Int J Theor Phys (2011) 50: 637–643
DOI 10.1007/s10773-010-0587-8

Plane Symmetric Vacuum Bianchi Type *III* Cosmology in $f(R)$ Gravity

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Received: 15 September 2010 / Accepted: 16 November 2010 / Published online: 1 December 2010
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Abstract The main purpose of this paper is to study the exact solution of Bianchi type *III* spacetime in the context of metric $f(R)$ gravity. The field equations are solved by taking expansion scalar θ proportional to shear scalar σ which gives $C = A^n$, where A and C are the metric coefficients. The physical behavior of the solution has been discussed using some physical quantities. Also, the function of the Ricci scalar is evaluated.

Keywords $f(R)$ gravity and Bianchi type *III*

1 Introduction

Among the various modifications of the general theory of relativity (GR), the $f(R)$ theory of gravity is treated most seriously during the last decade. In this theory, a general function of Ricci scalar is used instead of standard Einstein-Hilbert Lagrangian R. The $f(R)$ theory of gravity is considered most suitable due to cosmologically important $f(R)$ models. These models consist of higher order curvature invariants as functions of the Ricci scalar. Viable $f(R)$ gravity models [1] have been proposed which show the unification of early-time inflation and late-time acceleration. The problem of dark matter can also be addressed by using viable $f(R)$ gravity models. $f(R)$ theory of gravity has received considerable attention in the recent years [2–6]. There are some useful aspects [7] of this theory. It gives an easy unification of early time inflation and late time acceleration. It provides a natural gravitational alternative to dark energy. The explanation of cosmic acceleration is obtained just by introducing the term $1/R$ which is essential at small curvatures. It also describes the transition phase of the universe from deceleration to acceleration. Thus $f(R)$ theory of gravity seems attractive and a reasonable amount of work has been done in different contexts. A partial review of the literature, focusing on exact solutions in $f(R)$ theory of gravity is given below.

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Lobo and Oliveira [8] constructed wormhole geometries in the context of $f(R)$ theories of gravity. Cognola et al. [9] investigated $f(R)$ gravity at one-loop level in de-Sitter universe. It was found that one-loop effective action can be useful for the study of constant curvature black hole nucleation rate. Multamäki and Vilja [10] investigated spherically symmetric vacuum solutions in $f(R)$ theory. The same authors [11] also studied the perfect fluid solutions and showed that pressure and density did not uniquely determine $f(R)$. Capozziello et al. [12] explored spherically symmetric solutions of $f(R)$ theories of gravity via the Noether symmetry approach. Hollenstein and Lobo [13] analyzed exact solutions of static spherically symmetric spacetimes in $f(R)$ gravity coupled to non-linear electrodynamics. Azadi et al. [14] studied cylindrically symmetric vacuum solutions in this theory. Momeni and Gholizade [15] extended cylindrically symmetric solutions in a more general way. Reboucas and Santos [16] studied Gödel-type universes in $f(R)$ gravity. We have explored static plane symmetric vacuum solutions [17] in $f(R)$ gravity. Recently, Sharif and Kausar [18] investigated non-vacuum static spherically symmetric solutions in this theory.

Friedmann-Robertson-Walker (FRW) models, being spatially homogeneous and isotropic in nature, are best for the representation of the large scale structure of the present universe. However, the models with anisotropic background are the most suitable to describe the early stages of the universe. Bianchi type-*III* models are among the simplest models with anisotropic background. These models are interesting because they not only allow expansion but also shear and rotation. Many authors [19–24] explored these spacetimes in different contexts. Moussiaux et al. [25] investigated the exact solution for vacuum Bianchi type-*III* model with a cosmological constant. Lorenz-Petzold [19] studied exact Bianchi type-*III* solutions in the presence of electromagnetic field. Xing-Xiang [26] discussed Bianchi type *III* string cosmology with bulk viscosity. He assumed the expansion scalar proportional to the shear scalar to derive the solutions. Upadhaya [27] explored some magnetized Bianchi type-*III* massive string cosmological models in GR.

The investigation of Bianchi type models in alternative or modified theories of gravity is also an interesting discussion. Singh et al. [28] studied some Bianchi type-*III* cosmological models in scalar-tensor theory. Adhav et al. [29] obtained an exact solution of the vacuum Brans-Dicke field equations for the metric tensor of a spatially homogeneous and anisotropic model. Paul et al. [30] investigated FRW cosmologies in $f(R)$ gravity. We [31, 32] have studied the solutions of Bianchi types I and V spacetimes in the framework of $f(R)$ gravity. In a recent paper [33], the vacuum solutions of some Bianchi type cosmological models in $f(R)$ gravity has been discussed.

In this paper, we focus our attention to explore the vacuum solution of Bianchi types *III* spacetime in metric $f(R)$ gravity. The field equations are solved by taking expansion scalar θ proportional to shear scalar σ which gives $C = A^n$, where A and C are the metric coefficients. The paper is organized as follows: A brief introduction of the field equations in metric version of $f(R)$ gravity is given in Sect. 2. In Sect. 3, we present the field equations for Bianchi types *III* spacetime and define some physical parameters. Section 4 is used to find the exact solution of the field equations. In the last section, we discuss the results.

2 Field Equations in Metric $f(R)$ Gravity

The metric tensor plays an important role in GR. The dependence of Levi-Civita connection on the metric tensor is one of the main properties of GR. However, if we allow torsion in the theory, then the connection no longer remains the Levi-Civita connection and the dependence of connection on the metric tensor vanishes. This is the main idea behind different approaches of $f(R)$ theories of gravity.

When the connection is the Levi-Civita connection, we get metric $f(R)$ gravity. In this approach, we take variation of the action with respect to the metric tensor only. The action for $f(R)$ gravity is given by

$$S=\int\sqrt{-g}\left(\frac{1}{16\pi G}f(R)+L_m\right)d^4x, \tag{1}$$

where $f(R)$ is a general function of the Ricci scalar and L_m is the matter Lagrangian. The field equations resulting from this action are the following:

$$F(R)R_{\mu\nu}-\frac{1}{2}f(R)g_{\mu\nu}-\nabla_\mu\nabla_\nu F(R)+g_{\mu\nu}\Box F(R)=\kappa T_{\mu\nu}, \tag{2}$$

where $F(R)\equiv df(R)/dR,\ \Box\equiv\nabla^\mu\nabla_\mu,\ \nabla_\mu$ is the covariant derivative and $T_{\mu\nu}$ is the standard matter energy-momentum tensor derived from the Lagrangian L_m. Now contracting the field equations, it follows that

$$F(R)R-2f(R)+3\Box F(R)=\kappa T.$$

In vacuum, this reduces to

$$F(R)R-2f(R)+3\Box F(R)=0, \tag{3}$$

which implies that

$$f(R)=\frac{3\Box F(R)+F(R)R}{2}. \tag{4}$$

This gives an important relationship between $f(R)$ and $F(R)$ which may be used to simplify the field equations and to evaluate $f(R)$.

3 Bianchi Type *III* Spacetime

The line element of Bianchi type *III* spacetime is given by

$$ds^2=dt^2-A^2(t)dx^2-e^{-2mx}B^2(t)dy^2-C^2(t)dz^2, \tag{5}$$

where $A,\ B$ and C are cosmic scale factors. The corresponding Ricci scalar is

$$R=-2\left[\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}-\frac{m^2}{A^2}+\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{C}\dot{A}}{CA}\right], \tag{6}$$

where dot denotes derivative with respect to t. Using (4), the vacuum field equations take the form,

$$F(R)R_{\mu\nu}-\nabla_\mu\nabla_\nu F(R)=\left[\frac{F(R)R-\Box F(R)}{4}\right]g_{\mu\nu}. \tag{7}$$

One can view (7) as the set of differential equations for $F,\ A,\ B$ and C. Thus the subtraction of the 00-component and 11-component gives

$$-\frac{\ddot{B}}{B}-\frac{\ddot{C}}{C}-\frac{m^2}{A^2}+\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{C}\dot{A}}{CA}+\frac{\dot{A}\dot{F}}{AF}-\frac{\ddot{F}}{F}=0. \tag{8}$$

Similarly, we get two more independent equations

$$-\frac{\ddot{A}}{A}-\frac{\ddot{C}}{C}-\frac{m^2}{A^2}+\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{B}\dot{F}}{BF}-\frac{\ddot{F}}{F}=0, \tag{9}$$

$$-\frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}+\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{C}\dot{A}}{CA}+\frac{\dot{C}\dot{F}}{CF}-\frac{\ddot{F}}{F}=0. \tag{10}$$

Also, the 01-component can be written in the following form

$$mF\left[\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right]=0, \tag{11}$$

which gives

$$B=kA, \tag{12}$$

where k is an integration constant. For the sake of simplicity, we take $k=1$. We give definition of some physical quantities before solving these equations.

The average scale factor a and the volume scale factor V are defined as

$$a=\sqrt[3]{A^2C}, \quad V=a^3=A^2C. \tag{13}$$

The generalized mean Hubble parameter H is given in the form

$$H=\frac{1}{3}(H_1+H_2+H_3), \tag{14}$$

where $H_1=\frac{\dot{A}}{A}=H_2,\ H_3=\frac{\dot{C}}{C}$ are the directional Hubble parameters in the directions of $x,\ y$ and z axis respectively. Using (13) and (14), we obtain

$$H=\frac{1}{3}\frac{\dot{V}}{V}=\frac{1}{3}(H_1+H_2+H_3)=\frac{\dot{a}}{a}. \tag{15}$$

The expansion scalar θ and shear scalar σ are defined as follows

$$\theta=u^{\mu}_{;\mu}=2\frac{\dot{A}}{A}+\frac{\dot{C}}{C}, \tag{16}$$

$$\sigma^2=\frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}=\frac{1}{3}\left[\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right]^2, \tag{17}$$

where

$$\sigma_{\mu\nu}=\frac{1}{2}\left(u_{\mu;\alpha}h^{\alpha}_{\nu}+u_{\nu;\alpha}h^{\alpha}_{\mu}\right)-\frac{1}{3}\theta h_{\mu\nu}, \tag{18}$$

$h_{\mu\nu}=g_{\mu\nu}-u_{\mu}u_{\nu}$ is the projection tensor while $u_{\mu}=\sqrt{g_{00}}(1,0,0,0)$ is the four-velocity in co-moving coordinates.

4 Solution of the Field Equations

Using (12) in (8–10), it follows that

$$-\frac{\ddot{A}}{A}-\frac{\ddot{C}}{C}-\frac{m^2}{A^2}+\frac{\dot{A}^2}{A^2}+\frac{\dot{A}\dot{C}}{AC}+\frac{\dot{A}\dot{F}}{AF}-\frac{\ddot{F}}{F}=0, \tag{19}$$

$$-\frac{2\ddot{A}}{A}+\frac{2\dot{C}\dot{A}}{CA}+\frac{\dot{C}\dot{F}}{CF}-\frac{\ddot{F}}{F}=0. \tag{20}$$

Now we have two differential equations with three unknowns namely A, C and F. Thus we need one additional constraint to solve these equations. We use a physical condition that expansion scalar θ is proportional to shear scalar σ which gives

$$C = A^n. \tag{21}$$

Using this condition, (19–20) take the form

$$(n+1)\frac{\ddot{A}}{A} + \left(n^2 - 2n - 1\right)\frac{\dot{A}^2}{A^2} - \frac{\dot{A}\dot{F}}{AF} + \frac{\ddot{F}}{F} = -\frac{m^2}{A^2}, \tag{22}$$

$$-\frac{2\ddot{A}}{A} + 2n\frac{\dot{A}^2}{A^2} + n\frac{\dot{A}\dot{F}}{AF} - \frac{\ddot{F}}{F} = 0. \tag{23}$$

Adding these, we get

$$(n-1)\frac{\ddot{A}}{A} + \left(n^2 - 1\right)\frac{\dot{A}^2}{A^2} + (n-1)\frac{\dot{A}\dot{F}}{AF} = -\frac{m^2}{A^2}. \tag{24}$$

We solve this equation using power law relation between F and a [31],

$$F = ka^m, \tag{25}$$

where k is the constant of proportionality, m is any integer and a is given by

$$a = A^{\frac{n+2}{3}}. \tag{26}$$

Thus for $m = 3$, we obtain

$$F = kA^{n+2}. \tag{27}$$

Using this in (24), it follows that

$$\frac{\ddot{A}}{A} + (2n+3)\frac{\dot{A}^2}{A^2} = -\frac{m^2}{(n-1)A^2}. \tag{28}$$

Put $\dot{A} = f(A)$ in this equation, we get

$$\frac{df^2}{dA} + \frac{4n+6}{A}f^2 = -\frac{m^2}{(n-1)A^2}, \tag{29}$$

which leads to the solution

$$f^2 = \frac{c_1}{A^{4n+6}} - \frac{m^2}{(n-1)(2n+3)}, \tag{30}$$

where c_1 is an integration constant. Hence the solution becomes

$$ds^2 = \left(\frac{dt}{dA}\right)^2 dA^2 - A^2dx^2 - e^{-2mx}A^2dy^2 - A^{2n}dz^2, \tag{31}$$

which can be written as

$$ds^2 = \left(\frac{1}{\frac{c_1}{T^{4n+6}} - \frac{m^2}{(n-1)(2n+3)}}\right)dT^2 - T^2\left(dX^2 + e^{-2mX}dY^2\right) - T^{2n}dZ^2, \tag{32}$$

where $A = T,\ x = X,\ y = Y$ and $z = Z$.

The directional Hubble parameters H_i $(i = 1, 2, 3)$ take the form

$$H_1 = H_2 = \frac{c_1}{T^{4n+8}} - \frac{m^2}{(n-1)(2n+3)T^2},$$
$$H_3 = n\left[\frac{c_1}{T^{4n+8}} - \frac{m^2}{(n-1)(2n+3)T^2}\right]. \tag{33}$$

The mean generalized Hubble parameter becomes

$$H = \left(\frac{n+2}{3}\right)\left[\frac{c_1}{T^{4n+8}} - \frac{m^2}{(n-1)(2n+3)T^2}\right], \tag{34}$$

while the volume scale factor turns out to be

$$V = T^{n+2}. \tag{35}$$

The expansion scalar θ is given by

$$\theta = (n+2)\left[\frac{c_1}{T^{4n+8}} - \frac{m^2}{(n-1)(2n+3)T^2}\right]^{\frac{1}{2}}, \tag{36}$$

while the shear scalar σ becomes

$$\sigma^2 = \frac{1}{3}(n-1)^2\left[\frac{c_1}{T^{4n+8}} - \frac{m^2}{(n-1)(2n+3)T^2}\right]. \tag{37}$$

Moreover, the function of Ricci scalar, $f(R)$, can be found by using (4)

$$f(R) = \frac{k}{2}\left[T^{n+2}R - \frac{3m^2(2n^2+7n+6)}{2n^2+n-3}T^n\right]. \tag{38}$$

It follows from (6) that

$$R = 2\left[\frac{n^2+6n+5}{T^{4n+8}} + \frac{m^2}{T^2}\left(\frac{3n^2+2n-2}{2n^2+n-3}\right)\right], \tag{39}$$

which clearly indicates that $f(R)$ cannot be explicitly written in terms of R. However, by inserting this value of R, $f(R)$ can be written as a function of T, which is true as R depends upon T. For a special case when $n = -1$, $f(R)$ turns out to be

$$f(R) = \frac{5mk}{4}\sqrt{R}. \tag{40}$$

This gives $f(R)$ only as a function of R.

5 Summary and Conclusion

This purpose of this paper is to study Bianchi types *III* cosmology in metric $f(R)$ gravity. We have found exact solution of the vacuum field equations. Initially, the field equations look complicated but lead to a solution using some assumptions. The first assumption is that the expansion scalar θ is proportional and shear scalar σ. It gives $C = A^n$, where A, C are the metric coefficients and n is an arbitrary constant. Secondly, the power law relation between F and a is used to find the solution. Some important cosmological physical quantities for the solution such as expansion scalar θ, shear scalar σ^2 and mean anisotropy parameter A are evaluated. The general function of Ricci scalar, $f(R)$, is also constructed. As a special case, it is found that the $f(R)$ includes square root power of the Ricci scalar for the given model.

The model of the universe (32) is non-singular at $T = 0$. The physical parameters H_1, H_2, H_3, H, θ and σ are all infinite at this point but the volume scale factor V vanishes. The general function of the Ricci scalar is finite while the spatial part of the metric vanish at $T = 0$. The expansion stops for $n = -2$. The model also suggests that the expansion and shear scalar decrease for $-2 < n < 1$ with the passage of time. This indicates that after a large time the expansion will stop completely and the universe will achieve isotropy. The isotropy condition, i.e., $\frac{\sigma^2}{\theta} \to 0$ as $T \to \infty$, is also satisfied. Thus we can conclude from these observations that the model starts its expansion from zero volume at $T = 0$ and its volume increases with the passage of time.

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