Int J Theor Phys (2011) 50:3737–3749
DOI 10.1007/s10773-011-0748-4

Tense Operators on Basic Algebras

M. Botur · I. Chajda · R. Halaš · M. Kolařík

Received: 10 November 2010 / Accepted: 2 March 2011 / Published online: 18 March 2011

Abstract The concept of tense operators on a basic algebra is introduced. Since basic algebras can serve as an axiomatization of a many-valued quantum logic (see e.g. Chajda et al. in Algebra Univer. 60(1):63–90, 2009), these tense operators are considered to quantify time dimension, i.e. one expresses the quantification "it is always going to be the case that" and the other expresses "it has always been the case that". We set up the axiomatization and basic properties of tense operators on basic algebras and involve a certain construction of these operators for left-monotonous basic algebras. Finally, we relate basic algebras with tense operators with another quantum structures which are the so-called dynamic effect algebras.

Keywords Basic algebra · Tense operator · Left-monotonous basic algebra · Commutative basic algebra · Effect algebra

Propositional logics usually do not incorporate the dimension of time. To obtain a tense logic, we enrich a propositional logic by adding new unary operators (or connectives) which are usually denoted by G, H, F and P. We can define F and P by means of G and H as follows: $F(x) = \neg G(\neg x)$ and $P(x) = \neg H(\neg x)$. The semantical interpretation of these

This work is supported by the Research and Development Council of the Czech Government via the project MSM6198959214.

M. Botur · I. Chajda · R. Halaš (✉)
Department of Algebra and Geometry, Palacký University Olomouc, 17. listopadu 12, 771 46 Olomouc, Czech Republic
e-mail: halas@inf.upol.cz

M. Botur
e-mail: botur@inf.upol.cz

I. Chajda
e-mail: chajda@inf.upol.cz

M. Kolařík
Department of Computer Science, Palacký University Olomouc, 17. listopadu 12, 771 46 Olomouc, Czech Republic
e-mail: kolarik@inf.upol.cz

so-called **tense operators** is as follows. Consider a pair $(T; \leq)$ where T is a non-void set and $\leq$ is a partial order on T. Let $x \in T$ and $f(x)$ be a formula of our (given) propositional logic. We say that $G(f(t))$ **is valid** if for any $s \geq t$ the formula $f(s)$ is valid. Analogously, $H(f(t))$ is valid if $f(s)$ is valid for each $s \leq t$. Hence, $P(f(t))$ is valid if there exists $s \leq t$ such that $f(s)$ is valid and $F(f(t))$ is valid if there exists $s \geq t$ such that $f(s)$ is valid in the given logic. Thus the unary operations G and H constitute an algebraic counterpart of the tense operators "it is always going to be the case that" and "it has always been the case that".

It is worth saying that tense operators were firstly introduced for the classical propositional logic, see [6], as operators on the corresponding Boolean algebra satisfying the axioms

(B1) $G(1) = 1, \quad H(1) = 1$;
(B2) $G(x \wedge y) = G(x) \wedge G(y), \quad H(x \wedge y) = H(x) \wedge H(y)$;
(B3) $x \leq GP(x), \quad x \leq HF(x)$.

The axiom (B3) is equivalent (for Boolean algebras) to $G(x) \vee y = x \vee H(y)$.

The concept of basic algebra was introduced in [7] as a common generalization of an MV-algebra (an algebraic counterpart of the Łukasiewicz many-valued propositonal logic) and an orthomodular lattice (an algebraic counterpart of the logic of quantum mechanics). Recall that a **basic algebra** (see e.g. [7, 10]) is an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type (2, 1, 0) satisfying the following identities

(BA1) $x \oplus 0 = x$;
(BA2) $\neg\neg x = x$ (double negation);
(BA3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ (Łukasiewicz axiom);
(BA4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0$.

In a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$, we define the following term operations $x \to y = \neg x \oplus y$ and $x \odot y = \neg(\neg x \oplus \neg y)$.

A basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is called **commutative** if it satisfies the identity $x \oplus y = y \oplus x$ which is equivalent to $x \odot y = y \odot x$, see e.g. [4] or [5].

Every basic algebra bears a natural order relation defined by $x \leq y$ if and only if $\neg x \oplus y = 1$ (1 denotes $\neg 0$). With respect to this order, $(A; \leq)$ is a bounded lattice where $0 \leq x \leq 1$ for each $x \in A$ and the lattice operations $\vee$ and $\wedge$ are defined by $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$. The following **Correspondence Theorem** is known (see e.g. [7], Theorem 8.5.7).

Proposition (a) *Let $\mathcal{L} = (L; \vee, \wedge, (^a)_{a\in L}, 0, 1)$ be a lattice with sectional antitone involutions. Then the assigned algebra $\mathcal{A}(L) = (L; \oplus, \neg, 0)$, where*

$$x \oplus y = (x^0 \vee y)^y \quad \textit{and} \quad \neg x = x^0$$

is a basic algebra.

(b) *Conversely, given a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$, we can assign a bounded lattice with sectional antitone involutions $\mathcal{L}(A) = (A; \vee, \wedge, (^a)_{a\in A}, 0, 1)$, where $1 = \neg 0$,*

$$x \vee y = \neg(\neg x \oplus y) \oplus y, \qquad x \wedge y = \neg(\neg x \vee \neg y)$$

and for each $a \in A$, the mapping $x \mapsto x^a = \neg x \oplus a$ is an antitone involution on the principal filter $[a, 1]$, where the order is given by

$$x \leq y \quad \textit{if and only if} \quad \neg x \oplus y = 1.$$

(c) *The assignments are in a one-to-one correspondence*, *i.e.* $\mathcal{A}(\mathcal{L}(A)) = \mathcal{A}$ *and* $\mathcal{L}(\mathcal{A}(L)) = \mathcal{L}$.

Let us note that if a basic algebra $\mathcal{A}$ is commutative then the assigned lattice $\mathcal{L}(A)$ is distributive (see [7], Theorem 8.5.9).

The propositional logic corresponding to a commutative basic algebra was already described (see [3]). Our aim is to introduce tense operators G, H, F, P on any basic algebra in a way which corresponds to that for MV-algebras in [11]. The interpretation can be as follows. Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a given basic algebra and $(T; \leq)$ a non-empty poset. Let A^T denotes the set of all the functions $f : T \to A$. For $f, g \in A^T$ we define the operations $\oplus$ and $\neg$ componentwise and denote by 0 the constant function f with $f(t) = 0$ for any $t \in T$. Of course, $\mathcal{A}(T) = (A^T; \oplus, \neg, 0)$ is a basic algebra again which is in fact a direct power of $\mathcal{A}$. Now, we can define for any formula Φ of the (first order) language of basic algebras that

$$
\begin{array}{llllll}
G(\Phi(t)) & \text{is valid if} & \Phi(s) & \text{is valid for all} & s \geq t & \\
H(\Phi(t)) & \text{is valid if} & \Phi(s) & \text{is valid for all} & s \leq t & \\
P(\Phi(t)) & \text{is valid if there exists an} & s < t & \text{such that} & \Phi(s) & \text{is valid} \\
F(\Phi(t)) & \text{is valid if there exists an} & s > t & \text{such that} & \Phi(s) & \text{is valid}
\end{array}
$$

We can formalize this as follows. Assume for a moment that $\mathcal{L}(A)$ is a complete lattice. Define G, H, P, F as unary operators: $A^T \to A^T$ such that

$$G(\Phi(x)) = \bigwedge\{\Phi(y); x \leq y\},$$

$$H(\Phi(x)) = \bigwedge\{\Phi(y); y \leq x\}$$

and

$$P(\Phi(x)) = \neg H(\neg\Phi(x)),$$

$$F(\Phi(x)) = \neg G(\neg\Phi(x)).$$

One can check several interesting properties of these tense operators G, H, P, F which are explicitly captured in the following definition.

Definition 1 Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra, let G, H be unary operations on A satisfying

(1) $G(1) = 1, \quad H(1) = 1;$
(2) $G(x \to y) \leq G(x) \to G(y), \quad H(x \to y) \leq H(x) \to H(y);$
(3) $G(x) \oplus G(y) \leq G(x \oplus y), \quad H(x) \oplus H(y) \leq H(x \oplus y);$
(4) $G(x \wedge y) = G(x) \wedge G(y), \quad H(x \wedge y) = H(x) \wedge H(y);$
(5) $x \leq GP(x), \quad x \leq HF(x),$

where $x \to y$ stands for $y \oplus \neg x$ and $P(x) = \neg H(\neg x)$, $F(x) = \neg G(\neg x)$. Then the algebra $(\mathcal{A}, G, H) = (A; \oplus, \neg, 0, G, H)$ will be called a **tense basic algebra** and G, H will be called **tense operators**.

Remark One can easily check that if a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is a Boolean algebra (where $\oplus$ is equal to $\vee$, $\neg$ is the complementation) then the tense operators G, H on $\mathcal{A}$ satisfy axioms (B1), (B2), (B3) and thus our concept is sound. Moreover, if $\mathcal{A}$ is an MV-algebra then G, H satisfy axioms (A0)–(A3) and (A5) for tense operators on MV-algebras as introduced in [11].

We are ready to list elementary properties of tense basic algebras:

Lemma 1 *Let* $(\mathcal{A}, G, H)$ *be a tense basic algebra an* $x, y \in A$. *Then*

(a) $G(0) \le G(x)$ *and* $H(0) \le H(x)$;
(b) $x \le y$ *implies* $G(x) \le G(y)$ *and* $H(x) \le H(y)$;
(c) $PG(x) \le x$ *and* $FH(x) \le x$;
(d) $x \le y$ *implies* $P(x) \le P(y)$, $F(x) \le F(y)$.

Proof (a) Since $0 \to x = 1$ and, by (1), $G(1) = 1 = H(1)$, we get $1 = G(1) = G(0 \to x) \le G(0) \to G(x)$ thus $G(0) \to G(x) = 1$ which implies $G(0) \le G(x)$. Analogously the second inequality can be shown.

(b) If $x \le y$ then $x \to y = 1$ thus $1 = G(1) = G(x \to y) \le G(x) \to G(y)$ whence $G(x) \le G(y)$, analogously for H.

(c) By (5) we have $\neg x \le HF(\neg x) = H(\neg G(x))$ and hence $PG(x) = \neg H(\neg G(x)) \le x$, similarly $FH(x) \le x$.

(d) Clearly $x \le y$ implies $\neg y \le \neg x$. Applying (b) we have $H(\neg y) \le H(\neg x)$ and, consequently, $P(x) = \neg(H(\neg x)) \le \neg(H(\neg y)) = P(y)$. Analogously, the second inequality can be shown. □

Example 1 There are two extreme examples of tense operators on a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$:

Define $G = H$ such that $G(1) = 1$ and $G(x) = 0$ for $x \neq 1$. One can easily check that $(\mathcal{A}, G, H)$ is a tense basic algebra.

Another example of tense operators are identical mappings, i.e. $G(x) = x = H(x)$ for all $x \in A$.

In the following we present non-trivial tense operators on a basic algebra:

Example 2 Let $A = \{o, p, q, r, s, t, v, w, j\}$ and $\mathcal{A} = (A; \oplus, \neg, o)$ be a basic algebra whose operations are defined as follows:

x	o	p	q	r	s	t	v	w	j
$\neg x$	j	w	v	t	s	r	q	p	o

$\oplus$	o	p	q	r	s	t	v	w	j
o	o	p	q	r	s	t	v	w	j
p	p	q	q	s	t	t	w	j	j
q	q	q	q	t	t	t	j	j	j
r	r	s	t	s	w	j	v	w	j
s	s	t	t	w	j	j	w	j	j
t	t	t	t	j	j	j	j	j	j
v	v	w	j	v	w	j	v	w	j
w	w	j	j	w	j	j	w	j	j
j	j	j	j	j	j	j	j	j	j

Define the operators G, H by the table

x	o	p	q	r	s	t	v	w	j
$G(x)$	o	p	q	o	s	t	o	s	j
$H(x)$	o	o	o	r	s	s	v	w	j

Then $(\mathcal{A}; G, H)$ is a commutative tense basic algebra.

To prove our first result, we need to check several important properties of basic algebras. A basic algebra $\mathcal{A}$ is called a **chain basic algebra** if the assigned lattice $\mathcal{L}(A)$ is a chain. A basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is called **left-monotonous** if $x \leq y$ implies $z \odot x \leq z \odot y$ (which is equivalent to $x \leq y$ implies $z \oplus x \leq z \oplus y$) for every $x, y, z \in A$. Let us note that the right-monotonicity $x \leq y \Rightarrow x \odot z \leq y \odot z$ holds in every basic algebra and hence every commutative basic algebra is left-monotonous. However, there exist basic algebras which are not chain basic algebras and which are not left-monotonous (and hence not commutative), see e.g. [7], Example 8.5.2. If the assigned lattice $\mathcal{L}(A)$ of $\mathcal{A}$ is complete, we speak about a **complete** basic algebra.

Lemma 2 *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a complete left-monotonous basic algebra and $a_i, b_i \in I$ for $i \in I$. Then*

(i) $\bigwedge\{a_i \to b_i; i \in I\} \leq \bigwedge\{a_i; i \in I\} \to \bigwedge\{b_i; i \in I\}$;
(ii) $\bigwedge\{a_i; i \in I\} \oplus \bigwedge\{b_i; i \in I\} \leq \bigwedge\{a_i \oplus b_i; i \in I\}$;
(iii) *If $\mathcal{A}$ is, moreover, a chain basic algebra and fulfils the identity $x \oplus \bigwedge\{y_i; i \in I\} = \bigwedge\{x \oplus y_i; i \in I\}$, then*

$$\bigwedge\{a_i \oplus a_i; i \in I\} = \bigwedge\{a_i; i \in I\} \oplus \bigwedge\{a_i; i \in I\}.$$

Proof (i) By [2] (Lemma 2), every basic algebra is a left residuated groupoid, i.e.

$$x \leq y \to z \quad \text{if and only if} \quad x \odot y \leq z.$$

Hence, our identity (i) is equivalent to

$$\bigwedge\{a_i \to b_i; i \in I\} \odot \bigwedge\{a_i; i \in I\} \leq \bigwedge\{b_i; i \in I\}. \tag{i$'$}$$

Since $\mathcal{A}$ is left-monotonous and $(x \to y) \odot x = (y \oplus \neg x) \odot x = \neg(\neg(y \oplus \neg x) \oplus \neg x) = \neg(\neg y \vee \neg x) = \neg(\neg x \vee \neg y) = x \wedge y$, we infer

$$\bigwedge\{a_i \to b_i; i \in I\} \odot \bigwedge\{a_i; i \in I\} \leq (a_i \to b_i) \odot a_i = a_i \wedge b_i \leq b_i$$

for each $i \in I$ which yields (i') immediately.

(ii) It follows directly by the monotonicity of $\oplus$.

(iii) Suppose now that $\mathcal{A}$ is a chain basic algebra. Then $a_i \leq a_j$ or $a_i \geq a_j$ for every $i, j \in I$, i.e.

$$a_i \oplus a_j \geq a_i \oplus a_i \quad \text{or} \quad a_i \oplus a_j \geq a_j \oplus a_j.$$

This implies $\bigwedge\{a_i \oplus a_j; i, j \in I\} \geq \bigwedge\{a_i \oplus a_i; i \in I\}$. In every basic algebra, we have

$$\bigwedge\{y_i; i \in I\} \oplus x = \bigwedge\{y_i \oplus x; i \in I\}$$

(see e.g. [1]). Hence, applying the identity $x \oplus \bigwedge\{y_i; i \in I\} = \bigwedge\{x \oplus y_i; i \in I\}$, we obtain

$$\bigwedge\{a_i \oplus a_j; i, j \in I\} = \bigwedge\{a_i; i \in I\} \oplus \bigwedge\{a_i; i \in I\}.$$

Together, we conclude

$$\bigwedge\{a_i \oplus a_i; i \in I\} \leq \bigwedge\{a_i; i \in I\} \oplus \bigwedge\{a_i; i \in I\}.$$

In account of (ii), we obtain (iii). □

Adapting the terminology of [11], a **frame** is a couple (T, R), where T is a non-void set and R is a binary relation on T, i.e. $R \subseteq T \times T$. In what follows, we assume that R is non-empty. Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra. By $\mathcal{A}^T$ is meant the direct power of $\mathcal{A}$, i.e. the base set of $\mathcal{A}^T$ is the set of all functions from T to A and the operations $\oplus, \neg$ are defined pointwise.

Theorem 1 *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a complete left-monotonous basic algebra and (T, R) a frame with R reflexive. Define the operators G^*, H^* on $\mathcal{A}^T$ as follows*:

$G^*(p)(x) = \bigwedge\{p(y); xRy\}$
$H^*(p)(x) = \bigwedge\{p(y); yRx\}.$

Then G^*, H^* are tense operators on $\mathcal{A}^T$ such that $G^*(0) = 0$ and $H^*(0) = 0$.

Proof It is easy to see that G^*, H^* satisfy condition (1) of Definition 1.

To prove (2) for G^*, we need to show

$$\bigwedge\{p(y); xRy\} \to \bigwedge\{q(y); xRy\} \geq \bigwedge\{p(y) \to q(y); xRy\}$$

which follows by (i) of Lemma 2. Analogously, the inequality for H^* can be shown.

To prove (3), we apply (ii) of Lemma 2 in the same way. For (4) we use the definition of the infimum in a complete lattice.

It remains to prove (5). As R is reflexive, we compute for any $x, y \in A$ with xRy

$$(\neg H^*(\neg p))(y) = \neg \bigwedge\{\neg p(z); zRy\} = \bigvee\{p(z); zRy\} \geq p(x),$$

thus

$$G^*(\neg H^*(\neg p))(x) = \bigwedge\{(\neg H^*(\neg p))(y); xRy\} \geq p(x).$$

Since $0(x) = 0$ for each $x \in T$, it is evident that $\bigwedge\{0(y); xRy\} = 0 = \bigwedge\{0(y); yRx\}$ and hence $G^*(0) = 0 = H^*(0)$. □

Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra and (T, R) a frame with reflexive relation R. The tense operators G^*, H^* defined in Theorem 1 will be called **natural tense operators** on $\mathcal{A}^T$. One can mention that natural tense operators satisfy $G(0) = 0 = H(0)$.

It is known (see e.g. [6] or [11]) that if $(\mathcal{A}, G, H)$ is a tense Boolean algebra then there exists a frame (T, R) such that $(\mathcal{A}, G, H)$ can be isomorphically embedded into a tense Boolean algebra $(\mathcal{L}^T, G^*, H^*)$, where $\mathcal{L}$ is a two-element Boolean algebra. An analogous result for MV-algebras (where $\mathcal{L}$ should be an MV-chain) is not known. Let us remark that if $\mathcal{A} = \mathcal{L}^T$, where $\mathcal{L}$ is a chain basic algebra and $(\mathcal{A}, G, H)$ is a tense basic algebra then there

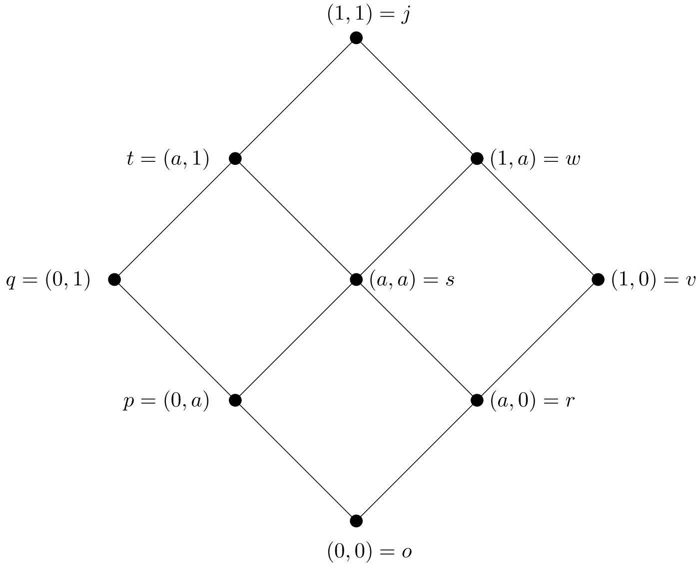

Fig. 1 The induced lattice of $\mathcal{A}$ from Example 3

need not exist a binary relation R on T such that $(\mathcal{A}, G, H)$ is isomorphic to $(\mathcal{L}^T, G^*, H^*)$, see the following

Example 3 Let $\mathcal{A}$ be the basic algebra of Example 2 and let G, H be defined as follows:

x	o	p	q	r	s	t	v	w	j
$H(x) = G(x)$	o	o	q	o	o	q	v	v	j

Then $\mathcal{A}$ is the direct product of two copies of a 3-element chain basic algebra $\mathcal{L}$ where $0 < a < 1$, see Fig. 1.

Let $H_L = G_L$ be defined on L as follows: $G_L(1) = 1$, $G_L(a) = G_L(0) = 0$. Then clearly G_L, H_L are tense operators on $\mathcal{L}$ and hence $G = G_L \times G_L$, $H = H_L \times H_L$ are tense operators on $\mathcal{A} = \mathcal{L} \times \mathcal{L}$ as well. It is an exercise to check that there does not exist a binary relation on $T = \{1, 2\}$ such that G and H are natural tense operators with respect to the frame (T, R).

On the other hand, if G, H are tense operators defined on $\mathcal{A}$ as given in Example 2 then they are natural and induced by the frame $(T, \leq)$ where $T = \{1, 2\}$ and $\leq$ is the natural order on the set $\{1, 2\}$.

By (5) and (c) of Lemma 1 it follows, that P, G as well as F, H form a Galois connection on A, i.e.

$$x \leq G(y) \text{ iff } P(x) \leq y$$

and

$$x \leq H(y) \text{ iff } F(x) \leq y.$$

Consequently, we have

Theorem 2 *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra and G, H its tense operators. Then G and H preserve arbitrary infima, whenever they exist.*

Theorem 3 *If $(\mathcal{A}, G, H)$ is a tense basic algebra then it satisfies*

$$G(x) \odot G(y) \leq G(x \odot y) \quad \text{and} \quad H(x) \odot H(y) \leq H(x \odot y).$$

Proof According to the left-residuation property we have $x \leq y \to (x \odot y)$, thus by (b) of Lemma 1 and (2) of Definition 1 we obtain

$$G(x) \leq G(y \to (x \odot y)) \leq G(y) \to G(x \odot y).$$

Finally, applying left residuation property once more we conclude $G(x) \odot G(y) \leq G(x \odot y)$.
Analogously the inequality for the tense operator H can be shown. □

Corollary 1 *If $(\mathcal{A}, G, H)$ is a tense basic algebra then*

$$F(x \oplus y) \leq F(x) \oplus F(y) \quad \text{and} \quad P(x \oplus y) \leq P(x) \oplus P(y).$$

Proof By Theorem 3 we have $G(\neg x) \odot G(\neg y) \leq G(\neg x \odot \neg y)$ thus $F(x \oplus y) = \neg G(\neg(x \oplus y)) = \neg G(\neg x \odot \neg y) \leq \neg(G(\neg x) \odot G(\neg y)) = \neg G(\neg x) \oplus \neg G(\neg y) = F(x) \oplus F(y)$ and analogously for P. □

By the proposition, to every basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ there is assigned a lattice $\mathcal{L}(A) = (A; \vee, \wedge, (^a)_{a\in A}, 0, 1)$ with sectional antitone involutions and, conversely, to every lattice $\mathcal{L} = (L; \vee, \wedge, (^a)_{a\in L}, 0, 1)$ with sectional antitone involutions there is assigned a basic algebra $\mathcal{A}(L) = (L; \oplus, \neg, 0)$. There is a natural question under which conditions given on tense operators G and H on $\mathcal{L}(A)$ the resulting algebra will be a tense basic algebra.

Theorem 4 *Let $(\mathcal{A}, G, H)$ be a tense basic algebra and $\mathcal{L}(A) = (A; \vee, \wedge, (^a)_{a\in L}, 0, 1)$ the assigned lattice with sectional antitone involutions. Then G and H satisfy the following conditions*

$$\begin{aligned} G((x \vee y)^y) &\leq (G(x) \vee G(y))^{G(y)} \\ H((x \vee y)^y) &\leq (H(x) \vee H(y))^{H(y)} \end{aligned} \tag{L1}$$

$$\begin{aligned} (\neg G(x) \vee G(y))^{G(y)} &\leq G((\neg x \vee y)^y) \\ (\neg H(x) \vee H(y))^{H(y)} &\leq H((\neg x \vee y)^y) \end{aligned} \tag{L2}$$

If $\mathcal{L} = (L; \vee, \wedge, (^a)_{a\in L}, 0, 1)$ is a lattice with sectional antitone involutions and $G, H : L \to L$ are mappings satisfying (1), (4), (5), (L1) *and* (L2) *then the assigned algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ endowed by the operators G, H is a tense basic algebra.*

Proof Since $(x \vee y)^y = x \to y$ and $(\neg x \vee y)^y = x \oplus y$, it is easy to verify (L1) and (L2) for any tense basic algebra $(\mathcal{A}, G, H)$. Also conversely, if (1), (4), (5) (L1) and (L2) are satisfied then it is straightforward to verify (2) and (3) of Definition 1. □

Definition 2 A tense basic algebra $(\mathcal{A}, G, H)$ is called a **strict tense basic algebra** if it satisfies

(1°) $G(0) = 0, \quad H(0) = 0;$
(2°) $G(x \oplus x) = G(x) \oplus G(x), \quad H(x \oplus x) = H(x) \oplus H(x).$

Lemma 3 *Let $\mathcal{C} = (C; \oplus, \neg, 0)$ be a complete commutative chain basic algebra and (T, R) a frame. Let G^*, H^* be the natural tense operators on $\mathcal{C}^T$. Then $(\mathcal{C}^T, G, H)$ is a strict tense basic algebra.*

Proof Since $\mathcal{C}$ is commutative, it satisfies the identity $x \oplus \bigwedge\{y_i; i \in I\} = \bigwedge\{x \oplus y_i; i \in I\}$. Consequently, applying (iii) of Lemma 2, G^*, H^* fulfil (2°). Clearly, every natural tense operators satisfy $G^*(0) = 0 = H^*(0)$. □

Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra and G, H tense operators on $\mathcal{A}$. Let $T \neq \emptyset$ be a set. Consider the operators G^T and H^T on the direct power $\mathcal{A}^T$ defined by $G^T(f) := G \circ f$ and $H^T(f) := H \circ f$ for all $f \in \mathcal{A}^T$. It is evident that G^T, H^T are again tense operators on $\mathcal{A}^T$. However the converse does not hold in general.

Moreover, we can show that the construction of operators G^*, H^* in Theorem 1 is possible only under the assumption that $\mathcal{A}$ is left-monotonous.

Example 4 Consider the four-element basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ where $A = \{0, a, b, 1\}$ and the operations are determined by the tables

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	1	b	1
b	b	a	1	1
1	1	1	1	1

x	0	a	b	1
$\neg x$	1	a	b	0

Let (T, R) be a frame such that $T = \{1, 2, 3\}$ and let R denote the natural order on T, i.e. $1 \leq 2 \leq 3$. Then G^* and H^* are determined by the table:

x	$G^*(x)$	$H^*(x)$
(000)	(000)	(000)
$(00a)$	$(00a)$	(000)
$(00b)$	$(00b)$	(000)
(001)	(001)	(000)
$(0a0)$	(000)	(000)
$(0aa)$	$(0aa)$	(000)
$(0ab)$	$(00b)$	(000)
$(0a1)$	$(0a1)$	(000)
$(0b0)$	(000)	(000)
$(0ba)$	$(00a)$	(000)
$(0bb)$	$(0bb)$	(000)
$(0b1)$	$(0b1)$	(000)
(010)	(000)	(000)
$(01a)$	$(0aa)$	(000)
$(01b)$	$(0bb)$	(000)
(011)	(011)	(000)
$(a00)$	(000)	$(a00)$
$(a0a)$	$(00a)$	$(a00)$
$(a0b)$	$(00b)$	$(a00)$
$(a01)$	(001)	$(a00)$
$(aa0)$	(000)	$(aa0)$
(aaa)	(aaa)	(aaa)

x	$G^*(x)$	$H^*(x)$
(aab)	$(00b)$	$(aa0)$
$(aa1)$	$(aa1)$	(aaa)
$(ab0)$	(000)	$(a00)$
(aba)	$(00a)$	$(a00)$
(abb)	$(0bb)$	$(a00)$
$(ab1)$	$(0b1)$	$(a00)$
$(a10)$	(000)	$(aa0)$
$(a1a)$	(aaa)	(aaa)
$(a1b)$	$(0bb)$	$(aa0)$
$(a11)$	$(a11)$	(aaa)
$(b00)$	(000)	$(b00)$
$(b0a)$	$(00a)$	$(b00)$
$(b0b)$	$(00b)$	$(b00)$
$(b01)$	(001)	$(b00)$
$(ba0)$	(000)	$(b00)$
(baa)	$(0aa)$	$(b00)$
(bab)	$(00b)$	$(b00)$
$(ba1)$	$(0a1)$	$(b00)$
$(bb0)$	(000)	$(bb0)$
(bba)	$(00a)$	$(bb0)$
(bbb)	(bbb)	(bbb)

x	$G^*(x)$	$H^*(x)$
$(bb1)$	$(bb1)$	(bbb)
$(b10)$	(000)	$(bb0)$
$(b1a)$	$(0aa)$	$(bb0)$
$(b1b)$	(bbb)	(bbb)
$(b11)$	$(b11)$	(bbb)
(100)	(000)	(100)
$(10a)$	$(00a)$	(100)
$(10b)$	$(00b)$	(100)
(101)	(001)	(100)
$(1a0)$	(000)	$(1a0)$
$(1aa)$	(aaa)	$(1aa)$
$(1ab)$	$(00b)$	$(1a0)$
$(1a1)$	$(aa1)$	$(1aa)$
$(1b0)$	(000)	$(1b0)$
$(1ba)$	$(00a)$	$(1b0)$
$(1bb)$	(bbb)	$(1bb)$
$(1b1)$	$(bb1)$	$(1bb)$
(110)	(000)	(110)
$(11a)$	(aaa)	$(11a)$
$(11b)$	(bbb)	$(11b)$
(111)	(111)	(111)

One can easily see that e.g.

$$G^*((aaa)) \oplus G^*((aba)) = (aaa) \oplus (00a) = (aa1)$$

but

$$G^*((aaa)) \oplus (aba)) = G^*(1b1) = (bb1).$$

Since the elements $(aa1)$ and $(bb1)$ are incomparable, axiom (3) of Definition 1 is violated. Of course, the algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is not left-monotonous since e.g. $0 \le a$ but $b \oplus 0 = b \not\le a = b \oplus a$.

When $(\{1, 2, 3\}, \le)$ is a frame then clearly $G^*(p)(3)$ can be considered as a true value of a given proposition in the future of p and $H^*(p)(1)$ as the same in the past.

This can be shown in the following

Example 5 Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra where $A = \{0, a, 1\}$ is a chain the operations of which are given by the following tables:

$\oplus$	0	a	1
0	0	a	1
a	a	1	1
1	1	1	1

x	0	a	1
$\neg x$	1	a	0

Since $\mathcal{A}$ is left-monotonous, our construction of G^*, H^* gives tense operators on $\mathcal{A}$.

Let $T = \{1, 2, 3\}$ and R be the natural order $1 \le 2 \le 3$ on T. Then G^*, H^* are as follows

x	$G^*(x)$	$H^*(x)$	x	$G^*(x)$	$H^*(x)$	x	$G^*(x)$	$H^*(x)$
(000)	(000)	(000)	$(a00)$	(000)	$(a00)$	(100)	(000)	(100)
$(00a)$	$(00a)$	(000)	$(a0a)$	$(00a)$	$(a00)$	$(10a)$	$(00a)$	(100)
(001)	(001)	(000)	$(a01)$	(001)	$(a00)$	(101)	(001)	(100)
$(0a0)$	(000)	(000)	$(aa0)$	(000)	$(aa0)$	$(1a0)$	(000)	$(1a0)$
$(0aa)$	$(0aa)$	(000)	(aaa)	(aaa)	(aaa)	$(1aa)$	(aaa)	$(1aa)$
$(0a1)$	$(0a1)$	(000)	$(aa1)$	$(aa1)$	(aaa)	$(1a1)$	$(aa1)$	$(1aa)$
(010)	(000)	(000)	$(a10)$	(000)	$(aa0)$	(110)	(000)	(110)
$(01a)$	$(0aa)$	(000)	$(a1a)$	(aaa)	(aaa)	$(11a)$	(aaa)	$(11a)$
(011)	(011)	(000)	$(a11)$	$(a11)$	(aaa)	(111)	(111)	(111)

One can easily see that for each element $x \in A^T$ we have $H^*(x)(1) = x(1)$ and $G^*(x)(3) = x(3)$ and hence the operators H^* or G^* describe the past or the future state of the proposition x. Moreover, it is evident that neither H^* nor G^* can be expressed componentwise, i.e. as $H^* = H_1 \times H_2 \times H_3$, $G^* = G_1 \times G_2 \times G_3$ for some tense operators G_i, H_i on $\mathcal{A}$ $(i = 1, 2, 3)$ since e.g. for $i = 2$ we would have $G_2(a) = G^*(aa0)(2) = (000)(2) = 0$ but $G_2(a) = G^*(aa1)(2) = (aa1)(2) = a$, a contradiction.

The concept of tense operators was already introduced for lattice effect algebras by the second and fourth author in [9]. Let us recall that an effect algebra is a system $\mathcal{E} = (E; +, 0, 1)$ where $+$ is a partial binary operation on E, 0 and 1 are distinguished elements of E and $\mathcal{E}$ satisfies the axioms

(E1) $x + y$ is defined if and only if $y + x$ is defined and then $x + y = y + x$.
(E2) $x + (y + z)$ is defined if and only if $(x + y) + z$ is defined and then $x + (y + z) = (x + y) + z$.

(E3) For each $x \in E$ there exists a unique element $y \in E$ such that $x + y = 1$; this element y is called the **supplement** of x and it is denoted by x'.
(E4) If $x + 1$ is defined then $x = 0$.

Let us recall that on every effect algebra $\mathcal{E}$ the induced order is defined by

$$x \leq y \quad \text{if} \quad z + x = y \quad \text{for some } z \in E.$$

It can be shown that $x + y$ is defined iff $x \leq y'$.

If $(E; \leq)$ is a lattice, $\mathcal{E}$ is called a **lattice effect algebra**.

The following concept was introduced in [9]. Let $\mathcal{E} = (E; +, 0, 1)$ be a lattice effect algebra. A triple $(\mathcal{E}; G, H)$ is called a **dynamic effect algebra** if G, H are mappings of E into itself such that

(T1) $G(1) = 1 = H(1)$.
(T2) $G(x \wedge y) = G(x) \wedge G(y)$, $H(x \wedge y) = H(x) \wedge H(y)$.
(T3) If $x + y$ is defined then $G(x) + G(y)$ and $H(x) + H(y)$ are defined and $G(x) + G(y) \leq G(x + y)$, $H(x) + H(y) \leq H(x + y)$.
(T4) $G(x') \leq G(x)'$, $H(x') \leq H(x)'$.
(T5) $x \leq GP(x)$, $x \leq HF(x)$ where $P(x) = H(x')'$ and $F(x) = G(x')'$.

Here $G(x)'$ or $H(x)'$ is an abbreviation for $(G(x))'$ or $(H(x))'$, respectively.

These G, H are called **tense operators** of $\mathcal{E}$.

Since every lattice effect algebra can be converted into a total algebra which is a basic algebra, see [8], we are interested in the question how the tense operators of these algebras are related.

A basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ satisfying the quasiidentity

$$x \leq \neg y \quad \text{and} \quad x \oplus y \leq \neg z \quad \Rightarrow \quad x \oplus (z \oplus y) = (x \oplus y) \oplus z$$

is called an **effect basic algebra** [8]. We define $1 = \neg 0$ and

$$x + y = x \oplus y \quad \text{if } x \leq \neg y$$

and otherwise $x + y$ is not defined. Then $\mathcal{E}(A) = (A; +, 0, 1)$ is a lattice effect algebra called the **lattice effect algebra induced** by $\mathcal{A}$.

Also conversely, for a lattice effect algebra $\mathcal{E} = (E; +, 0, 1)$ we can define $\neg x = x'$ and

$$x \oplus y = (x \wedge y') + y.$$

Then $\oplus$ is a total operation on E and $\mathcal{A}(E) = (E; \oplus, \neg, 0)$ is an effect basic algebra called the **effect basic algebra corresponding** to $\mathcal{E}$. For details of these constructions, the reader is referred to [8]. Now we can answer our question as follows.

Lemma 4 *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be an effect basic algebra and G, H tense operators on $\mathcal{A}$. Let $\mathcal{E}(A)$ be the induced effect algebra. If $G(0) = 0 = H(0)$ then $(\mathcal{E}(A); G, H)$ is a dynamic effect algebra.*

Proof Assume $G(0) = 0 = H(0)$. It is evident that G, H satisfy axioms (T1), (T2) and (T5).

We are going to verify (T4). We have

$$G(x') = G(\neg x) = G(x \to 0) \leq G(x) \to G(0) = G(x) \to 0 = G(x)'$$

and, analogously, $H(x') \leq H(x)'$.

To prove (T3), assume $x+y$ is defined, i.e. $x \leq y'$. Then $x+y=x\oplus y$ and by (3) of Definition 1, $G(x)\oplus G(y)\leq G(x\oplus y)=G(x+y)$. It suffices to show that $G(x)+G(y)$ is defined. Since G is monotone by (b) of Lemma 1, applying (T4) we conclude $G(x)\leq G(y')\leq G(y)'$. This proves that $G(x)\oplus G(y)$ is defined. The second part of (T3) can be verified in a similar way.

Thus G, H are tense operators on $\mathcal{E}(A)$. □

To show the converse, we have to introduce the following inequalities:

$$\begin{array}{l}(G(x')\wedge G(y)')+G(y)\leq G((x'\wedge y')+y)\leq (G(x)'\wedge G(y)')+G(y)\\ (H(x')\wedge H(y)')+H(y)\leq H((x'\wedge y')+y)\leq (H(x)'\wedge H(y)')+H(y).\end{array} \tag{L3}$$

Theorem 5 *Let $(\mathcal{E}; G, H)$ be a dynamic effect algebra and $\mathcal{A}(E)$ the corresponding effect basic algebra. Then G, H are tense operators on $\mathcal{A}(E)$ if and only if condition (L3) holds in $(\mathcal{E}; G, H)$.*

Proof Assume (L3) holds in $(\mathcal{E}; G, H)$. Then for $x\oplus y=(x\wedge y')+y$ and $\neg x=x'$ we have $x\to y=\neg x\oplus y=(x'\wedge y')+y$ and, due to (L3), we compute

$$G(x)\oplus G(y)=(G(x)\wedge G(y)')+G(y)\leq G((x\wedge y')+y)=G(x\oplus y)$$

$$G(x\to y)=G((x'\wedge y')+y)\leq (G(x)'\wedge G(y)')+G(y)=G(x)\to G(y)$$

and, analogously, we obtain $H(x)\oplus H(y)\leq H(x\oplus y)$ and $H(x\to y)\leq H(x)\to H(y)$ proving axioms (2) and (3) of Definition 1. The other axioms are evident and hence G, H are tense operators on the basic algebra $\mathcal{A}(E)$.

Conversely, assume that $(\mathcal{E}; G, H)$ is a dynamic effect algebra and that G, H are simultaneously tense operators on the corresponding effect basic algebra $\mathcal{A}(E)$. Then

$$G((x'\wedge y')+y)=G(x\to y)\leq G(x)\to G(y)=(G(x)'\wedge G(y)')+G(y)$$

and

$$(G(x')\wedge G(y)')+G(y)=G(x')\oplus G(y)\leq G(x'\oplus y)=G((x'\wedge y')+y).$$

Analogously we can prove (L3) for the operator H. □

References

1. Botur, M.: An example of a commutative basic algebra which is not an MV-algebra. Math. Slovaca **60**(2), 171–178 (2010)
2. Botur, M., Chajda, I., Halaš, R.: Are basic algebras residuated structures? Soft Comput. **14**(3), 251–255 (2009)
3. Botur, M., Halaš, R.: Commutative basic algebras and non-associative fuzzy logics. Arch. Math. Log. **48**(3–4), 243–255 (2009)
4. Botur, M., Halaš, R.: Complete commutative basic algebras. Order **24**, 89–105 (2007)
5. Botur, M., Halaš, R.: Finite commutative basic algebras are MV-algebras. J. Mult.-Valued Log. Soft Comput. **14**(1–2), 69–80 (2008)
6. Burges, J.: Basic tense logic. In: Gabbay, D.M., Günther, F. (eds.) Handbook of Philosophical Logic, vol. II, pp. 89–139. Reidel, Dordrecht (1984)

7. Chajda, I., Halaš, R., Kühr, J.: Semilattice Structures, Heldermann, Lemgo (2007). ISBN:978-3-88538-230-0
8. Chajda, I., Halaš, R., Kühr, J.: Many-valued quantum algebras. Algebra Univers. **60**(1), 63–90 (2009)
9. Chajda, I., Kolařík, M.: Dynamic effect algebras. Math. Slovaca (submitted)
10. Chajda, I., Kolařík, M.: Independence of axiom system of basic algebras. Soft Comput. **13**(1), 41–43 (2009)
11. Diaconescu, D., Georgescu, G.: Tense operators on MV-algebras and Łukasiewicz-Moisil algebras. Fundam. Inform. **81**, 1–30 (2007)