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The Modified Bell Inequality and Its Physical Implications in the ESR Model

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Abstract The *extended semantic realism* (*ESR*) *model* proposes a theoretical perspective which reinterprets quantum probabilities as conditional on detection rather than absolute and embodies the mathematical formalism of standard (Hilbert space) quantum mechanics (QM) in a noncontextual, hence local, framework. The assumptions needed to prove the *Bell inequality* therefore hold in the ESR model, but we show that the Bell inequality must be substituted in it by the *modified Bell inequality* and that the standard quantum expectation values, when reinterpreted as proposed by the ESR model, do not violate the latter inequality. Hence the long-standing conflict between "local realism" and QM is settled in the ESR model. Finally we provide an elementary example of a prediction that might be used to check whether the ESR model is correct.

Keywords Quantum mechanics · Quantum probabilities · Bell's inequalities · Local realism

1 Introduction

We have recently presented an improved version of the *Extended Semantic Realism* (ESR) model [1] and a mathematical representation of the physical entities that occur in it, recovering the mathematical apparatus of quantum mechanics (QM) as a part of our formalism [2]. We have expounded in many papers the reasons that led us to contrive this model, hence we limit ourselves here to recall the essentials of our arguments.

As many physicists we were puzzled at the beginning of our research by the paradoxes of QM and by the *objectification problem* which occurs in the quantum theory of measurement

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when a (minimal) realistic position is adopted [3]. The common root of these difficulties can be traced back to the *contextuality* and *nonlocality* of QM, which imply nonobjectivity of physical properties and are commonly maintained to be unavoidable features of QM because of known "no-go" theorems (mainly the Bell–Kochen–Specker [4, 5] and the Bell [6] theorems). We therefore carried out a critical analysis of these theorems and realized that their proofs require a strong assumption on the range of validity of empirical quantum laws (*metatheoretical classical principle*, or MCP) which is usually left implicit and yet is problematic both from a quantum and from an epistemological point of view [7–11]. If MCP is replaced by a weaker assumption (*metatheoretical generalized principle*, or MGP) the proofs of the aforesaid theorems cannot be completed and the possibility of providing an interpretation of QM that restores some forms of objectivity of properties cannot be excluded. An interpretation of this kind was then worked out (*Semantic Realism*, or SR, interpretation [7, 12, 13]) and some models were devised to show its consistency, among which the ESR model [14, 15]. This model was successively developed and enriched, and it can now be considered as an autonomous theoretical proposal that modifies and extends the original SR interpretation, maintaining, however, its fundamental assumptions (in particular, MGP).

Basically, the ESR model can be considered as a hidden variables (h.v.) theory for QM which recovers noncontextuality, hence locality, embedding the mathematical formalism of QM into a more general mathematical framework and circumventing the no-go theorems by reinterpreting quantum probabilities as conditional on detection instead of absolute. This new perspective has many interesting consequences that we have explored in some recent papers [1, 2, 16, 17]. In particular, it avoids quantum paradoxes and provides some predictions that do not coincide with the predictions of QM (hence it is, in principle, *falsifiable*). We have also proved that, though "local realism" holds in the ESR model, the *Bell-Clauser-Horne-Shimony-Holt* (*BCHSH*) *inequality* [18] must be substituted by a *modified BCHSH inequality* whenever macroscopic physical properties are considered which does not imply any conflict between local realism and the (reinterpreted) formalism of QM [1]. The underlying reason of this achievement is that the standard derivation of the BCHSH inequality from local realism requires the assumption, usually left implicit, that all examples of a physical system are detected when an *ideal* measurement is performed, which is a variant of MCP and does not hold in the ESR model.

We want to show in this paper that the original *Bell inequality* can be dealt with in a similar way. The situation with this inequality is however more complicated because its proof requires using a physical law (the *perfect correlation* law), besides local realism and MCP, which does not occur in the case of the BCHSH inequality. After reporting the essentials of the ESR model in Sect. 2 we therefore discuss in Sect. 3 how the *modified Bell inequality* can be obtained in the ESR model, comparing its deduction with the standard deduction of the Bell inequality in the literature. We then consider the consequences following from these inequalities in Sect. 4, showing that in the ESR model the modified Bell inequality does not imply a contradiction between local realism and the (reinterpreted) formalism of QM. We add that it provides instead some predictions that can, in principle, be empirically checked to confirm or disprove the ESR model.

2 The ESR Model

To make the present paper self-consistent we report in this section the essentials of the ESR model, together with some results that are needed in the following. More detailed treatments can be found in [1, 2, 16].

As anticipated in Sect. 1, the ESR model is a noncontextual h.v. theory for QM with reinterpretation of quantum probabilities. It therefore consists of a macroscopic part, where some physical entities occur that can be operationally defined and related with standard physical entities of QM, a microscopic part, in which h.v. are introduced, and some assumptions which establish a link between the microscopic and the macroscopic part, allowing the reinterpretation of quantum probabilities.

In the macroscopic part of the ESR model a physical system Ω is considered which is characterized by the sets $\mathcal{S}$ and $\mathcal{O}$ of its *macroscopic states* and *macroscopic generalized observables*, respectively. Following standard procedures [19], each state $S \in \mathcal{S}$ is operationally defined as a class of probabilistically equivalent *preparing devices*, and every preparing device $\pi \in S$, when constructed and activated, performs a preparation of an individual example x of Ω (*physical object*: one briefly says that "x is in the state S" in this case). Each generalized observable $A_0 \in \mathcal{O}$ is operationally defined as a class of probabilistically equivalent *measuring apparatuses*, and it is obtained in the ESR model by considering an observable A of QM with set of possible values Ξ on the real line $\Re$ and adding a further outcome $a_0 \in \Re \setminus \Xi$ (*no-registration outcome* of A_0), so that the set of all possible values of A_0 is $\Xi_0 = \{a_0\} \cup \Xi$.[1] Hence, the set $\mathcal{F}_0$ of all (*macroscopic*) *properties* of Ω is defined by

$$\mathcal{F}_0 = \{(A_0, X) | A_0 \in \mathcal{O}, X \in \mathbb{B}(\Re)\}, \tag{1}$$

where $\mathbb{B}(\Re)$ is the σ-algebra of all Borel subsets of $\Re$, and the subset $\mathcal{F} \subset \mathcal{F}_0$ of all properties associated with observables of QM is defined by

$$\mathcal{F} = \{(A_0, X) | A_0 \in \mathcal{O}, X \in \mathbb{B}(\Re), a_0 \notin X\}. \tag{2}$$

In the microscopic part of the ESR model a set $\mathcal{E}$ of theoretical entities is introduced called *microscopic properties*. These are the *dichotomic* h.v. of the model, and for every physical object x, each $f \in \mathcal{E}$ either takes a value interpreted as "f is possessed by x" or a value interpreted as "f is not possessed by x". The set of all microscopic properties possessed by x constitutes the *microscopic state* of x. We denote by $\mathcal{M} = (S^i)_{i \in I}$ the set of all microscopic states, with I a set of indexes that we assume to be discrete, for the sake of simplicity. It is then apparent that assigning a microscopic state S^i to x is equivalent to assigning the values of all h.v. on x, hence we find it convenient to consider S^i as a value of a global hidden variable whose values range over $\mathcal{M}$ in the following. Moreover, a microscopic state can be assigned to every physical object in the macroscopic state S, and we denote by $p(S^i|S)$ the conditional probability that such a microscopic state be S^i.

Finally the link between microscopic and macroscopic entities is established by assuming that a bijective mapping φ exists which associates every property $f \in \mathcal{E}$ with a property $F \in \mathcal{F}$.

Let us come to measurements in the ESR model. An *idealized measurement* of a property $F = (A_0, X)$ on a physical object x in the state S is described as a *registration* performed by means of a *dichotomic registering device* (which may be constructed by using one of the apparatuses associated with A_0) whose outcomes are denoted by *yes* and *no*. The measurement yields outcome yes/no (equivalently, *x displays/does not display F*) if the value of A_0 belongs/does not belong to X. We can now use the link between microscopic and macroscopic properties established by φ to interpret the result of a measurement of a macroscopic

[1] We assume here, for the sake of simplicity, that $\Re \setminus \Xi$ is non-void, which is not restrictive. Indeed, if $\Xi = \Re$, one can choose a bijective Borel function $f : \Re \rightarrow \Xi'$ such that $\Xi' \subset \Re$ (*e.g.*, $\Xi' = \Re^+$) and replace A by $f(A)$.

property $F = (A_0, X)$ on a physical object x. First of all, we assume that the set of all microscopic properties of x, that is, the microscopic state S^i of x, determines whether x is detected when F is measured (if we want to construct a *deterministic* ESR model) or, more generally, the probability that x be detected (if we want to construct a *probabilistic* ESR model). Then we assume that a measurement of F provides actual information about the microscopic properties possessed by x according to the following scheme. Let $F \in \mathcal{F}$, hence $a_0 \notin X$. If the yes outcome is obtained, the microscopic property $f = \varphi^{-1}(F)$ is possessed by x. If instead the no outcome is obtained, one knows that the value of A_0 belongs to $\Re \setminus X$ but one cannot conclude that x does not possess f, for x could have been non-detected because $a_0 \in \Re \setminus X$. Similarly, let $F \in \mathcal{F}_0 \setminus \mathcal{F}$, hence $a_0 \notin \Re \setminus X$. If the no outcome is obtained, the microscopic property $f^c = \varphi^{-1}(F^c)$, with $F^c = (A_0, \Re \setminus X) \in \mathcal{F}$, is possessed by x. If instead the yes outcome is obtained, one knows that the value of A_0 belongs to X but one cannot conclude that x does not possess f^c, for x could have been non-detected because $a_0 \in X$.[2]

By using the definitions and assumptions resumed above one gets that, whenever $F = (A_0, X) \in \mathcal{F}$, the overall probability $p_S^t(F)$ that a physical object x in the state S display F is given by

$$p_S^t(F) = p_S^d(F) p_S(F). \tag{3}$$

The symbol $p_S^d(F)$ in (3) denotes the probability that x be detected whenever it is in the state S (*detection probability*) and F is measured, which is not necessarily fixed for a given A_0 but it may depend on the property F, hence on the Borel set X. Since the measurement is idealized, $p_S^d(F)$ does not depend instead on the features of the apparatus measuring F or on the environment. The symbol $p_S(F)$ in (3) denotes the probability that x display F when it is detected. Then, the basic assumption of the ESR model states that, *if S is a* pure *state, the probability $p_S(F)$ can be evaluated by using the same rules that yield the probability of F in the state S according to QM*. Because of this assumption one can recover the formalism of QM in the framework of the ESR model, but the standard interpretation of quantum probabilities is modified. Indeed, according to the ESR model, if S is pure the quantum rules yield the probability that a physical object x display the property F whenever it is selected in the subset of all objects in the state S that are detected (hence a probability that is *conditional* on detection), not the probability that x display the property F whenever it is selected in the set of all objects in the state S (*absolute* probability).[3]

One can now provide different expressions for the overall probability $p_S^t(F)$ that a physical object x in the state S display a property $F = (A_0, X) \in \mathcal{F}_0 \setminus \mathcal{F}$ by introducing the reasonable assumption that $p_S^d(F^c) = p_S^d(\tilde{F})$, with $\tilde{F} = (A_0, X \setminus \{a_0\}) \in \mathcal{F}$, as follows.

$$p_S^t(F) = 1 - p_S^d(F^c) p_S(F^c) = 1 - p_S^d(\tilde{F})(1 - p_S(\tilde{F})) = 1 - p_S^d(\tilde{F}) + p_S^t(\tilde{F}). \tag{4}$$

Let us come to the mathematical representation of generalized observables. Let $A_0 \in \mathcal{O}$ be obtained from the observable A of QM represented by the self-adjoint operator $\widehat{A}$.

[2] It is apparent that the assumption that φ maps $\mathcal{E}$ bijectively onto $\mathcal{F}$ and not onto $\mathcal{F}_0$ plays a crucial role in the description of the measurement process and, in some sense, characterizes the ESR model. Indeed it implies that, for every measurement of a macroscopic property F on a physical object x, the outcome *no* (if $F \in \mathcal{F}$) or *yes* (if $F \in \mathcal{F}_0 \setminus \mathcal{F}$) does not provide information about the microscopic property possessed by x, which allows us to reinterpret standard quantum probabilities as conditional on detection rather than absolute, as we show in the following.

[3] Note that the ESR model would coincide with QM if the detection probability $p_S^d(F)$ were equal to 1 for every pure state S and physical property F. We show in Sect. 4 that such a coincidence cannot occur.

Then A_0 can be represented by the family of commutative positive operator valued (POV) measures [2, 16]

$$\left\{T_\psi^{\widehat{A}} : X \in \mathbb{B}(\Re) \longrightarrow T_\psi^{\widehat{A}}(X) \in \mathcal{B}(\mathcal{H})\right\}_{|\psi\rangle \in \mathcal{V}}, \tag{5}$$

where $\mathcal{B}(\mathcal{H})$ is the set of all bounded operators on $\mathcal{H}$ and $\mathcal{V}$ is the set of all unit vectors of $\mathcal{H}$. The mapping $T_\psi^{\widehat{A}}$ is defined by

$$T_\psi^{\widehat{A}}(X) = \begin{cases} \int_X p_\psi^d(\widehat{A}, \lambda) \mathrm{d}P_\lambda^{\widehat{A}} & \text{if } a_0 \notin X, \\ I - \int_{\Re \setminus X} p_\psi^d(\widehat{A}, \lambda) \mathrm{d}P_\lambda^{\widehat{A}} & \text{if } a_0 \in X, \end{cases} \tag{6}$$

where $P^{\widehat{A}}$ is the spectral projection valued (PV) measure associated with $\widehat{A}$ and $p_\psi^d(\widehat{A}, \lambda)$ is such that, for every $|\psi\rangle \in \mathcal{V}$, $\langle\psi|p_\psi^d(\widehat{A}, \lambda)\frac{dP_\lambda^{\widehat{A}}}{d\lambda}|\psi\rangle$ is a measurable function on $\Re$. By using this representation one can evaluate the probability $p_S^t((A_0, X))$ that the outcome of an idealized measurement of A_0 on a physical object x in the pure state S represented by the unit vector $|\psi\rangle$ lie in the Borel set X, or, equivalently, the probability that a measurement of a property $F = (A_0, X) \in \mathcal{F}_0$ yield the yes outcome. One gets

$$p_S^t((A_0, X)) = \langle\psi|T_\psi^{\widehat{A}}(X)|\psi\rangle. \tag{7}$$

The representation in (5) also suggests one to introduce the following *generalized projection postulate*.

GPP Let S be a pure state represented by the unit vector $|\psi\rangle$, and let a nondestructive idealized measurement of a macroscopic property $F = (A_0, X) \in \mathcal{F}_0$ be performed on a physical object x in the state S.

Let the measurement yield the yes outcome. Then, the state S_F of x after the measurement is a pure state represented by the unit vector

$$|\psi_F\rangle = \frac{T_\psi^{\widehat{A}}(X)|\psi\rangle}{\sqrt{\langle\psi|T_\psi^{\widehat{A}\dagger}(X)T_\psi^{\widehat{A}}(X)|\psi\rangle}}. \tag{8}$$

Let the measurement yield the no outcome. Then, the state S'_F of x after the measurement is a pure state represented by the unit vector

$$|\psi'_F\rangle = \frac{T_\psi^{\widehat{A}}(\Re \setminus X)|\psi\rangle}{\sqrt{\langle\psi|T_\psi^{\widehat{A}\dagger}(\Re \setminus X)T_\psi^{\widehat{A}}(\Re \setminus X)|\psi\rangle}}. \tag{9}$$

The above results, that hold for pure states only, have been recently extended to mixtures [2]. However, we do not need this extension for our present purposes. It is instead expedient to resume some consequences of (5)–(9) in the special case of discrete generalized observables and composite systems. Let us therefore consider a discrete observable A of QM represented by the self-adjoint operator $\widehat{A}$, let $\{a_1, a_2, \ldots\}$ be the set of all its possible outcomes, let A_0 be a generalized observable obtained from A, with set of possible outcomes $\{a_0\} \cup \{a_1, a_2, \ldots\}$, and let us consider the property $F_n = (A_0, \{a_n\})$. By briefly setting $p_S^d(F_n) = p_{\psi n}^d(\widehat{A})$ one gets from (7)

$$p_S^t(F_n) = \begin{cases} p_{\psi n}^d(\widehat{A})\langle\psi|P_n^{\widehat{A}}|\psi\rangle & \text{if } n \neq 0, \\ \sum_{m \in \mathbb{N}}(1 - p_{\psi m}^d(\widehat{A}))\langle\psi|P_m^{\widehat{A}}|\psi\rangle & \text{if } n = 0. \end{cases} \tag{10}$$

Furthermore, let S_n be the pure state of x after a nondestructive idealized measurement of A_0 yielding the outcome a_n. Then S_n is represented by the unit vector

$$|\psi_n\rangle = \begin{cases} \frac{P_n^{\widehat{A}}|\psi\rangle}{\sqrt{\|P_n^{\widehat{A}}|\psi\rangle\|^2}} & \text{if } n \neq 0, \\ \frac{\sum_{m\in\mathbb{N}}(1-p^d_{\psi m}(\widehat{A}))P_m^{\widehat{A}}|\psi\rangle}{\sqrt{\sum_{m\in\mathbb{N}}(1-p^d_{\psi m}(\widehat{A}))^2\|P_m^{\widehat{A}}|\psi\rangle\|^2}} & \text{if } n = 0 \end{cases} \tag{11}$$

because of GPP.

Let us consider another discrete observable B of QM represented by the self-adjoint operator $\widehat{B}$ with set of possible outcomes $\{b_1, b_2, \ldots\}$, and let B_0 be a generalized observable obtained from B, with set of possible outcomes $\{b_0\} \cup \{b_1, b_2, \ldots\}$. Let us assume that nondestructive idealized measurements of A_0 and B_0 are performed. By using GPP we can calculate the probability $p^t_S(a_n, b_p)$ (with $n, p \in \mathbb{N}_0$) of obtaining the pair of outcomes (a_n, b_p) when firstly measuring A_0 and then B_0 on a physical object x in the state S represented by the unit vector $|\psi\rangle$. Whenever $n \neq 0 \neq p$ we get

$$p^t_S(a_n, b_p) = p^d_{\psi n}(\widehat{A}) p^d_{\psi_n p}(\widehat{B}) \langle\psi| P_n^{\widehat{A}} P_p^{\widehat{B}} P_n^{\widehat{A}} |\psi\rangle. \tag{12}$$

where $|\psi_n\rangle$ is given by (11).

Let now Ω be a composite system made up of two subsystems Ω_1 and Ω_2, associated with the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, in QM, so that Ω is associated with the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, and let the discrete observables A and B considered above be identified with observables $A(1)$ of Ω_1 and $B(2)$ of Ω_2, respectively. Whenever simultaneous measurements of the generalized observables $A_0(1)$ and $B_0(2)$ obtained from $A(1)$ and $B(2)$, respectively, are performed on a physical object x (individual example of the whole system Ω) in a pure state S such that Ω_1 and Ω_2 are spatially separated, the noncontextuality of the ESR model implies that the transformation of S induced by a measurement of $A_0(1)$ must not affect the detection probability associated with the measurement of $B_0(2)$. If S is represented by the unit vector $|\Psi\rangle$, one gets in this case

$$p^d_{\Psi_n p}(\widehat{B}(2)) = p^d_{\Psi p}(\widehat{B}(2)). \tag{13}$$

Hence, (12) yields

$$p^t_S(a_n, b_p) = p^d_{\Psi n}(\widehat{A}(1)) p^d_{\Psi p}(\widehat{B}(2)) \langle\Psi| P_n^{\widehat{A}(1)} P_p^{\widehat{B}(2)} |\Psi\rangle, \tag{14}$$

because $P_n^{\widehat{A}(1)}$ and $P_p^{\widehat{B}(2)}$ commute.

We can now define the expectation value $E(A_0(1), B_0(2))$ of the product of the generalized observables $A_0(1)$ and $B_0(2)$ in the state S as follows,

$$E(A_0(1), B_0(2)) = \sum_{n,p\in\mathbb{N}} a_n b_p p^t_S(a_n, b_p) + \sum_{n\in\mathbb{N}} a_n b_0 p^t_S(a_n, b_0) + \sum_{p\in\mathbb{N}} a_0 b_p p^t_S(a_0, b_p) + a_0 b_0 p^t_S(a_0, b_0). \tag{15}$$

By using (15) and restricting to generalized observables such that $a_0 = 0 = b_0$ (hence, for every $n, p \in \mathbb{N}$, $a_n \neq 0 \neq b_p$)[4] we get from (14)

$$E(A_0(1), B_0(2)) = \sum_{n,p \in \mathbb{N}} a_n b_p p^d_{\Psi n}(\widehat{A}(1)) p^d_{\Psi p}(\widehat{B}(2)) \langle \Psi | P_n^{\widehat{A}(1)} P_p^{\widehat{B}(2)} | \Psi \rangle. \quad (16)$$

3 The Modified Bell Inequality

We have seen in Sect. 1 that the ESR model can be considered as a new kind of noncontextual hidden variables theory with reinterpretation of quantum probabilities as conditional on detection. When the ESR model is deterministic, the following statements hold in it.

R: *the values of all observables of a physical system in a given state are predetermined for any measurement context*,

LOC: *if measurements are made at places remote from one another on parts of a physical system which no longer interact, the specific features of one of the measurements do not influence the results obtained with the others*.

The join of R and LOC is often called *local realism* in the literature.[5] By assuming local realism Bell derived his famous inequality [6] and many similar inequalities have been obtained afterwards. As we have anticipated in Sect. 1, we have recently proved [1, 2] that, if one describes the physical situation that led Clauser, Horne, Shimony and Holt to obtain their BCHSH inequality [18] from the point of view of the ESR model, one obtains a modified BCHSH inequality. We intend to show in this section that an analogous conclusion can be drawn if one takes into account the original Bell inequality.

Let us summarize the Bell argument. Let Ω be a composite physical system made up of two far away subsystems Ω_1 and Ω_2, let $A(\mathbf{a})$ and $B(\mathbf{b})$ be dichotomic observables of Ω_1 and Ω_2, respectively, depending on the experimentally adjustable parameters $\mathbf{a}$ and $\mathbf{b}$ and taking either value -1 or $+1$. Because of R and LOC, the expectation value of the product of $A(\mathbf{a})$ and $B(\mathbf{b})$ in a state S is given by

$$E(A(\mathbf{a}), B(\mathbf{b})) = \int_\Lambda d\rho A(\lambda, \mathbf{a}) B(\lambda, \mathbf{b}), \quad (17)$$

where λ is a deterministic *hidden variable* whose values range over a measurable space Λ when measurements on different examples of Ω in the state S are considered, ρ is a probability measure over Λ, $A(\lambda, \mathbf{a})$ and $B(\lambda, \mathbf{b})$ are values of $A(\mathbf{a})$ and $B(\mathbf{b})$, respectively. If one assumes now that S is such that the *perfect correlation* (PC) law holds for every physical object in the state S, that is, for (almost) every value of the hidden variable λ,

$$A(\lambda, \mathbf{a}) = -B(\lambda, \mathbf{a}), \quad (18)$$

[4] Note that, for every generalized observable A_0, with $a_0 \neq 0$, one can construct a new observable whose no-registration outcome is 0. Indeed, one can select a Borel function χ on $\Re$ which is bijective on Ξ_0 and such that $\chi(a_0) = 0$, and consider the generalized observable $\chi(A_0)$ obtained from $\chi(A)$ by adjoining the outcome 0 and putting, for every $\lambda \in \Re$, $p^d_\psi(\chi(\widehat{A}), \lambda) = p^d_\psi(\widehat{A}, \chi^{-1}(\lambda))$.

[5] Norsen has recently criticized this name, for local realism does not comply with any definition of realism in the philosophical literature [20]. We maintain it here because its use is widespread.

then one gets from (17)

$$E(A(\mathbf{a}), B(\mathbf{b})) = -\int_{\Lambda} d\rho A(\lambda, \mathbf{a}) A(\lambda, \mathbf{b}) \tag{19}$$

hence, easily, the Bell inequality

$$|E(A(\mathbf{a}), B(\mathbf{b})) - E(A(\mathbf{a}), B(\mathbf{c}))| \leq 1 + E(A(\mathbf{b}), B(\mathbf{c})). \tag{20}$$

Let us come to the ESR model. It is important to note that the proof of (20) requires the assumption, usually left implicit, that all physical objects that are prepared are detected when ideal measurements are performed. This condition does not hold in the ESR model, where the dichotomic observables $A(\mathbf{a})$, $B(\mathbf{b})$, $A(\mathbf{b})$ and $B(\mathbf{c})$ must be substituted by the trichotomic generalized observables $A_0(\mathbf{a})$, $B_0(\mathbf{b})$, $A_0(\mathbf{b})$ and $B_0(\mathbf{c})$, respectively, in each of which a no-registration outcome is adjoined to the outcomes $+1$ and -1. Hence, the reasonings leading to (20) must be modified if the perspective introduced by the ESR model is adopted. To this aim let us agree to consider only trichotomic observables whose no-registration outcomes coincide with 0 (which is not restrictive, see footnote 4). Then we recall from Sect. 2 that the range of values of the global hidden variable introduced in the ESR model is the set $\mathcal{M}$ of all microscopic states. Hence the expectation value in (17) must be substituted by the expectation value of the product of the trichotomic generalized observables $A_0(\mathbf{a})$ and $B_0(\mathbf{b})$

$$E(A_0(\mathbf{a}), B_0(\mathbf{b})) = \sum_{i \in I} p(S^i|S) A_0(S^i, \mathbf{a}) B_0(S^i, \mathbf{b}), \tag{21}$$

where $A_0(S^i, \mathbf{a})$ and $B_0(S^i, \mathbf{b})$ denote the values of $A_0(\mathbf{a})$ and $B_0(\mathbf{b})$, respectively, when the hidden variable takes value S^i (which implies that we must consider an ESR model which is deterministic in the sense explained in Sect. 2). We can now modify the procedures outlined above, as follows.

First of all, let us assume that the PC law holds for every physical object in a microscopic state S^i such that $p(S^i|S) \neq 0$, $A(S^i, \mathbf{a}) \neq 0 \neq B(S^i, \mathbf{a})$, so that we have in this case

$$A_0(S^i, \mathbf{a}) = -B_0(S^i, \mathbf{a}). \tag{22}$$

By using (22) we obtain from (21)

$$E(A_0(\mathbf{a}), B_0(\mathbf{b})) = -\sum_{i \in I} p(S^i|S) A_0(S^i, \mathbf{a}) A_0(S^i, \mathbf{b}), \tag{23}$$

hence we get

$$\begin{aligned} |E(A_0(\mathbf{a}), B_0(\mathbf{b})) - E(A_0(\mathbf{a}), B_0(\mathbf{c}))| &= \left| \sum_{i \in I} p(S^i|S) A_0(S^i, \mathbf{a})(A_0(S^i, \mathbf{b}) - A_0(S^i, \mathbf{c})) \right| \\ &\leq \sum_{i \in I} p(S^i|S) |A_0(S^i, \mathbf{a})| |A_0(S^i, \mathbf{b}) - A_0(S^i, \mathbf{c})|. \end{aligned} \tag{24}$$

Let us then observe that the following equation holds

$$|A_0(S^i, \mathbf{b}) - A_0(S^i, \mathbf{c})| = |A_0(S^i, \mathbf{m})||1 - A_0(S^i, \mathbf{m}) A_0(S^i, \mathbf{n})| \tag{25}$$

where $\mathbf{m} \neq \mathbf{n}$, $\mathbf{m} = \mathbf{b}$ if $A_0(S^i, \mathbf{b}) = \pm 1$ and $A_0(S^i, \mathbf{c}) = 0$, $\mathbf{m} = \mathbf{c}$ if $A_0(S^i, \mathbf{c}) = \pm 1$ and $A_0(S^i, \mathbf{b}) = 0$, $\mathbf{m} = \mathbf{b}$ or, indifferently, $\mathbf{m} = \mathbf{c}$ if $A_0(S^i, \mathbf{b})$ and $A_0(S^i, \mathbf{c})$ are both 0 or ± 1. Moreover we have

$$|A_0(S^i, \mathbf{a}) A_0(S^i, \mathbf{m})| \leq 1, \tag{26}$$

$$1 - A_0(S^i, \mathbf{m}) A_0(S^i, \mathbf{n}) \geq 0, \tag{27}$$

and

$$\sum_{i \in I} p(S^i | S) = 1. \tag{28}$$

Hence we get

$$\begin{aligned} &\sum_{i \in I} p(S^i | S) |A_0(S^i, \mathbf{a})| |A_0(S^i, \mathbf{b}) - A_0(S^i, \mathbf{c}))| \\ &\quad = \sum_{i \in I} p(S^i | S) |A_0(S^i, \mathbf{a}) A_0(S^i, \mathbf{m})| |1 - A_0(S^i, \mathbf{m}) A_0(S^i, \mathbf{n}))| \\ &\quad \leq \sum_{i \in I} p(S^i | S)(1 - A_0(S^i, \mathbf{m}) A_0(S^i, \mathbf{n})) \\ &\quad \leq 1 - \sum_{i \in I} p(S^i | S) A_0(S^i, \mathbf{m}) A_0(S^i, \mathbf{n}). \end{aligned} \tag{29}$$

By using (21) and (24) we obtain from (29) the modified Bell inequality

$$|E(A_0(\mathbf{a}), B_0(\mathbf{b})) - E(A_0(\mathbf{a}), B_0(\mathbf{c}))| \leq 1 + E(A_0(\mathbf{b}), B_0(\mathbf{c})), \tag{30}$$

that replaces (20) in the ESR model.

4 Physical Interpretation

We intend to compare in this section the orthodox position about the Bell inequality with the interpretation of the modified Bell inequality in the conceptual framework of the ESR model. To this end, let us firstly resume the standard view.

Let Ω_1 and Ω_2 be spin-$\frac{1}{2}$ particles and let S be the singlet spin state represented by the unit vector $|\eta\rangle = \frac{1}{\sqrt{2}}(|+,-\rangle - |-,+\rangle)$ (for the sake of simplicity we omit a factor $\frac{\hbar}{2}$ here and in the following). Let $A(\mathbf{a})$ be the quantum observable "spin of Ω_1 along the direction $\mathbf{a}$" represented by the self-adjoint operator $\sigma_a(1)$, and let $B(\mathbf{b})$ be the quantum observable "spin of Ω_2 along the direction $\mathbf{b}$" represented by the self-adjoint operator $\sigma_b(2)$. One can calculate the expectation value in (17) according to QM and get

$$E(A(\mathbf{a}), B(\mathbf{b})) = \langle \eta | \sigma_a(1) \sigma_b(2) | \eta \rangle = -\mathbf{a} \cdot \mathbf{b}. \tag{31}$$

Whenever $\mathbf{a} = \mathbf{b}$ one obtains from (17) and (31)

$$E(A(\mathbf{a}), B(\mathbf{a})) = \int_\Lambda d\rho A(\lambda, \mathbf{a}) B(\lambda, \mathbf{a}) = -1. \tag{32}$$

It follows that, for every $\lambda \in \Lambda$ (to be precise, for every $\lambda \in \Lambda \setminus \Lambda_0$, with Λ_0 such that $\int_{\Lambda_0} d\rho = 0$), $A(\lambda, \mathbf{a}) = -B(\lambda, \mathbf{a})$, that is, the PC law holds. But, then, a contradiction occurs. In fact one gets from (20), using (31),

$$|-\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}| \leq 1 - \mathbf{b}\cdot\mathbf{c}, \tag{33}$$

which does not hold if $\mathbf{a}\cdot\mathbf{b} = -\frac{\sqrt{2}}{2}$, $\mathbf{a}\cdot\mathbf{c} = \frac{\sqrt{2}}{2}$ and $\mathbf{b}\cdot\mathbf{c} = 0$.

The seemingly unavoidable conclusion is that the assumptions from which the inequality is deduced, that is R, LOC and PC, are not consistent with QM. Since PC cannot contradict QM because it is a law of this theory, R or LOC or both must be at odds with QM, which implies that local realism must be rejected (one usually also shows that "not R" implies "not LOC" if QM is assumed, hence one concludes that QM necessarily is a nonlocal theory).

The conflict between R and LOC, on one side, and QM, on the other side, has been a conundrum for physicists for a long time. In principle, one can contrive experimental tests to check whether the Bell inequality or the predictions of QM are correct. Most physicists then maintain that the experimental data that have been obtained [21–24] show that local realism is violated and confirm QM.[6]

Let us come to the modified Bell inequality. We firstly note that one can suitably particularize (16) and substitute it into (30). The resulting equation, however, is still too general and complicate for our present purposes. Therefore, let us denote by $\mathcal{O}_R$ the set of trichotomic generalized observables such that, for every $A_0 \in \mathcal{O}_R$, the detection probability in a given state depends on A_0 but not on its specific value, let us assume that $\mathcal{O}_R$ is non-void, and let us consider only observables in $\mathcal{O}_R$. Hence we can drop the dependence on n and p of the detection probabilities that appear in (16). Moreover the generalized observables that we are considering can take only values $+1$, 0, -1. Thus we get from (16)

$$\begin{aligned} E(A_0(\mathbf{a}), B_0(\mathbf{b})) &= p_\Psi^d(\widehat{A}(\mathbf{a}))p_\Psi^d(\widehat{B}(\mathbf{b}))[\langle\Psi|P_1^{\widehat{A}(\mathbf{a})}P_1^{\widehat{B}(\mathbf{b})}|\Psi\rangle - \langle\Psi|P_1^{\widehat{A}(\mathbf{a})}P_{-1}^{\widehat{B}(\mathbf{b})}|\Psi\rangle \\ &\quad - \langle\Psi|P_{-1}^{\widehat{A}(\mathbf{a})}P_1^{\widehat{B}(\mathbf{b})}|\Psi\rangle + \langle\Psi|P_{-1}^{\widehat{A}(\mathbf{a})}P_{-1}^{\widehat{B}(\mathbf{b})}|\Psi\rangle] \\ &= p_\Psi^d(\widehat{A}(\mathbf{a}))p_\Psi^d(\widehat{B}(\mathbf{b}))\langle\widehat{A}(\mathbf{a})\widehat{B}(\mathbf{b})\rangle_\Psi. \end{aligned} \tag{34}$$

According to the ESR model, the term $\langle\widehat{A}(\mathbf{a})\widehat{B}(\mathbf{b})\rangle_\Psi$ is interpreted as a *conditional expectation value*, that is, as the expectation value of the product of the trichotomic observables $A_0(\mathbf{a})$ and $B_0(\mathbf{b})$ whenever only detected objects are taken into account, and formally coincides with the quantum expectation value, in the pure state S represented by the unit vector $|\Psi\rangle$, of the product of the quantum observables $A(\mathbf{a})$ and $B(\mathbf{b})$ from which $A_0(\mathbf{a})$

[6] The experimental tests that have been performed do not refer to the Bell inequality but, rather, to similar inequalities, as the BCHSH inequality (which can be proven without using the PC law). Real measurements, however, are not ideal, hence empirical tests actually check derived inequalities, obtained by adding further assumptions (*e.g.*, *fair sampling*) to R and LOC [21–24]. The reliability of these assumptions is disputed by many authors, who therefore uphold that the empirical data showing that the derived inequalities are violated do not prove that R and LOC do not hold [25–32]. In particular, some authors show that violations can be predicted in quantum physics, or in quantum states in which they would not occur if ideal measurements were performed, as a consequence of thresholds in real measurements that imply violations of fair sampling [33–35]. These violations, however, have nothing to do with the modification of the Bell inequality introduced by the ESR model. The latter indeed holds in the case of idealized measurements and depends on h.v. associated with the physical object that is measured, not on contingent features of the measuring apparatuses (we refer to [1, 2] for a more detailed analysis of this topic and comparison with the perspective introduced by the ESR model).

and $B_0(\mathbf{b})$, respectively, are obtained. Since similar equations hold if we consider $A_0(\mathbf{a})$ and $B_0(\mathbf{c})$ or $A_0(\mathbf{b})$ and $B_0(\mathbf{c})$, we obtain from (30) and (34)

$$\begin{aligned}&|p_\Psi^d(\widehat{A}(\mathbf{a}))p_\Psi^d(\widehat{B}(\mathbf{b}))\langle\widehat{A}(\mathbf{a})\widehat{B}(\mathbf{b})\rangle_\Psi - p_\Psi^d(\widehat{A}(\mathbf{a}))p_\Psi^d(\widehat{B}(\mathbf{c}))\langle\widehat{A}(\mathbf{a})\widehat{B}(\mathbf{c})\rangle_\Psi|\\ &\quad\le 1 + p_\Psi^d(\widehat{A}(\mathbf{b}))p_\Psi^d(\widehat{B}(\mathbf{c}))\langle\widehat{A}(\mathbf{b})\widehat{B}(\mathbf{c})\rangle_\Psi.\end{aligned} \tag{35}$$

Equation (35) contains four detection probabilities and three conditional expectation values. The expectation values can be calculated by using the rules of QM because of the basic assumption in Sect. 2, and formally coincide with expectation values of QM. If one puts them into (35) the inequality does not imply, *a priori*, any contradiction, but must be interpreted as a condition that has to be fulfilled by the detection probabilities in the ESR model. We have as yet no theory allowing us to calculate the values of those probabilities, but should one be able to perform actual measurements that are close to idealized measurements, they could be determined experimentally and then inserted into (35). Two possibilities occur.

(i) There exist pure states and observables such that the conditional expectation values violate (35). In this case the ESR model (hence R and LOC) is called into question.
(ii) For every choice of pure states and observables the conditional expectation values fit in with (35). In this case the ESR model is supported by the experimental data.

The above alternatives show explicitly that the ESR model is, in principle, falsifiable, as we have stated in Sect. 1.[7]

To illustrate our conclusions let $A(\mathbf{a})$ and $B(\mathbf{b})$ be the spin observables introduced above, so that $A_0(\mathbf{a})$ and $B_0(\mathbf{b})$, respectively, are the generalized observables obtained from them, assume that both $A_0(\mathbf{a})$ and $B_0(\mathbf{b})$ belong to $\mathcal{O}_R$ for any choice of the parameters $\mathbf{a}$ and $\mathbf{b}$, and let S be the singlet state represented by $|\eta\rangle$. By applying GPP we then obtain that the PC law holds for every physical object in a microstate S^i such that $p(S^i|S)\neq 0$, $A_0(S^i,\mathbf{a})\neq 0\neq B_0(S^i,\mathbf{a})$.[8] Therefore we can use the second equality in (31) and obtain from (35)

$$|-p_\eta^d(\sigma_a(1))p_\eta^d(\sigma_b(2))\mathbf{a}\cdot\mathbf{b}+p_\eta^d(\sigma_a(1))p_\eta^d(\sigma_c(2))\mathbf{a}\cdot\mathbf{c}|\le 1-p_\eta^d(\sigma_b(1))p_\eta^d(\sigma_c(2))\mathbf{b}\cdot\mathbf{c}. \tag{36}$$

If Ω_1 and Ω_2 move freely in opposite directions and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ lie in the plane that is orthogonal to the momenta, the rotational invariance of $|\eta\rangle$ and the choice of the observables suggest that the four detection probabilities in (36) have the same value, say p_η. Hence we get from (36) in this case

$$p_\eta^2\le\frac{1}{|-\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}|+\mathbf{b}\cdot\mathbf{c}}. \tag{37}$$

[7] We stress the words "in principle". Actual experiments may be very difficult to perform (see footnote 6). In our case further difficulties occur, because one has to count the physical objects that are actually prepared in the pure state S, even if they are not detected by the measurement of F, to determine $p_S^d(F)$. Furthermore, one must contrive a way to distinguish empirically $p_S^d(F)$ from the lack of efficiency of any actual measuring device.

[8] Note that the projection postulate has not been applied to obtain the PC law in the orthodox approach to the Bell inequality reported above. This postulate is indeed problematic in QM because of nonobjectivity of physical properties (Sect. 1), hence it is reasonable to avoid using it as far as possible in QM. This problem does not occur in the ESR model, where all physical properties are objective, hence we can deduce the PC law directly from GPP.

Since p_η does not depend on **a**, **b**, **c**, (37) implies that p_η^2 has an upper bound that coincides with the minimum value of the term on the right, which is 2/3. Hence we get

$$p_\eta \leq \sqrt{\frac{2}{3}} \approx 0.8165. \tag{38}$$

Equation (38) implies that no spin measurement on 1 or 2, even if idealized, can have a detection efficiency greater than 0.8165. Should a real measurement have a bigger efficiency the join of the ESR model together with the additional assumptions that we have introduced to attain (38) would be falsified (but, of course, one should still decide whether this falsification refers to the ESR model, to the additional assumptions, or both). If not, one can consider this result as a clue that the ESR model is correct.[9]

Finally, we observe that the example above shows that special cases of violation of (35) occur if the values of all detection probabilities in it coincide with 1. This implies that the detection probability $p_S^d(F)$ in (3) cannot be equal to 1 for every pure state S and physical property F. Therefore the ESR model cannot coincide with QM (see footnote 3) and necessarily yields some predictions that are different from the predictions of QM and other predictions that coincide with those of QM only if the latter are suitably reinterpreted (Sect. 2).

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[9]Equation (38) provides a limit which is similar to the limits obtained in other h.v. theories for QM (see, *e.g.*, [27, 28, 31]), but its interpretation is quite different. Indeed, most standard h.v. theories maintain the orthodox perspective according to which local realism contradicts QM, and establish a lower limit for the efficiency of any real measurement intended to decide whether local realism or QM is correct. Should the efficiency be smaller than this limit, the measurement could not be suitable for distinguishing the two alternatives. On the contrary local realism and QM coexist in the ESR model, and p_η constitutes an upper limit for the efficiency of every measurement, even if idealized.

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