangle + \nu|\gamma\rangle|\delta\rangle. \qquad (2.10)
$$

Theorem *The general form of the two-qubit state equation* (2.10) *constructed by linearly independent coherent states in real field with requirement* $\langle\alpha|\gamma\rangle = \langle\beta|\delta\rangle$, *is maximally entangled if and only if one of the following conditions holds*

(a) $\nu = 1$ *and* $\lambda + \rho = -2\langle\alpha|\gamma\rangle$,
(b) $\lambda = \rho$ *and* $\nu + 1 = -2\lambda\langle\alpha|\gamma\rangle$.

In order to prove the theorem, we take the following orthonormal basis

$$
|0\rangle = |\alpha\rangle, \qquad |1\rangle = \frac{|\gamma\rangle - p_1|\alpha\rangle}{N_1} \quad \text{for system 1,}
$$
$$
|0\rangle = |\delta\rangle, \qquad |1\rangle = \frac{|\beta\rangle - p_2|\delta\rangle}{N_2} \quad \text{for system 2,}
$$
$$\tag{2.11}$$

where

$$
p_1 = \langle\alpha|\gamma\rangle, \quad N_1 = \sqrt{1 - |p_1^2|},
$$
$$
p_2 = \langle\delta|\beta\rangle, \quad N_1 = \sqrt{1 - |p_2^2|}.
$$
$$\tag{2.12}$$

Taking these new bases we can rewrite $|\psi\rangle$ as

$$
|\psi\rangle = \frac{(p_2 + \lambda + \rho p_1 p_2 + \nu p_1)|00\rangle + N_2(1 + \rho p_1)|01\rangle + N_1(\nu + \rho p_2)|10\rangle + \rho N_1 N_2|11\rangle}{N}.
$$
$$\tag{2.13}$$

For a general two-qubit pure state

$$
|\varphi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle, \qquad (2.14)
$$

the concurrence is given by

$$
C = 2|ad - bc|. \qquad (2.15)
$$

Using (2.13) and (2.15), the concurrence of the state (2.10) becomes

$$
C = \frac{2|\nu - \lambda\rho|(\sqrt{1 - |p_1^2|})(\sqrt{1 - |p_2^2|})}{N^2}. \qquad (2.16)
$$

We just survey the case $\langle \alpha | \gamma \rangle = \langle \beta | \delta \rangle$ in detail. Then (2.6) for α, $\gamma \in \mathbb{R}$ becomes

$$0 < \langle \alpha | \gamma \rangle = e^{\frac{-1}{2}(\alpha - \gamma)^2} < 1. \tag{2.17}$$

By solving (2.16) for $C = 1$ which is equivalent to maximal entanglement of the state (2.10), we get

$$2|\nu - \lambda \rho|(1 - x^2) = (1 + \lambda^2 + \rho^2 + \nu^2) + 2(\lambda + \rho + \nu\lambda + \rho\nu)x + 2(\nu + \lambda\rho)x^2. \tag{2.18}$$

We divide this equation into two parts as follows

Case 1: Let $\nu > \lambda \rho$, then we have

$$(1 - \nu)^2 + (\lambda + \rho)^2 + 2(\lambda + \rho)(1 + \nu)x + 4\nu x^2 = 0, \tag{2.19}$$

this equation is a second order polynomial with respect to x and has solutions if and only if $\nu = 1$ and $\lambda + \rho = -2x$ (see appendix).

For example, the following states are maximally entangled states and belong to this class

$$|\psi\rangle_{max} = \frac{1}{\sqrt{2(1 - e^{-(\alpha - \gamma)^2})}} \left(|\alpha\rangle|\beta\rangle - e^{\frac{-1}{2}(\alpha - \gamma)^2}|\alpha\rangle|\delta\rangle - e^{\frac{-1}{2}(\alpha - \gamma)^2}|\gamma\rangle|\beta\rangle + |\gamma\rangle|\delta\rangle \right), \tag{2.20}$$

$$|\psi\rangle_{max} = \frac{1}{\sqrt{2(1 - e^{-(\alpha - \gamma)^2})}} \left(|\alpha\rangle|\beta\rangle - 2e^{\frac{-1}{2}(\alpha - \gamma)^2}|\alpha\rangle|\delta\rangle + |\gamma\rangle|\delta\rangle \right), \tag{2.21}$$

$$|\psi\rangle_{max} = \frac{1}{\sqrt{2(1 + 2e^{-(\alpha - \gamma)^2}) - 3e^{-2(\alpha - \gamma)^2}}} \left(|\alpha\rangle|\beta\rangle + e^{\frac{-1}{2}(\alpha - \gamma)^2}|\alpha\rangle|\delta\rangle \right.$$
$$\left. - 3e^{\frac{-1}{2}(\alpha - \gamma)^2}|\gamma\rangle|\beta\rangle + |\gamma\rangle|\delta\rangle \right), \tag{2.22}$$

Taking $\beta = \alpha$ and $\delta = \gamma$ in the above states, we get

$$|\psi\rangle_{max} = \frac{1}{\sqrt{2(1 - e^{-(\alpha - \gamma)^2})}} \left(|\alpha\rangle|\alpha\rangle - e^{\frac{-1}{2}(\alpha - \gamma)^2}|\alpha\rangle|\gamma\rangle - e^{\frac{-1}{2}(\alpha - \gamma)^2}|\gamma\rangle|\alpha\rangle + |\gamma\rangle|\gamma\rangle \right), \tag{2.23}$$

$$|\psi\rangle_{max} = \frac{1}{\sqrt{2(1 + 2e^{-(\alpha - \gamma)^2}) - 3e^{-2(\alpha - \gamma)^2}}} \left(|\alpha\rangle|\alpha\rangle + e^{\frac{-1}{2}(\alpha - \gamma)^2}|\alpha\rangle|\gamma\rangle \right.$$
$$\left. - 3e^{\frac{-1}{2}(\alpha - \gamma)^2}|\gamma\rangle|\alpha\rangle + |\gamma\rangle|\gamma\rangle \right), \tag{2.24}$$

$$|\psi\rangle_{max} = \frac{1}{\sqrt{2(1 - e^{-(\alpha - \gamma)^2})}} \left(|\alpha\rangle|\alpha\rangle - 2e^{\frac{-1}{2}(\alpha - \gamma)^2}|\alpha\rangle|\gamma\rangle + |\gamma\rangle|\gamma\rangle \right). \tag{2.25}$$

Case 2: Let $\nu < \lambda \rho$, then we have

$$(1 + \nu)^2 + (\lambda - \rho)^2 + 2(\lambda + \rho)(1 + \nu)x + 4\lambda\rho x^2 = 0. \tag{2.26}$$

The above equation has solutions if and only if $\lambda = \rho$ and $\nu + 1 = -2\lambda x$. For example the following states are maximally entangled states and belong to this class

$$|\psi\rangle_{max} = \frac{1}{\sqrt{2(1 - e^{-(\alpha - \gamma)^2})}} \left(|\alpha\rangle|\beta\rangle - |\gamma\rangle|\delta\rangle \right), \tag{2.27}$$

$$|\psi\rangle_{max} = \frac{e^{\frac{-1}{2}(\alpha-\gamma)^2}}{\sqrt{2(1-e^{-(\alpha-\gamma)^2})}} \left(|\alpha\rangle|\beta\rangle - e^{\frac{1}{2}(\alpha-\gamma)^2}|\alpha\rangle|\delta\rangle - e^{\frac{1}{2}(\alpha-\gamma)^2}|\gamma\rangle|\beta\rangle + |\gamma\rangle|\delta\rangle \right), \quad (2.28)$$

$$|\psi\rangle_{max} = \frac{e^{\frac{-1}{2}(\alpha-\gamma)^2}}{\sqrt{2(4-7e^{-(\alpha-\gamma)^2}+3e^{-2(\alpha-\gamma)^2})}} \left(|\alpha\rangle|\beta\rangle - 2e^{\frac{1}{2}(\alpha-\gamma)^2}|\alpha\rangle|\delta\rangle \right.$$
$$\left. -2e^{\frac{1}{2}(\alpha-\gamma)^2}|\gamma\rangle|\beta\rangle + 3|\gamma\rangle|\delta\rangle \right), \quad (2.29)$$

$$|\psi\rangle_{max} = \frac{e^{\frac{-1}{2}(\alpha-\gamma)^2}}{\sqrt{2(1+2e^{-(\alpha-\gamma)^2}-3e^{-2(\alpha-\gamma)^2})}} \left(|\alpha\rangle|\beta\rangle + e^{\frac{1}{2}(\alpha-\gamma)^2}|\alpha\rangle|\delta\rangle \right.$$
$$\left. +e^{\frac{1}{2}(\alpha-\gamma)^2}|\gamma\rangle|\beta\rangle - 3|\gamma\rangle|\delta\rangle \right), \quad (2.30)$$

$$|\psi\rangle_{max} = \frac{\sqrt{2}e^{\frac{-1}{2}(\alpha-\gamma)^2}}{\sqrt{(1-e^{-(\alpha-\gamma)^2})}} \left(|\alpha\rangle|\beta\rangle - \frac{1}{2}e^{\frac{1}{2}(\alpha-\gamma)^2}|\alpha\rangle|\delta\rangle - \frac{1}{2}e^{\frac{1}{2}(\alpha-\gamma)^2}|\gamma\rangle|\beta\rangle \right). \quad (2.31)$$

The first state is the antisymmetric maximally entangled state obtained in [11]. The proof of the inverse of the theorem, is straightforward and can be verified directly by putting the above conditions in the concurrence measure.

Now let us consider some special cases of the theorem. For example when $\langle\alpha|\gamma\rangle \to 0$ (satisfied when $|\alpha-\gamma| \to \infty$) we get from the condition (**a**) that the following state is maximal

$$\frac{1}{\sqrt{2(1+\lambda^2)}}(\lambda|00\rangle + |01\rangle + |10\rangle - \lambda|11\rangle), \quad (2.32)$$

and taking $\lambda = 0$, the state reduces to one of the Bell states, i.e.,

$$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle). \quad (2.33)$$

However taking the condition (**b**) we have

$$\frac{1}{\sqrt{2(1+\lambda^2)}}(\lambda|00\rangle + |01\rangle - |10\rangle + \lambda|11\rangle), \quad (2.34)$$

which taking $\lambda = 0$, reduces to another Bell state, i.e.,

$$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \quad (2.35)$$

Remark The state (2.8) with $\mu = 1$ is separable (C = 0) if and only if $\nu = \lambda\rho$.

We note that the separability conditions hold in complex field too. For example the following states, up to normalization factors, are separable

$$|\psi\rangle_{sep} = |\alpha\rangle|\beta\rangle + |\alpha\rangle|\delta\rangle + |\gamma\rangle|\beta\rangle + |\gamma\rangle|\delta\rangle, \quad (2.36)$$

$$|\psi\rangle_{sep} = |\alpha\rangle|\beta\rangle + \lambda|\alpha\rangle|\delta\rangle + \rho|\gamma\rangle|\beta\rangle + \lambda\rho|\gamma\rangle|\delta\rangle, \quad (2.37)$$

$$|\psi\rangle_{sep} = |\alpha\rangle|\beta\rangle - |\alpha\rangle|\delta\rangle + |\gamma\rangle|\beta\rangle - |\gamma\rangle|\delta\rangle, \tag{2.38}$$

$$|\psi\rangle_{sep} = |\alpha\rangle|\beta\rangle + |\alpha\rangle|\delta\rangle + \rho|\gamma\rangle|\beta\rangle + \rho|\gamma\rangle|\delta\rangle. \tag{2.39}$$

In summary, we investigated the entanglement of two-qubit coherent states $|\psi\rangle = \mu|\alpha\rangle|\beta\rangle + \lambda|\alpha\rangle|\delta\rangle + \rho|\gamma\rangle|\beta\rangle + \nu|\gamma\rangle|\delta\rangle$, with real parameters, assuming $\langle\alpha|\gamma\rangle = \langle\beta|\delta\rangle$. This problem has been solved for the special case $|\varphi\rangle = \mu|\alpha\rangle|\beta\rangle + \nu|\gamma\rangle|\delta\rangle$ in complex field in Ref. [7]. However, finding all maximal entangled regions of (2.8) in complex field, is challenging and open for debate.

Appendix

In this Appendix, we show that the equation

$$(1 - \nu)^2 + (\lambda + \rho)^2 + 2(\lambda + \rho)(1 + \nu)x + 4\nu x^2 = 0, \tag{1}$$

with constraint $0 < x < 1$ has solution iff both $\nu = 1$ and $\lambda + \rho = -2\langle\alpha|\gamma\rangle$, hold. We note that, if all the coefficients of the left hand side vanish, the equality holds identically and it would become independent of x, but here we can not take all the coefficients to be zero. Now we suppose that $\nu = 0$, then

$$x = \frac{1 + (\lambda + \rho)^2}{-2(\lambda + \rho)}, \tag{2}$$

applying $0 < x < 1$, yields $(1 + \lambda + \rho)^2 < 0$ which is impossible. Thus $\nu \neq 0$. The above second order polynomial has solutions

$$x_\pm = \frac{-(\lambda + \rho)(1 + \nu) \pm \sqrt{(1 - \nu)^2((\lambda + \rho)^2 - 4\nu)}}{4\nu}, \tag{3}$$

Since $x_\pm$ are real parameters, then $(1 - \nu)^2((\lambda + \rho)^2 - 4\nu) \geq 0$. We discuss equality and inequality cases separately.
I:
Let $(1 - \nu)^2((\lambda + \rho)^2 - 4\nu) = 0$, then $(\lambda + \rho)^2 = 4\nu$, or $\nu = 1$. If $(\lambda + \rho)^2 = 4\nu$ then

$$0 < x_\pm = \frac{-(\lambda + \rho)(1 + \nu)}{4\nu} < 1 \implies (\lambda + \rho) < 0,$$

which implies that $(1 - \nu)^2 < 0$ and this is impossible in the real field. Thus we turn to the other possibility and assume that

$$\nu = 1 \implies 0 < x_\pm = \frac{-(\lambda + \rho)}{2} < 1 \implies , \lambda + \rho = -2\langle\alpha|\gamma\rangle,$$

which is the result (**a**) of the main theorem.
II:
Now let us survey the case $(1 - \nu)^2((\lambda + \rho)^2 - 4\nu) > 0$ or equivalently

$$(\lambda + \rho)^2 > 4\nu \quad \text{and} \quad \nu \neq 1 \tag{4}$$

We discuss two solutions $x_\pm$ for the case $\nu > 0$. The same discussion holds if $\nu < 0$. First let us concentrate on x_-, thus

$$x_- > 0 \quad \implies \quad (\lambda + \rho) < -2\sqrt{\nu}, \tag{5}$$

$$x_- < 1 \quad \implies \quad -((\lambda + \rho)(1 + \nu) + 4\nu) < \sqrt{(1 - \nu)^2((\lambda + \rho)^2 - 4\nu)}. \tag{6}$$

If $-((\lambda + \rho)(1 + \nu) + 4\nu)$, belongs to the following bounds

$$-\sqrt{(1 - \nu)^2((\lambda + \rho)^2 - 4\nu)} < -((\lambda + \rho)(1 + \nu) + 4\nu) < \sqrt{(1 - \nu)^2(\lambda + \rho)^2 - 4\nu)}, \tag{7}$$

then one easily gets $(1 + \lambda + \rho)^2 < 0$ which is a contradiction. Therefore we must restrict ourselves to the following bound

$$-((\lambda + \rho)(1 + \nu) + 4\nu) \le -\sqrt{(1 - \nu)^2((\lambda + \rho)^2 - 4\nu)}. \tag{8}$$

This equation implies that

$$(\lambda + \rho) > \frac{-4\nu}{1 + \nu}, \tag{9}$$

Two equations (5) and (9) yield $(\nu - 1)^2 < 0$ and consequently, we have no solution for x_- in this case.

Now we turn our attention to x_+, thus

$$0 < x_+ = \frac{-(\lambda + \rho)(1 + \nu) + \sqrt{(1 - \nu)^2((\lambda + \rho)^2 - 4\nu)}}{4\nu} < 1, \tag{10}$$

The upper bound yields

$$x_+ < 1 \quad \implies \quad (\lambda + \rho) > \frac{-4\nu}{1 + \nu} \tag{11}$$

which together with (4) reads

$$(\lambda + \rho) > 2\sqrt{\nu},$$

and the lower bound of (10) yields

$$(\lambda + \rho)(1 + \nu) < \sqrt{(1 - \nu)^2((\lambda + \rho)^2 - 4\nu)}, \tag{12}$$

which leads to

$$(\lambda + \rho)^2 + (1 - \nu)^2 < 0. \tag{13}$$

which is evidently a contradiction.

One can pursue the same line as above and consider the solutions of (2.26) in order to obtain the result (**b**) of the main theorem.

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