

Cosmic Strings in a Model of Non-relativistic Gravity

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Received: 8 October 2010 / Accepted: 30 December 2010 / Published online: 9 January 2011
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Abstract Hořava proposed a non-relativistic renormalizable theory of gravitation, which is reduced to general relativity (GR) in large distances (infra-red regime (IR)). It is believed that this theory is an ultra-violet (UV) completion for the classical theory of gravitation. In this paper, after a brief review of some fundamental features of this theory, we investigate it for a static cylindrical symmetric solution which describes *Cosmic string* as a special case. We have also investigated some possible solutions, and have seen that how the classical GR field equations are modified for generic potential $V(g)$. In one case there is an algebraic constraint on the values of three coupling constants. Finally as a pioneering work we deduce the most general *cosmic string* in this theory. We explicitly show that how the *coupling constants* distort the mass parameter of *cosmic string*. We deduce an explicit function for mass per unit length of the space-time as a function of the *coupling constants*. We compare this function with another which Aryal et al. (Phys. Rev. D 34:2263, 1986) have found in GR. Also we calculate the self-force on a massive particle near Hořava-Lifshitz straight string and we give a typical order for the *coupling constant* g_9 . This order of magnitude proposes a cosmological test for validity of this theory.

Keywords Cosmic string · Hořava-Lifshitz gravity · Exact solutions

1 Introduction

In January 2009, a power-counting renormalizable UV complete theory of gravity was proposed by Hořava [1–3]. Quantum gravity models based on an “anisotropic scaling” of the space and time dimensions have recently attracted significant attention [4, 5]. In particular, Hořava-Lifshitz point gravity [1] might be has desirable features, but in its original incarnation one is forced to accept a non-zero cosmological constant of the wrong sign to be compatible with observation [6]. There are four different versions of this theory: with

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(or without) projectability condition and with (or without) detailed balance. In first look it seems that this non relativistic model for quantum gravity has a well defined IR limit and it reduces to GR. But as it was first indicated by Mukohyama [38], HL theory mimics GR plus Dark matter (a pressureless fluid). This theory has a scale invariant power spectrum which describes inflation. This theory is strongly coupled and must be modified for escaping from an unphysical extra mode. This time this theory has been improved. This theory is renormalizable in the sense that the effective coupling constant in the UV is dimensionless. Cosmology in Hořava theory has been studied by several authors [7–9]. Homogeneous vacuum solutions in this theory were got in [10]. The cosmological evolution in Hořava gravity with scalar-field was intensively studied, and the matter bounce scenario in Hořava theory was investigated [11].

Hořava theory has at least two important properties. The first one is it's UV renormalizability, While the second one is most interesting in cosmology. The fact that the speed of light diverges in the UV implies that exponential inflation is not necessary for solving the horizon problem. Moreover, the short distance structure of perturbations in Hořava-Lifshitz theory is different from standard inflation in GR. Especially, in UV limit, the scalar field perturbation is essentially scale-invariant and it is insensitive to the expansion rate of the universe, as it has been addressed in [6]. In Hořava theory time and space are treated in an unequal footing, with four-dimensional general coordinations invariance emerges as an accidental symmetry in large distance. In the present form of Hořava-Lifshitz cosmology, one combines the aforementioned modified gravitational background with a scalar field which reproduces (dark) matter. Doing so we obtain a dark-matter universe, with the appearance of a cosmological constant and an effective “dark radiation” term. Although these terms are interesting cosmological artifacts of the novel features of Hořava-Lifshitz gravitational background, they restricted the possible scenarios of Hořava-Lifshitz cosmology. Formulating Hořava-Lifshitz cosmology in a way that an effective dark energy, with a varying equation-of-state parameter, will emerge is discussed by Saridakis in [12]. Calcagni found vacuum solutions and argue that bouncing solutions exist and avoid the big bang singularity [9].¹

The general renormalizable actions for the scalar field and gauge field are proposed in [13]. They provide a possible explanation for the time delays in Gamma-Ray bursts due to the modification of the dispersion relation. Also it has been shown that the Hořava theory for the completion of General Relativity at UV scales can be interpreted as a gauge fixed Tensor-Vector theory, and it can be extended to an invariant theory under the full group of four-dimensional diffeomorphisms [14]. Charmousis et al. showed that Hořava gravity suffers from strong coupling problems, with and without detailed balance, and therefore it is unable to reproduce General Relativity in the IR [15]. Myung and Kim studied Hořava-Lifshitz black hole solutions and its thermodynamic properties [16]. Mukohyama presented a simple scenario to generate almost scale-invariant, super-horizon curvature perturbations [17]. Also Mukohyama and et al. pointed out that the radiation energy density in the UV epoch is proportional to a^{-6} and, thus, it decays faster than where in the IR epoch or in relativistic theories. This leads to intriguing cosmological consequences such as enhancement of baryon asymmetry and stochastic gravity waves. They might also discussed current observational constrains on the dispersion relation [18]. Topological (charged) black holes in

¹Solutions with Euclidean signature are asymptotically de Sitter and in qualitative agreement with the CDT scenario. On the other hand, inhomogeneous scalar perturbations against the classical background, generated by quantum fluctuations of an inflationary Lifshitz field, are unable to yield a scale-invariant spectrum [9].

Hořava-Lifshitz theory is discussed in [19]. There is a few exact solutions in Hořava theory and it is a considerable problem to investigate our familiar GR objects in the context of this new theory. Some exact solutions may be found in [100–108]. Azeyanagi et al. present type IIB super gravity solutions which are expected to be dual in comparison with certain Lifshitz-like fixed points with anisotropic scale invariance [47]. Mann found a class of black hole solutions to a $(3 + 1)$ dimensional theory gravity coupled to abelian gauge fields with negative cosmological constant that has been proposed as a dual theory to a Lifshitz theory describing critical phenomena in $(2 + 1)$ dimensions [48]. Ohta and collaborators discovered new solutions and discussed their properties [73, 74]. Orlando and Reffert studied the renormalization properties of HL gravity beyond power counting arguments [63]. In fact, their results confirm its renormalizability by certain conditions. They make use of the fact that (super) HL gravity can be taken to the stochastic quantization of topologically massive gravity. This argument relies on the renormalizability of the latter, which thought is even not strictly proven and it is thought to be hold [64]. Other readable and momentous papers listed in [65–72].

Wormhole solutions to Hořava theory in vacuum are discussed in [33]. The black hole and cosmological solutions for *arbitrary* cosmological constant was obtained [34]. One of the best works on thermodynamics of Hořava space times is the paper of Wang and Wu. They studied thermodynamics of cosmological models in the Hořava-Lifshitz theory of gravity, and systematically investigated that the evolution of the universe filled with a perfect fluid that has the equation of state $p = w\rho$, where p and ρ denote, respectively, the pressure and energy density of the fluid, and w is an arbitrary real constant [35]. Brane cosmology in the Hořava-Witten heterotic M-theory discussed by Wu, Gong and Wang in [46]. Too Minamitsuji classified the cosmological evolutions [36]. The timelike geodesic motion in the Hořava-Lifshitz spacetime studied by Chen and Wang [37]. Dynamics of a component which behaves like pressureless dust emerges as an *integration constant* of dynamical equations investigated by Mukohyama [38]. Saridakis noted that Hořava-Lifshitz cosmology with an additional scalar field leads to an effective dark energy sector [39]. The properties of strong field gravitational lensing in the deformed Hořava-Lifshitz black hole studied by Chen and Jing [40]. Too Yamamoto et al. studied the spectral tilt of primordial perturbations in Hořava-Lifshitz cosmology [49].

But later, Blas et al. [75] listed inconsistencies of the Hořava-Lifshitz gravity as a complete description of Quantum gravity. They addressed *the consistency of Hořava's proposal for the theory of quantum gravity from the low-energy perspective*. A peculiarity of the new mode is that *it satisfies an equation of motion that is of first order in time derivatives*. In linear level this extra mode manifests only around spatially inhomogeneous and time-dependent backgrounds. They found two serious problems associated with this mode. First, *the mode develops very fast exponential instabilities at short distances*. Second, *it becomes strongly coupled at an extremely low cutoff scale*. They also discussed the *projectable* version of Hořava's proposal and argue that this version can be understood as a certain limit of the ghost condensate model. The theory still problematic since the additional field generically forms caustics and, again, has a very low strong coupling scale. Also they clarify some subtleties that arise in the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature.

One of the most important topological objects is the cosmic string, discussed both in $f(R)$ gravity and scalar field theories by the author [21–23]. Our main purpose of this short paper is the investigation of the special cylindrically symmetric spacetimes which describes the cosmic strings. Question that we want to answer is that “Why and when the specific properties of the cosmic string defected by the new coupling constants in the new kind of the

non relativistic quantum gravity?”. In this work we show that the near axis limit for a cosmic string in the HL theory has a quantum mechanical origin. It means that we have a minimum mass scale for the cosmic strings, which enforced us that we must limited ourselves only to the region of the space near the location of the string.

Also we derived the general force exerted by string to a test particle and by comparing the results from the orbit motion around the string and comparing our calculations with the known data, we present some new order magnitudes for the coupling constants in this model which role as the Lorentz breaking terms in the UV limit.

2 The Metric Due to an Infinite Straight String in GR

The metric due to an infinite straight cosmic string in vacuum is in its distributional form, arguably the simplest non-empty solution of the Einstein field equations. The weak-field version (which is virtually identical to the full solution) was first derived by Vilenkin [50]. The full metric was independently discovered by Gott [51] and Hiscock [52], who matched a vacuum exterior solution to a simple interior solution containing fluid with the equation of state

$$T_t^t = T_z^z = \epsilon$$

(ϵ a constant) and then let the radius of the interior solution go to zero. The Gott’s work followed directly from a study of the gravitational field of point particles in 2 + 1 dimensions [53]. A more general class of interior solutions was subsequently constructed by Linet [54]. The Gott-Hiscock solution is constructed by first assuming a static, cylindrically-symmetric line element with the general form:

$$ds^2 = -e^{2\chi} dt^2 + e^{2\psi} (dr^2 + dz^2) + e^{2\omega} d\varphi^2$$

where χ, ψ and ω are functions of “ r ” alone, and “ r ” and φ are standard polar coordinates on R^2 . The metric is generated by solving the non vacuum Einstein equations

$$G_v^\mu = -8\pi T_v^\mu$$

The only constraints are that the ω should be positive and that the solution should be regular on the axis $r = 0$, so that $e^\omega \sim r$ for small “ r ”s. Gott [51] and Hiscock [52] both assumed ϵ to be a constant ϵ_0 . The more general situation where ϵ varies has been discussed by Linet [54]. The exterior metric is the solution of the vacuum Einstein equations

$$G_v^\mu = 0$$

It was shown that the most general statics, cylindrical-symmetric vacuum line element is which was first discovered by *Tullio Levi-Civita* [62],

$$ds^2 = -r^{2m} c^2 dt^2 + r^{2m(m-1)} b^2 (dz^2 + dr^2) + r^{2(1-m)} a_0^2 d\varphi^2$$

It is always possible to set b and c to 1 by suitable rescaling t, z and r , but for present purposes it is more convenient to retain them as arbitrary integration constants. The interior and exterior solutions can be matched at any nominated value r_0 for r in the interior solution. It was shown that [45]

$$a = 1 - 4\eta$$

by noting that the total mass per unit length η on each surface for constant t and z in the interior solution, it is possible to endow the interior solution with an equation of state more general than that considered by Gott, Hiscock and Linet [51, 52, 54] while preserving the form of the exterior metric. The mass per unit length in the interior solution is then typical not equal to the metric parameter $\frac{1}{4}(1 - a)$.

3 Review of Hořava-Lifshitz Gravity with Detailed Balance Condition

Following from the ADM decomposition of the metric [32], and the Einstein equations, the fundamental objects of interest are the fields $N(t, x)$, $N_i(t, x)$, $g_{ij}(t, x)$ corresponding to the *lapse*, *shift* and *spatial metric* of the ADM decomposition. In the (3 + 1)-dimensional ADM formalism, where the metric can be written as

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{1}$$

and for a spacelike hyper surface with a fixed time, its extrinsic curvature K_{ij} is

$$K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i)$$

where a dot denotes a derivative with respect to “ t ” and covariant derivatives defined with respect to the spatial metric g_{ij} , the action of Hořava-Lifshitz theory for $z = 3$ is

$$S = \int_M dt d^3x \sqrt{g} N (\mathcal{L}_K - \mathcal{L}_V) \tag{2}$$

we define the space-covariant derivative on a covector v_i as $\nabla_i v_j \equiv \partial_i v_j - \Gamma^l_{ij} v_l$ where Γ^l_{ij} is the spatial Christoffel symbol. The g is the determinant of the 3-metric and $N = N(t)$ is a dimensionless homogeneous gauge field. The kinetic term is

$$\mathcal{L}_K = \frac{2}{\kappa^2} \mathcal{O}_K = \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2)$$

Here N_i is a gauge field with scaling dimension $[N_i] = z - 1$.

The ‘potential’ term \mathcal{L}_V of the (3 + 1)-dimensional theory is determined by the *principle of detailed balance* [3], requiring \mathcal{L}_V to follow, in a precise way, from the gradient flow generated by a 3-dimensional action W_g . This principle was applied to gravitation with the result that the number of possible terms in \mathcal{L}_V are drastically reduced with respect to the broad choice available in an ‘potential’ is

$$\mathcal{L}_V = \alpha_6 C_{ij} C^{ij} - \alpha_5 \epsilon_i^{jj} R_{im} \nabla_j R^{ml} + \alpha_4 \left[R_{ij} R^{ij} - \frac{4\lambda - 1}{4(3\lambda - 1)} R^2 \right] + \alpha_2 (R - 3\Lambda_w) \tag{3}$$

where in it C_{ij} is the *Cotton tensor* [3] which is defined as,

$$C^{ij} = \epsilon^{kl(i} \nabla_k R_l^{j)}$$

The kinetic term could be rewrite in terms of the *de Witt metric* as:

$$\mathcal{L}_K = \frac{2}{\kappa^2} K_{ij} G^{ijkl} K_{kl}$$

where we have introduced the *de Witt metric*

$$G^{ijkl} = \frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) - \lambda g^{ij}g^{kl}$$

The inverse of this metric is given by

$$G_{ijkl} = \frac{1}{2}(g_{ik}g_{jl} + g_{il}g_{jk}) - \tilde{\lambda}g_{ij}g_{kl}$$

$$\tilde{\lambda} = \frac{\lambda}{3\lambda - 1}$$

Inspired by methods which are used in quantum critical systems and non equilibrium critical phenomena, Hořava restricts the large class of possible potentials using the principle of detailed balance outlined above. This requires that the potential (3) takes the form

$$\mathcal{L}_V = \frac{\kappa^2}{8} E^{ij} G_{ijkl} E^{kl}$$

Note that by constructing E^{ij} as a functional derivative it automatically transverse within the foliation slice, $\nabla_i E^{ij} = 0$. The equations of motion were obtained in [7].

4 About the Inconsistency of Hořava Gravity

The action in the ADM formalism contains only first order time derivatives, which allows to circumvent the problems with the ghosts appearing in covariant higher order gravity theories [76]. The higher derivative terms naively become irrelevant in the infrared and Hořava was argued that the theory reduces to GR at large distances. However as Blas et al. showed, the consistency of the above proposal is far from being clear. The main concern comes from the fact that the introduction of a preferred foliation explicitly breaks the gauge group of GR down to the group of space-time diffeomorphisms preserving this foliation. As already pointed out by Hořava, this breaking is expected to introduce extra degrees of freedom compared to GR. The new degrees of freedom can be persisted down to the infrared and be leaded to various pathologies (instabilities, strong coupling) that may invalidate the theory.² There have been several controversial claims about the properties of the extra freedom degrees [83]. In [77–79] the new mode was identified among the perturbations around a static spatially homogeneous background in the presence of matter. The mode was argued to be strongly coupled to matter in the limit when the theory is expected to approach GR, making it hard to believe that a GR limit exists. It is worth noting that the mode found in [80] is not propagating: its equation of motion does not contain time derivatives [80]. Thus it remains unclear from this analysis whether this mode corresponds to a real degree of freedom or can be integrated out as unphysical. The observation that the extra mode is non-propagating was generalized in [81] to the case of cosmological backgrounds. The interpretation of this result given in [81] is that actually the Hořava gravity is free from additional freedom degrees. Also it was claimed that the strong coupling is alleviated by the expansion of the Universe. Finally, the non-linear Hamiltonian analysis performed in [82] shows that the phase space

²An illustration of this phenomenon is provided by theories of massive gravity where special care is needed to make the additional degrees of freedom well-behaved [76].

of Hořava gravity is 5-dimensional. This result is puzzling: a normal degree of freedom corresponds to a 2-dimensional phase space; so the result of [82] suggests that the number of degrees of freedom in Hořava gravity is two and a half. Two of these freedom degrees are naturally identified with the two helicities of graviton. But the physical meaning of the extra “*halfmode*” is obscure.

Blas et al. showed that Hořava gravity does possess an additional light scalar mode. For a general background the equation of motion for this mode contains time derivatives implying that the mode is propagating. The peculiarity of Hořava gravity is that the equation for the extra mode is first order in time derivatives. The solution still corresponds to waves with a background dependent dispersion relation and is fixed once a single function of spatial coordinations is determined as the initial condition in the Cauchy problem. This explains why this mode corresponds to a single direction in the phase space. Next they addressed the consistency of the Hořava proposal by study the infrared properties of the extra mode. They found that its dynamics exhibits a number of bad features. First, the mode becomes singular for static or spatial homogeneous backgrounds. Namely, the mode frequency diverges in that limit. This explains why this mode has been overlooked in the previous analysis of perturbations in Hořava gravity. Second, for certain (background-dependent) values of spatial momentum the mode becomes unstable. Again, the rate of the instability diverges if one takes the static/spatially homogeneous limit for the background metric. Third, they show that at energies above the certain scale the extra mode is strongly coupled to itself, and not only to matter. Also they found that the strong coupling scale is background dependent and goes to zero for *flat-cosmological* backgrounds. Hence, the model suffers from a much more severe strong coupling problem than pointed out in, where the dependence of the strong coupling scale on the background curvature was ignored. Because of the strong coupling the Hořava model can be trusted only in a narrow window of very small energies, way below the Planck scale. This implies that the Hořava model cannot be considered as consistent theory of quantum gravity.

Later Charmousis et al. [80] showed that Hořava gravity suffers from strong coupling problems, with and without detailed balance, and is therefore unable to reproduce General Relativity in the infra-red. They considered the perturbative theory about the vacuum, yielding two important results. The first considered the role of detailed balance in these models. As the breaking terms go zero, They find that the linearized gravitational Hamiltonian constraint vanishes off-shell.³ By comparing field equations to their counterparts in General Relativity, Charmousis et al. showed that the “*emergent*” Planck length actually diverges in the limit of detailed balance, in contrast to the original claims of Hořava. This strong coupling behavior means that the theory with detailed balance does not have any sort of perturbative infra-red limit, explaining the results of Lu et al. [10]. Indeed, from the point of view of spherical symmetric solutions one sees that the putative higher order terms in the IR are just as important as the “lower” order terms. In summary, with detailed balance, one can never hope to recover GR in the infra-red for the following reason: General Relativity admits an effective linearized description beyond the Schwarzschild radius of a source, but in Hořava gravity with detailed balance, strong coupling prevents an effective linearized description on any scale. Further discussions may be found it [98, 99].

³This means that linearized theory breaks down in this limit, just as it does for the Chern-Simons limit of Gauss-Bonnet gravity.

5 Thermodynamics and Dark Energy in Hořava Gravity

If we assume that the cosmological scenario of a universe governed by Hořava-Lifshitz gravity, it is a natural problem to an investigation of its thermodynamic properties, and in particular of the generalized second thermodynamic law. The validity of the generalized second law of thermodynamics in a universe governed by Hořava-Lifshitz gravity has been discussed by Jamil et al. in [109]. They calculated the entropy time-variation for the matter fluid and, using the modified entropy relation, that of the apparent horizon itself and found that under detailed balance the generalized second law was generally valid for flat and closed geometry and it was conditionally valid for an open universe, while beyond detailed balance it was only conditionally valid for all curvatures. They followed the exact and robust approach, that is they used the modified entropy relation as it has been calculated in the specific context of Hořava-Lifshitz gravity. Under the equilibrium assumption between the universe interior and the horizon, which is expected to be valid at late cosmological times, they found that the generalized second law is only conditionally valid. The possible violation of the generalized second thermodynamical law in Hořava-Lifshitz cosmology could lead to various conclusions. Further the dark energy was discussed by setare et al. [20]. They nicely investigated the holographic dark energy scenario with a varying gravitational constant in a flat background in the context of Hořava-Lifshitz gravity and immediately, extracted and determined the evolution of the dark energy density parameter. Also they discussed some non trivial cosmological implications of this holographic model. Also they evaluated the dark energy equation of state for low redshifts even when the model contains a time varying gravitational constant.

6 Exact Solutions

Considering the non-static, cylindrically symmetric solutions with the metric ansatz

$$ds^2 = -N(t)^2 dt^2 + \frac{1}{N(t)^2} [dr^2 + \Phi(r)^2 dz^2 + \Psi(r)^2 d\varphi^2] \quad (4)$$

For simplicity we decompose the spatial metric as a conformal by another diagonal simple form:

$$g_{ij} dx^i dx^j = \frac{1}{N(t)^2} \gamma_{ij} dx^i dx^j$$

$$\gamma_{ij} = \text{diag}(1, \Phi(r)^2, \Psi(r)^2) \quad (5)$$

This may or may not represent a cosmic string, and it may have singularities and/or event horizons.

In synchronous time t , the Cylindrical *ADM* metric has $N_i = 0$, where $\Phi(r)$, $\Psi(r)$ are the usual Weyl metric functions. Too in GR and only for vacuum solutions the metric function $\Psi(r) = r$ and in another cases is determined from a quadrature on another metric functions $\Phi(r)$ [27]. There is another general choose for metric form but since in this paper our main goal is to investigate static cosmic strings in analogous for usual GR samples, we limited ourselves to this simple but applicable gauge. On this background,

$$K_{ij} = -\frac{\dot{N}}{N^4} \gamma_{ij} \quad (6)$$

$$K = K_i^i = -3 \frac{\dot{N}}{N^2} \tag{7}$$

$$K_{ij} K^{ij} = 3 \left(\frac{\dot{N}}{N^2} \right)^2 \tag{8}$$

Following Sotiriou and et al. [24] we use from a general full classical action,

$$S = \int [T(K) - V(g)] \sqrt{g} N d^3x dt \tag{9}$$

where

$$T(K) = g_K (K^{ij} K_{ij} - K^2) + \xi K^2 \tag{10}$$

This is the general kinetic term corresponds to the limit $\xi \rightarrow 0$. Since the kinetic action is (by definition) chosen to be dimensionless, we have set the critical exponent $z = 3$ to make g_K dimensionless, then provided g_K is positive one can without loss of generality rescale the time and/or space coordinates to set $g_K \rightarrow 1$. Now consider the following form for potential term:

$$V(g) = g_0 \zeta^6 + g_1 \zeta^4 R + g_2 \zeta^2 R^2 + g_3 \zeta^2 R_{ij} R^{ij} + g_4 R^3 + g_5 R (R_{ij} R^{ij}) + g_6 R_j^i R_k^j R_i^k + g_7 R \nabla^2 R + g_8 \nabla_i R_{jk} \nabla^i R^{jk} \tag{11}$$

As in [25, 26, 31] was stated, suitable factors of ζ are introduced to ensure the coupling g_a are all dimensionless. Without loss of generality we can rescale the time and space coordinations to set both of the $g_K \rightarrow 1$ and $g_1 \rightarrow -1$. From normalization of the Einstein–Hilbert term, we see that in physical units $c \rightarrow 1$

$$(16\pi G_{Newton})^{-1} = \zeta^2$$

$$\Lambda = \frac{g_0 \zeta^2}{2}$$

so that ζ is identified as the Planck scale. The cosmological constant is determined by the free parameter g_0 , and obviously $g_0 \sim 10^{-123}$. In particular, the way Sotiriou had set this up, now we are free to choose the Newton constant and cosmological constant independently (and so to be compatible with observation). In contrast, in the original model presented in [3], a non-zero Newton constant requires a non-zero cosmological constant, of the wrong sign to be compatible with cosmological observations [7, 32].

For a special choose of our metric (4) the Ricci scalar and non-vanishing components of Ricci Tensor are:

$$R_{11} = \frac{\Phi''}{\Phi} + \frac{\Psi''}{\Psi} \tag{12}$$

$$R_{22} = \Phi^2 \left(\frac{\Phi''}{\Phi} + \frac{\Phi' \Psi'}{\Phi \Psi} \right)$$

$$R_{33} = \Psi^2 \left(\frac{\Psi''}{\Psi} + \frac{\Phi' \Psi'}{\Phi \Psi} \right)$$

$$R = 2N(t)^2 \left(\frac{\Phi''}{\Phi} + \frac{\Psi''}{\Psi} + \frac{\Phi' \Psi'}{\Phi \Psi} \right) \tag{13}$$

Here, a prime denotes a derivative with respect to r . By substituting (6), (7), (8) in (10) and (12), (13) in (11) and all in (9) we obtain the following form of action:

$$\begin{aligned}
 S &= \int dt d^3x N^3 \Phi \Psi \left[3(3\xi - 2) \left(\frac{\dot{N}}{N^2} \right)^2 - f(R) - (g_3 \zeta^2 + g_5 R)H - g_6 W - g_7 B - g_8 Y \right] \\
 &\equiv \int dt d^3x \Xi
 \end{aligned}
 \tag{14}$$

where in it,

$$\begin{aligned}
 f(R) &= g_0 \zeta^6 - \zeta^4 R + g_2 \zeta^2 R^2 + g_4 R^3 \\
 H &= R_{ij} R^{ij} = N^4 \left[\left(\frac{\Phi''}{\Phi} + \frac{\Psi''}{\Psi} \right)^2 + \left(\frac{\Phi''}{\Phi} + \frac{\Phi' \Psi'}{\Phi \Psi} \right)^2 + \left(\frac{\Psi''}{\Psi} + \frac{\Phi' \Psi'}{\Phi \Psi} \right)^2 \right] \\
 W &= R^i_j R^j_k R^k_i = N^6 \left[\left(\frac{\Phi''}{\Phi} + \frac{\Psi''}{\Psi} \right)^3 + \left(\frac{\Phi''}{\Phi} + \frac{\Phi' \Psi'}{\Phi \Psi} \right)^3 + \left(\frac{\Psi''}{\Psi} + \frac{\Phi' \Psi'}{\Phi \Psi} \right)^3 \right] \\
 B &= R \nabla^2 R = 4N^4 \left(\frac{\Phi''}{\Phi} + \frac{\Psi''}{\Psi} + \frac{\Phi' \Psi'}{\Phi \Psi} \right) \left(\frac{\Phi'}{\Phi} R' + \frac{\Psi'}{\Psi} R' + R'' \right) \\
 Y &= \nabla_i R_{jk} \nabla^i R^{jk} = N^6 \left[\left(\left(\frac{\Phi''}{\Phi} + \frac{\Psi''}{\Psi} \right)' \right)^2 \right. \\
 &\quad \left. + \left[\left(\Phi^2 \left(\frac{\Phi''}{\Phi} + \frac{\Phi' \Psi'}{\Phi \Psi} \right) \right)' \left(\frac{1}{\Phi^2} \left(\frac{\Phi''}{\Phi} + \frac{\Phi' \Psi'}{\Phi \Psi} \right) \right)' \right] + [\Phi \rightarrow \Psi] \right]
 \end{aligned}
 \tag{15}$$

The extreme functions are the solutions of the Euler-Lagrange equations that are obtained by setting the all variational derivatives of the functional with respect to each function $X \equiv (\Phi(r), \Psi(r), N(t))$ equal to zero. The Ritz variational principle affords a powerful technique for the approximate solution of (9). The result is an upper bound on the corresponding eigenvalue and optimal values for the parameters of (9).⁴ Variational Bound can also be used to extremize general functional given appropriate trial functions. We remind that if we consider a system with the Lagrangian with linear terms of curvature R we must recover the GR solutions i.e., a non static cylindrical solution with cosmological constant. However It is proper, if we make an analytic continuation of coordinations r, t , namely, we obtain the Tian solution [28] in a special coordinations or a non static solution which is *pure radiation field* generated from a flat space-time and has a Weyl tensor of type N . The metric of this space-time is described by the *Rao* line element [29],

$$ds^2 = e^{k(t-r)} (dr^2 - dt^2) + r^2 d\varphi^2 + dz^2
 \tag{16}$$

This is a special case of non static Weyl gauges which we used in witting (4). For this metrics with null vector fields it was shown that [30] in our notations $\Psi(r) = r$. The equations of motion due to the variation of metric functions are more complicated and we do not present them here. As a simple but physically important case we seeking only those solutions which can be described the cosmic strings. We choose a very restricted gauge as,

$$\Phi(r) = 1, \quad N(t) = const
 \tag{17}$$

⁴Variational Bound.

In GR these constraints leads to a static *cosmic string* and also in *metric $f(R)$ gravity* [22, 23]. The resulting metric with new parameters (ζ, g_0, \dots, g_8) may be so interesting. In order to obtain the solution, let us substitute the metric ansatz (4) with constraints (17) into the action, and then vary the function $\Psi(r)$. The same process was done in spherical symmetry in [16]. This is possible because the metric ansatz shows all the allowed singlets which are compatible with the $SO(3)$ action on the $S^2 \times R$. The resulting reduced Lagrangian, up to an overall scaling constant, is given by

$$L_0 = \Psi(r)[-f_0 - (g_3\zeta^2 + g_5R_0)H_0 - g_6W_0 - g_7B_0 - g_8Y_0] \tag{18}$$

where

$$\begin{aligned} R_0 &= 2 \frac{\Psi''}{\Psi} \\ f_0 &= f(R_0) \\ H_0 &= \frac{1}{2} R_0^2 \\ W_0 &= \frac{1}{4} R_0^3 \\ B_0 &= 2R_0 \left(R_0' \frac{\Psi'}{\Psi} + R_0'' \right) \\ Y_0 &= R_0^2 + (\Psi^2 R_0)' \left(\frac{R_0}{\Psi^2} \right)' \end{aligned}$$

The functional (14) with reduced Lagrangian (18) is in the form,

$$S = (t_2 - t_1) \int d^3 \Pi(\Psi, \Psi', \dots, \Psi^6) \tag{19}$$

where

$$\Psi^a = \frac{d^a \Psi}{dr^a}$$

By using a general variational principle applied to this higher order function [25, 26] we can write all equations of motion for metric function Ψ ,

$$\frac{dL_0}{d\Psi} + \sum_{a=1}^6 (-1)^a \frac{d^a}{dr^a} \left[\frac{\partial L_0}{\partial \Psi^a} \right] = 0 \tag{20}$$

Because the equation of motion contain up to eight derivative terms, it is difficult to find the exact solutions. In order to understand the behavior of solutions in Hořava gravity we try to solve the equations of motion in a special case. By our inspirations from GR we know that a cosmic string has a linear functionality as $\Psi = ar + b$. The constant b may be turned to zero by changing scales along the t and z axes and choosing the zero point of the r coordinate. Since our model essentially is not GR, we expected that an ad hoc assumption for metric function as

$$\Psi = (Ar + B)^m \tag{21}$$

For a general value of A, m, B the solution to the equations of motion is consistent only with the following cases. Considering m as a real parameter, one must note that for $m = 1, B = 0, A^2 = 1 - 4\eta$ (η mass per length of string) we have a cosmic string. By considering this constraint we find the field equation (20) with solution (21) is satisfied identically.

6.1 Solutions with $g_8 = 0, g_7 = 0, B = 0$

In this case by substituting (21) in field equation (20) we obtain

$$m_i = -3, 4, \quad i = 1, 2$$

Consequently we can set $B = 0$ in metric function (21). Thus the most general solution for (4) is one of the two possible functions

$$\Psi(r) = (Ar)^{m_i} \tag{22}$$

Thus,

$$ds^2 = -dt^2 + dr^2 + dz^2 + (Ar)^{2m_i} d\varphi^2 \tag{23}$$

In this case there is an arbitrarily in choosing another coupling coefficients ($g_2 \dots g_6$). The Ricci scalar is given by

$$R = 2 \frac{m_i(m_i - 1)}{r^2} \tag{24}$$

The solution has a curvature singularity at $r = 0$ for general $m_i \neq 0, 1$. The only non zero component of the Riemann Tensor is:

$$R_{r\varphi r\varphi} = \frac{(Ar)^{2m_i} m_i(m_i - 1)}{r^2}$$

The a singularity structure of the solution (23) is apparent from its Kretschmann scalar:

$$R = \frac{(Ar)^{4m_i} m_i^2(m_i - 1)^2}{r^4}$$

then the Kretschmann scalar has a naked singularity at $r = 0$. In GR, the cosmological horizon(s) for cylindrically symmetric spacetimes is discussed in detail by Wang [110].

6.2 Solutions with $g_8 = 0, g_7 = 0, g_2 = 0, B \neq 0$

In this case as the previous section, substituting a general form of (21) in (20) leads to the similar values for m . It is particular interest to investigate the $m = -3$ solution, in which case, the coupling coefficients ($g_3 \dots g_6$) constrained to $g_6 = 2g_5 + 4g_4$. This is one of the good results of this paper. We obtained a restriction on some constants of the model. In this case the general solution can be written as

$$ds^2 = -dt^2 + dr^2 + dz^2 + (Ar + B)^{2m_i} d\varphi^2 \tag{25}$$

The solution has a curvature singularity at $r = \frac{-B}{A}$ for general $B < 0$. We mention here that the cases $m = 0, 1$ is satisfied field equations without any limitation. Specially the case $m = 1$ is so interesting. Since it represents the usual familiar line element of a static cosmic

string. In this case there is an arbitrarily in choosing another *coupling coefficients* ($g_2 \dots g_6$). The Ricci scalar is given by

$$R = 2A^2 \frac{m_i(m_i - 1)}{(Ar + B)^2} \tag{26}$$

The solution has a curvature singularity at $r = \frac{-B}{A}$ for general $m_i \neq 0, 1$. The only non zero component of the Riemann Tensor is:

$$R_{r_z r_z} = m(m - 1)(Ar + B)^{2m-2} A^2$$

The singularity structure of the solution (25) is apparent from its Kretschmann scalar:

$$R = (m(m - 1))^2 (Ar + B)^{4m-4} A^4$$

then the Kretschmann scalar has a naked singularity at $r = \frac{-B}{A}$.

6.3 Solutions with $g_8 = 0, g_7 = 0, g_4 + \frac{1}{2}g_5 + \frac{1}{4}g_6 = 0, g_2 + \frac{1}{2}g_3 = 0$

The Lagrangian (18) in this case has only three independent coupling constants and reduces to the following form:

$$L_0 = \zeta^4 (-g_0 \Psi + 2\zeta^2 \Psi'') \tag{27}$$

the field equation (20) will be

$$g_0 = 0 \tag{28}$$

This term has a significant physical meaning. If we refer to the previous relations between parameters of model we observe that if this condition holds, the cosmological constant must be vanished. Thus this model describes a non classical (for appearance of a second order derivatives of matter field in action) system with no potential term. Indded the action (19) may be integrated to obtain:

$$S = 4\zeta^6 \pi (t_2 - t_1) l \int dr \Psi'' r^2 \tag{29}$$

where in it we assumed that the cylindrical coordinations z is bounded in interval $(0, l)$. If we carry out a part by part integration on the radial part of this integral and by assumption that our Potential function ψ may be bounded and posses suitable boundary conditions (a well posses function) entirely away from the origin of radial coordinations $r = 0$ to infinity. Note that the action (29) can be written as

$$S = 8\zeta^6 \pi (t_2 - t_1) l \int dr \Psi r + B$$

where B is a surface term, which must be chosen so that the action has an extreme under variations of the fields with appropriate boundary conditions. One demands that the fields approach the classical solutions at infinity. Varying the action, we find the boundary term

$$\delta B = -(t_2 - t_1) N_0 \delta M$$

The boundary term B is the conserved charge associated to the *improper gauge transformations* produced by time evolution. Here M and N_0 are conjugate pairs. Therefore when one varies M , N_0 must be fixed. Thus the boundary term should be in the form

$$B = -(t_2 - t_1)N_0M + B_0$$

where B_0 is an arbitrary constant, which should be fixed by some physical considerations; for example, in *topological black hole* case with arbitrary constant scalar curvature horizon. Mass vanishes when black hole’s horizon goes to zero [19]. For details, see [43]. We will not go further in detail. The dynamics of the metric function in this case is not determined without more mathematical features of variational calculus which is found in any textbook in this field as which is discussed in [25, 26].

7 The Real Cosmic String

Now let we impose the next constraints,

$$g_8 = 0, \quad g_7 = 0, \quad g_4 + \frac{1}{2}g_5 + \frac{1}{4}g_6 = 0, \quad g_2 + \frac{1}{2}g_3 \neq 0 \tag{30}$$

From the action, we can obtain the equation of motion as

$$-g_0\zeta^6 + 2g_9R - 6g_9\frac{d^2R^2}{dr^2} = 0 \tag{31}$$

where $R = 2\frac{\Psi''}{\Psi}$ is the Ricci scalar. The function $R(r)$ can be obtained as

$$R(r) = \beta + \frac{1}{C^2}sn(Ar, D)^2 \tag{32}$$

where the *Jacobi elliptic functions* sn is in turn defined in terms of the amplitude function *JacobiAM* [41, 42]

$$sn(z, k) \equiv \text{JacobiSN}(z, k) = \sin(\text{JacobiAM}(z, k))$$

and

$$\phi = \text{JacobiAM}\left(\int_0^\phi \frac{d\phi}{(1 - k^2 \sin^2(\theta))^2}, k\right), \quad \phi \in (-3/2, 3/2)$$

where in (32)

$$C = \xi e^{i\theta} = 0.19 + 0.73i \tag{33}$$

$$D = \eta e^{i\rho} = 0.86 + 0.50i \tag{34}$$

$$A = \pm \sqrt{\frac{\beta}{15}} \frac{1}{2\xi} e^{i\theta} \tag{35}$$

$$\beta^3 = \frac{15}{2}\alpha \tag{36}$$

$$\alpha = \frac{g_0\zeta^6}{6g_9}, \quad g_9 = g_4 + \frac{1}{2}g_5 + \frac{1}{4}g_6 \tag{37}$$

As in [24] is proved that the $\zeta = M_{pl}, g_0 = \frac{2\Lambda}{M_{pl}^2}$. For metric (4) with (17) the Ricci scalar is

$$R(r) = 2 \frac{\Psi''}{\Psi} \tag{38}$$

The general solution for (38) with (32) is so complicated. Instead of doing that, we focused ourselves only to the near axis $r \approx 0$ behavior of (31). It is adequate to define a very important physical radial scale

$$r_0 = \frac{5.670}{M_{pl}} \left(\frac{g_9}{g_0} \right)^{\frac{1}{6}} \tag{39}$$

With this length scale the meaning of *near axis* limit is thinkable as the following expression

$$r \ll r_0 \tag{40}$$

Not that in this limit we do not tend to the origin. In this good reasonable physical approximation the differential equation (38) (albeit after expansion by series (31) in terms of $\frac{r}{r_0}$) can be solved easily. The solution is written in terms of *Whittaker* functions M, W [41, 42],

$$\begin{aligned} \Psi(r) = \frac{1}{\sqrt{r}} \left[c_1 \text{Whittaker}M \left(-\frac{\beta r_0}{4} \sqrt{2}, \frac{1}{4}, \sqrt{2} \frac{r^2}{r_0} \right) \right. \\ \left. + c_2 \text{Whittaker}W \left(-\frac{\beta r_0}{4} \sqrt{2}, \frac{1}{4}, \sqrt{2} \frac{r^2}{r_0} \right) \right] \end{aligned} \tag{41}$$

once again we impose the near axis limit on (41). The result up to order one is simple⁵

$$\Psi(r) = ar + b \tag{42}$$

$$a \equiv \left(\frac{\sqrt{2}}{r_0} \right)^{3/4} \left(c_1 - 2 \frac{\sqrt{\pi} c_2}{\Gamma(\frac{1}{4} + \beta r_0 \frac{\sqrt{2}}{4})} \right) \tag{43}$$

$$b \equiv \frac{\sqrt{\pi} (\frac{\sqrt{2}}{r_0})^{1/4} c_2}{\Gamma(\frac{3}{4} + \beta r_0 \frac{\sqrt{2}}{4})} \tag{44}$$

Again, c_1, c_2 are integration constants and c_2 could be set to zero. Similar to the case of GR, we find the metric of a *cosmic string*

$$ds^2 = -dt^2 + dr^2 + dz^2 + (ar)^2 d\varphi^2 \tag{45}$$

The metric (45) is locally flat and can be brought to a *Minkowski* form in any region not surrounding the string. This implies that the presence of the string has no effect on physical process in such a region. In particular, a test particle which is initially at rest relative to the string will remain at rest and will not experience any gravitational force. Although the space around the string is locally fiat its global structure is different from that of Euclidean space.

⁵ $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt.$

The parameter a in (45) can be expressed in terms of mass per unit length of spacetime η [44, 45],

$$a = 1 - 4\eta \quad (46)$$

The mass per unit length of spacetime η is found to be

$$\eta = 0.25 - 0.54109c_1(M_{pl})^{\frac{1}{2}} \left(\frac{\Lambda}{g_4 + \frac{1}{2}g_5 + \frac{1}{4}g_6} \right)^{\frac{1}{8}} \quad (47)$$

Clearly $\eta < 0.25$ is for all positive values of c_1 . Now we compare this function with another one which was found by Aryal et al. [61]. They constructed a solution which describes two Schwarzschild black holes held apart by a system of cosmic strings, by generalizing a solution, due to Bach and Weyl. Their metric is vacuum at all points away from the axis $r = 0$ and describes two black holes with masses m_1 and m_2 located on the z axis, and so separated by a z -coordination distance $2d$. The black holes are held in place by conical singularities along the different axial segments. The effective mass per unit length η of any of the conical segments depends only on the limitation of the value of the metric function on the axis. For all positive values of m_1 , m_2 and d . In the original Bach-Weyl solution the mass per unit length η_{ext} of the exterior segments was assumed to be zero and this forces $\eta_{int} < 0$ and the interior Bach-Weyl segment are normally characterized as a ‘strut’ rather than a string. However in (46), all segments will have non-negative masses per unit length if

$$c_1 > 0.46203(M_{pl})^{-\frac{1}{2}} \left(\frac{\Lambda}{g_4 + \frac{1}{2}g_5 + \frac{1}{4}g_6} \right)^{-\frac{1}{8}}$$

In the particular case

$$c_1 = 0.46203(M_{pl})^{-\frac{1}{2}} \left(\frac{\Lambda}{g_4 + \frac{1}{2}g_5 + \frac{1}{4}g_6} \right)^{-\frac{1}{8}}$$

the interior segment vanishes.

It should be also noted that the parameter η appearing in this derivation plays two essential independent roles:

One as a measurement of the strength of the gravitational field in the exterior metric and a second one as the integrated mass per unit length of the interior solution. In GR it is possible to endow the interior solution with an equation of state more general than that considered by Gott, Hiscock and Linet [51, 52, 54] while we preserve the form of the exterior metric. The mass per unit length in the interior solution is not typically equal to the metric parameter $\frac{1}{4}(1 - a)$. For this reason the symbol η reflects a geometric property of the exterior metric which only will be referred as the *gravitational mass per unit length* of the spacetime.

The metric (45) with (46), (47) describes Minkowski spacetime with a wedge of angular extent that $\Delta\varphi = 8\pi\eta$ has removed from each of the constant surfaces t and z . The apex of each wedge lies on the axial plane $r = 0$, and the sides of the wedge are *glued* together by forming what is sometimes referred as conical spacetime. The fact that the metric (45) is locally Minkowskian implies that the Riemann tensor is zero everywhere outside the axial plane, and therefore when a test particle moving through the metric would experience no tidal forces. In particular, such a particle would not be accelerated towards the string. Therefore a local observer should be unable to distinguish a preferred velocity in the z -direction; where any gravitational force in the radial direction would destroy this symmetry. When

it is combined with the other symmetries of the metric, this property forbids gravitational acceleration in any direction. Incidentally, Mark Hindmarsh and Andrew Wray [55] shown, by detailed analysis of the geodesics in a general Levi-Civita spacetime, that *gravitational lensing* with a well-defined angular separation between the images is possibly only in the specialized string case $m = 0$. When $\Lambda = 0$, η goes back to 0.25, the effect of higher derivative terms disappear. As one want, General Relativity is not recovered because the extra freedom degrees which are presented in the full theory all are not decouple. On the contrary, one of those freedom degrees becomes strongly coupled, and one recovers General Relativity with an additional strongly coupled scalar. It is difficult to see how this would correspond to a better choice since we move away from testable regions of GR. Therefore, evidence of this mass function absent in classical local tests of general relativity which it's implement may be weak and moving slowly sources. In Hořava gravity we have seen that we have no reliable linearized theory to work with due to strong coupling of an extra scalar degree of freedom. Even if it was tractable, it seems unlikely that a non-linear analysis could recover the successes of the General Relativity for cosmic strings in this case. One can then easily be seen that three of the coupling constants g_i , $i = 4, 5, 6$ cannot be set to zero. Thus, in general, there are cases where what appears should be violation of a symmetry is just a new choice of mass function for cosmic string. The same happens with Hořava theory. It looks that it violates four-dimensional covariance but this is just because it is written in a specific gauge, specified by the ADM frame, which can be used just because of the four-dimensional covariance of the theory (and the corresponding constraints).

8 About the Existence of Non-static Cylindrical Solutions

The cylindrical symmetric strings are not the single class of cosmic strings. As was shown by several authors, in GR there are both local and global non static cosmic strings in the context of Lyra geometry [84], static and non-static plane symmetric cosmic strings in Lyra manifold [85–88], non static self gravitating fluids [89] and non static cylindrical vacuum solutions [90]. Lyra [84] proposed a modification of Riemannian geometry by introducing a gauge function in to the structure less manifold, as a result of which the cosmological constant arises naturally from the geometry. Several authors have studied cosmological models based on this manifold with a constant gauge vector in the time direction. Non-static plane symmetric cosmic string model is quite similar to the non-static plane symmetric Zeldovich model $p = \rho$ obtained by Reddy and Innaiah [91] and Reddy [92] in general relativity. This model reduces to empty space-time discussed by Bera [93] in general relativity. In Lyra geometry there is a global string, the energy momentum tensor components are calculated from the action density for a complex scalar field ψ along with a typical potential. But finding these classes of solutions in Hořava gravity are most complicated and we can not present them here. But as good problems for further considerations we can treat them in future plan.

9 The Self-Force on a Massive Particle Near a Hořava-Lifshitz Straight String

Observational constraints in HL theory have been discussed by several authors [94–97]. In the last sections we examined some possible constraints on the parameters by calculating the self force in the field of a cosmic string which was obtained in the previous section. In GR we know that a charged particle at rest in the spacetime experiences a repulsive self-force [56], while fluctuations of the quantum vacuum near a straight string have a non-zero

stress-energy tensor and can induce a range of interesting effects [57–59]. In the weak-field approximation, the gravitational field due to a particle of mass “ m ” at rest at a distance “ a ” from a straight string is the most convenient which is calculated by transforming to the *Minkowski* form (45) of the metric and for fixing the coordinations so that the particle lies at $z = 0$ and $\varphi = \varphi_0 = \pi(1 - 4\eta)$. Now we want to generate a meaningful expression for the self-force on the particle. The formula for the gravitational self-force was first derived by *Dmitri Gal'tsov* in [60], although the electrostatic case, which is formally identical, was analyzed by Bernard Linet four years earlier [56]. By following *Gal'tsov*, it is instructive to write the self-force in the form

$$\vec{F} = -\frac{Gm^2\eta}{a^2} f(\eta)\hat{r} \tag{48}$$

where

$$f(\eta) = \frac{1}{4\pi\eta} \int_0^\infty \left[\frac{\sinh(\pi u/\varphi_0)\pi/\varphi_0}{\cosh(\pi u/\varphi_0) - 1} - \frac{\sinh(u)}{\cosh(u) - 1} \right] \frac{du}{\sinh(\frac{u}{2})} \tag{49}$$

10 The Value of the Coupling Constants g_4, g_5, g_6 Obtained from Analysis of Bound Circular Orbits

In GR, the fact that the self-force \vec{F} is central has given rise to the common misapprehension that bound circular orbits exist for massive particles in the neighborhood of a straight cosmic string. It is true that if (47) should be continue to hold for a moving particle, then circular orbits would exist with the standard Newtonian dependence of the orbital speed

$$v_{circ} = \sqrt{\frac{Gm\eta f(\eta)}{a}} \tag{50}$$

Thus, for example [45], a body with $m = 7 \times 10^{22}$ kg (roughly equal to the mass of the Moon) could orbit a GUT string with $\eta = 10^{-6}$ at a distance $a = 4 \times 10^8$ m (the mean Earth-Moon distance) if $v_{circ} \approx 0.1 \text{ ms}^{-1}$, which is about 1/10000 th of the Moon’s actual orbital speed around the Earth. Substituting this approximated value of η in (47) we obtain

$$g_9 \approx 481.55927(c_1^2 M_{pl})^4 \Lambda \tag{51}$$

Remember that c_1 is fixed by invoke quantum theory of gravity. The relation (51) is fundamentally important. It is related between quantum gravity and the cosmological observations.

11 Conclusion

In conclusion, we found cylindrical symmetric solutions with arbitrary scalar curvature in Hořava-Lifshitz theory, by generalizing the static cylindrical symmetric solutions in GR. We found that there exist solutions only in special choose of the coupling constants. One of the solutions has a near axis behavior as cosmic string. For this solution we can define a finite mass per unit length spacetime. Such an explicit term occurs in the occasion of considering quantum corrections to cosmic string line element. In our mass per unit length expression, there is a undetermined constant c_1 . To fix the constant c_1 , one has to invoke quantum theory

of gravity. The self-force on a massive particle near a Hořava-Lifshitz straight string is re-calculated. By analyzing bound circular orbits we derived a new value for the coupling constants of Hořava theory which seems that there is a new observational method for estimating the validity of the Hořava model in the context of cosmology.

Acknowledgements D. Momeni thanks the editor of IJTP, Professor *Heinrich Saller* and the anonymous referees made excellent observations and suggestions which resulted in substantial improvements of the presentation and the results. Also the author thanks Anzhong Wang, Jared Greenwald (Baylor-USA), Antonios Papazoglou (University of Portsmouth), Sourish Dutta (Vanderbilt University), Domenico Orlando (IPMU-Japan), Lorenzo Iorio (Istituto Nazionale di Fisica Nucleare (INFN)-Italy), Miao Li, Yi Pang (ITP-China), M.R. Setare (Department of Science, Bijar, Iran), E.N. Saridakis (University of Athens), and Nobuyoshi Ohta (Kinki University-Japan) for useful comments and valuable suggestions. The author would like to thank Shahram Khosravi (Tarbiat Moa'lem University, Tehran) for kind hospitality and support. This work is supported by Tarbiat Moa'lem University, Tehran.

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