Int J Theor Phys (2011) 50:1993–2001
DOI 10.1007/s10773-010-0522-z

Rosen–Morse Potential and Its Supersymmetric Partners

Samuel Domínguez-Hernández · David J. Fernández C.

Received: 18 September 2010 / Accepted: 29 September 2010 / Published online: 14 October 2010
© Springer Science+Business Media, LLC 2010

Abstract A set of supersymmetric partners of the Rosen–Morse potential is generated. The corresponding physical properties are studied—in particular, the change in intensity of the singularities at the boundaries of the domain.

Keywords Supersymmetric quantum mechanics · Rosen–Morse potentials · Shape invariance

1 Introduction

The study of exactly solvable potentials in quantum mechanics for years has received a lot of attention (see, e.g., [1, 2]). For solvable models, the simplest generation technique is supersymmetric quantum mechanics (SUSY QM) [3–6], which is equivalent to the factorization method, the intertwining technique and the Darboux transformation method [7, 8]. It is well known that the spectrum of the generated Hamiltonian differs little from the initial one. This suggests a way to realize, in practice, the spectral design [9, 10]: (i) one starts from a potential having a spectrum close to the desired one; (ii) then, by appropriately moving, creating or deleting a certain set of levels, and iterating the method as many times as needed to achieve the required spectrum, one will arrive at a Hamiltonian (or a set of Hamiltonians), which could model the situation under study.

SUSY techniques have been applied successfully to several one-dimensional Hamiltonians, e.g., the harmonic oscillator, infinite well and Pöschl–Teller potentials (trigonometric or hyperbolic) [6, 11–17]. For these systems the generic discrete energy level E_n is a second-order polynomial of the index n. This implies that there is an intrinsic algebraic structure

S. Domínguez-Hernández
Ciencias Básicas, UPIITA-IPN, Av. IPN 2508, CP 07340, México DF, Mexico
e-mail: sdominguezh@ipn.mx

D.J. Fernández C. (✉)
Departamento de Física, Cinvestav, A.P. 14-740, 07000 México DF, Mexico
e-mail: david@fis.cinvestav.mx

(IAS) of Lie type involving the number, annihilation and creation operators, since the commutator between the ladder operators, which coincides with $E_{n+1} - E_n$, is linear in n [16]. It is important to study potentials with a different dependence for E_n, such that the IAS is not longer of Lie type. One of them is the Rosen–Morse potential: since E_n is the sum of a term quadratic in n plus another one proportional to n^{-2} [18], the IAS becomes nonlinear (another system with similar E_n dependence appears in [19]). The Rosen–Morse potentials can be used also to illustrate the spectral manipulation possibilities offered by the second-order SUSY QM [20–26]. In addition, since they have been employed recently as quark models [27], it is natural to assume that their SUSY partners could as well be useful in high energy physics.

In this paper we are going to generate the first- and second-order SUSY partners of the Rosen–Morse potentials. The seeds employed to implement the transformations will be just the physical eigenfunctions of the initial Hamiltonian. We are going to show that the first-order transformation involving the ground state leads once again to a Rosen–Morse potential (with just one of the parameters displaced), which is the celebrated property of shape invariance [3, 18]. On the other hand, for second-order transformations in which two consecutive eigenfunctions of the initial Hamiltonian are used, it will be shown that the SUSY partner potentials in general are not shape invariant.

The paper is organized as follows. In Sects. 2 and 3 we will introduce respectively the first- and second-order SUSY QM. In Sect. 4 we will use the SUSY techniques to generate new exactly solvable Hamiltonians departing from the Rosen–Morse potentials. Our conclusions will be found in Sect. 5.

2 Standard Supersymmetric Quantum Mechanics

Let us consider the following intertwining relationships:

$$H_1 A_1^+ = A_1^+ H_0, \qquad H_0 A_1 = A_1 H_1, \tag{1}$$

where the intertwiners A_1 and A_1^+ are first-order differential operators of the form

$$A_1 = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \alpha(x)\right), \qquad A_1^+ = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x} + \alpha(x)\right), \tag{2}$$

$\alpha(x)$ is a real function of x, and H_0, H_1 are two Schrödinger type Hamiltonians,

$$H_0 = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_0(x), \qquad H_1 = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_1(x), \tag{3}$$

such that $V_0(x)$ and $V_1(x)$ are real potentials. It is well known that $V_0(x)$, $V_1(x)$ and $\alpha(x)$ are related by [5, 6]

$$V_1(x) = V_0(x) - \alpha'(x), \tag{4}$$

$$\alpha' + \alpha^2 = 2\big[V_0(x) - \epsilon\big]. \tag{5}$$

Thus, if the initial potential $V_0(x)$ is given and we use a solution of the Riccati equation (5) associated to a certain factorization energy ϵ, the new potential $V_1(x)$ is completely determined from (4). Alternatively, by expressing $\alpha(x) = u'/u = [\ln u]'$, (4) and (5) become

$$V_1(x) = V_0(x) - [\ln u]'', \tag{6}$$

$$-\frac{u''}{2} + V_0 u = H_0 u = \epsilon u. \tag{7}$$

Moreover, the formal eigenfunction u of H_0 satisfies as well $A_1^+ u = 0$.

Let us notice that (4), (5) lead to the factorizations of H_0 and H_1:

$$H_0 = A_1 A_1^+ + \epsilon, \tag{8}$$

$$H_1 = A_1^+ A_1 + \epsilon. \tag{9}$$

Suppose now that H_0 is a solvable Hamiltonian with normalized eigenfunctions $\psi_n(x)$ and eigenvalues E_n, $n = 0, 1, \ldots$ We would like to implement the first-order SUSY transformation by using one of the physical eigenfunctions. Since $u(x)$ has to be nodeless in order to avoid the arising of new singularities in the potential $V_1(x)$ (see (6)), the only possibility is to choose the ground state $\psi_0(x)$ of H_0. With this choice it turns out that

$$V_1(x) = V_0(x) - \big[\ln \psi_0(x)\big]''. \tag{10}$$

The normalized eigenfunctions $\psi_n^{(1)}(x)$ of H_1 thus acquire the form

$$\psi_n^{(1)}(x) = \frac{A_1^+ \psi_n(x)}{\sqrt{E_n - E_0}}, \quad n = 1, 2, \ldots. \tag{11}$$

Notice that there is a formal eigenfunction of H_1 associated to E_0, given by $\psi_0^{(1)}(x) \propto 1/\psi_0(x)$ and such that $H_1 \psi_0^{(1)}(x) = E_0 \psi_0^{(1)}(x)$, $A_1 \psi_0^{(1)}(x) = 0$. However, it is not square-integrable; thus $E_0 \notin \mathrm{Sp}(H_1)$, and therefore

$$\mathrm{Sp}(H_1) = \{E_n, n = 1, 2, \ldots\}. \tag{12}$$

Thus, during the first-order SUSY transformation the level E_0 of H_0 was deleted for generating the new potential $V_1(x)$.

3 Second-Order Supersymmetric Quantum Mechanics

Let us now suppose that

$$H_2 B_2^+ = B_2^+ H_0, \tag{13}$$

$$H_i = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_i(x), \quad i = 0, 2, \tag{14}$$

$$B_2^+ = \frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \eta(x)\frac{\mathrm{d}}{\mathrm{d}x} + \gamma(x)\right), \tag{15}$$

where $V_0(x)$ and $V_2(x)$ are again real. A calculation similar to the one for the first-order case leads to [5, 6]

$$V_2 = V_0 - \eta', \qquad \gamma = \frac{\eta'}{2} + \frac{\eta^2}{2} - 2V_0 + d, \tag{16}$$

$$\frac{\eta\eta''}{2} - \frac{\eta'^2}{4} + \frac{\eta^4}{4} + \eta^2\eta' - 2V_0\eta^2 + d\eta^2 + c = 0, \quad d, c \in \mathbb{R}. \tag{17}$$

Suppose now that $V_0(x)$ is given; to determine $V_2(x)$ and $\gamma(x)$ we need to find a solution $\eta(x)$ to the nonlinear differential equation (17). This is done through the ansatz

$$\eta' = -\eta^2 + 2\beta\eta + 2\xi, \tag{18}$$

where $\beta(x)$ and $\xi(x)$ are to be determined. By plugging this ansatz into (17) one obtains

$$\xi^2 \equiv c, \qquad \epsilon = (d + \xi)/2, \tag{19}$$

$$\beta' + \beta^2 = 2[V_0 - \epsilon]. \tag{20}$$

In this paper we are going to restrict ourselves to the case with $c > 0$, for which there are two real values for ξ, $\xi_{1,2} = \pm\sqrt{c}$; then there will be two different real factorization energies ϵ, $\epsilon_{1,2} = (d \pm \sqrt{c})/2$ as well as two associated solutions $\beta_{1,2}$ of the Riccati equation (20), which will be taken also as real. Each of them leads to the same η-function,

$$\eta' = -\eta^2 + 2\beta_1\eta + 2(\epsilon_1 - \epsilon_2), \tag{21}$$

$$\eta' = -\eta^2 + 2\beta_2\eta - 2(\epsilon_1 - \epsilon_2). \tag{22}$$

By subtracting these two equations and solving for η we finally arrive at

$$\eta = -\frac{2(\epsilon_1 - \epsilon_2)}{\beta_1 - \beta_2}. \tag{23}$$

Alternatively, by expressing $\beta_i = [\ln u_i]'$, $i = 1, 2$, η can be expressed in terms of two stationary Schrödinger solutions u_1, u_2 of H_0, $H_0 u_i = \epsilon_i u_i$, namely

$$\eta = \frac{2(\epsilon_1 - \epsilon_2)u_1 u_2}{W(u_1, u_2)} = \frac{W'(u_1, u_2)}{W(u_1, u_2)}. \tag{24}$$

Notice that $u_{1,2}$ are also annihilated by the intertwiner B_2^+: $B_2^+ u_{1,2} = 0$.

Similarly to the first-order case, in the second-order situation, (16) and (17) lead as well to two factorizations involving H_0 and H_2 [5, 6]:

$$B_2 B_2^+ = (H_0 - \epsilon_1)(H_0 - \epsilon_2), \tag{25}$$

$$B_2^+ B_2 = (H_2 - \epsilon_1)(H_2 - \epsilon_2). \tag{26}$$

Let us assume that H_0 is a solvable Hamiltonian with normalized eigenfunctions $\psi_n(x)$ and eigenvalues E_n, $n = 0, 1, \ldots$ We implement a second-order SUSY transformation by using as seeds two physical eigenfunctions of H_0. Since their Wronskian has to be nodeless in order to avoid the situation that new singularities appear in $V_2(x)$ (see (16) and (24)), let us choose them as two consecutive eigenfunctions $\psi_j(x)$, $\psi_{j+1}(x)$ (it has been shown that if u_1, u_2 have alternating zeros, then $W(u_1, u_2)$ will be nodeless, which is satisfied by ψ_j and ψ_{j+1} [5, 23]). With this choice it turns out that

$$V_2(x) = V_0(x) - \left[\ln W(\psi_j, \psi_{j+1})\right]''. \tag{27}$$

The normalized eigenfunctions $\psi_n^{(2)}(x)$ of H_2 thus acquire the form

$$\psi_n^{(2)}(x) = \frac{B_2^+ \psi_n(x)}{\sqrt{(E_n - E_j)(E_n - E_{j+1})}}, \quad n \neq j, j+1. \tag{28}$$

Notice that there are two formal eigenfunctions of H_2, $\psi_j^{(2)}$ and $\psi_{j+1}^{(2)}$, associated to E_j and E_{j+1}, which are also annihilated by B_2 and are given by

$$\psi_j^{(2)} \propto \frac{\psi_{j+1}}{W(\psi_j, \psi_{j+1})}, \qquad \psi_{j+1}^{(2)} \propto \frac{\psi_j}{W(\psi_j, \psi_{j+1})}. \tag{29}$$

However, they are not square-integrable, i.e., $E_j, E_{j+1} \notin \mathrm{Sp}(H_2)$, and therefore

$$\mathrm{Sp}(H_2) = \{E_0, \ldots, E_{j-1}, E_{j+2}, \ldots\}. \tag{30}$$

Thus, during the second-order SUSY transformation, the levels E_j, E_{j+1} of H_0 were deleted for generating the new potential $V_2(x)$.

4 Rosen–Morse Potentials

Let us consider the Rosen–Morse potentials

$$V_0(x) = \frac{a(a+1)}{2}\csc^2(x) - b\cot(x), \quad a > 0. \tag{31}$$

We need to find the solutions of the stationary Schrödinger equation:

$$-\frac{1}{2}\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + \left[\frac{a(a+1)}{2}\csc^2(x) - b\cot(x)\right]\Psi = E\Psi, \tag{32}$$

for any $E \in \mathbb{R}$, $0 \le x \le \pi$. Let us propose in the first place that

$$\Psi(x) = e^{-\frac{\alpha}{2}x} F(x). \tag{33}$$

Thus, $F(x)$ satisfies the following differential equation:

$$\frac{\mathrm{d}^2 F}{\mathrm{d}x^2} - \alpha\frac{\mathrm{d}F}{\mathrm{d}x} + \left[2b\cot(x) - a\,(a+1)\csc^2(x) + \frac{\alpha^2}{4} + 2E\right]F = 0. \tag{34}$$

Let us make now the change of variable $z = \cot(x)$. Thus:

$$\begin{aligned}&\left(1+z^2\right)^2\frac{\mathrm{d}^2 F}{\mathrm{d}z^2} + (2z+\alpha)\left(1+z^2\right)\frac{\mathrm{d}F}{\mathrm{d}z}\\ &\quad + \left[2bz - a\,(a+1)\left(1+z^2\right) + \frac{\alpha^2}{4} + 2E\right]F = 0.\end{aligned} \tag{35}$$

The additional factorization

$$F(z) = \left(1+z^2\right)^{\frac{\beta-1}{2}} f(z) \tag{36}$$

leads to

$$\left(1+z^2\right)\frac{\mathrm{d}^2 f}{\mathrm{d}z^2}+(\alpha+2\beta z)\frac{\mathrm{d}f}{\mathrm{d}z}+\Bigg[\beta(\beta-1)-a(a+1) + \frac{(\alpha\beta-\alpha+2b)\,z+\frac{\alpha^2}{4}+2E-(\beta-1)^2}{1+z^2}\Bigg]f=0. \tag{37}$$

Let us demand now that α, β are such that

$$\alpha(\beta-1)+2b=0, \qquad \frac{\alpha^2}{4}+2E-(\beta-1)^2=0, \tag{38}$$

whose solutions can be chosen as

$$\alpha=\frac{2b}{\sqrt{E+\sqrt{E^2+b^2}}}, \qquad \beta=1-\sqrt{E+\sqrt{E^2+b^2}}. \tag{39}$$

By making, finally, the new change of variable $\xi=(1-iz)/2$, (37) becomes

$$\xi(1-\xi)\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2}+\left[\left(\beta+i\frac{\alpha}{2}\right)-2\beta\xi\right]\frac{\mathrm{d}f}{\mathrm{d}\xi}+\left[\beta(1-\beta)+a(a+1)\right]f=0, \tag{40}$$

which is the hypergeometric equation with parameters

$$A=\beta+a, \qquad B=\beta-a-1, \qquad C=\beta+i\alpha/2. \tag{41}$$

There are several ways of choosing the two linearly independent (LI) solutions of the hypergeometric equation, and we choose those called type III and IV in the literature [28], namely:

$$f_1\propto\xi^{-A}\,{}_2F_1\left(A,A+1-C;A+1-B;\xi^{-1}\right), \tag{42}$$

$$f_2\propto\xi^{-B}\,{}_2F_1\left(B+1-C,B;B+1-A;\xi^{-1}\right). \tag{43}$$

We finally get the two real LI solutions of the Schrödinger equation for the Rosen–Morse potentials:

$$\Psi_1(x)=\mathrm{e}^{-[\frac{\alpha}{2}+\mathrm{i}(\beta+a)]x}\sin^{1+a}(x)\,{}_2F_1\left(\beta+a,a+1-\frac{\mathrm{i}\alpha}{2};2a+2;2\mathrm{i}\mathrm{e}^{-\mathrm{i}x}\sin(x)\right), \tag{44}$$

$$\Psi_2(x)=\mathrm{e}^{-[\frac{\alpha}{2}+\mathrm{i}(\beta-1-a)]x}\sin^{-a}(x)\,{}_2F_1\left(\beta-1-a,-a-\frac{\mathrm{i}\alpha}{2};-2a;2\mathrm{i}\mathrm{e}^{-\mathrm{i}x}\sin(x)\right). \tag{45}$$

In order to find the physical solutions, first of all $\Psi_2(x)$ is ruled out, since it diverges at $x=0$ and $x=\pi$. In the second place, it is required that the hypergeometric series of $\Psi_1(x)$ becomes a polynomial, which is achieved by taking $\beta+a=-n$. This implies that $\alpha=2b/(n+a+1)$ and therefore

$$E_n=\frac{1}{2}(n+a+1)^2-\frac{b^2}{2(n+a+1)^2}, \quad n=0,1,\ldots. \tag{46}$$

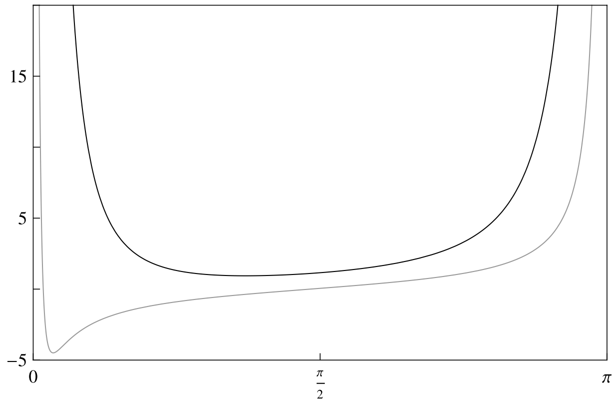

Fig. 1 First-order SUSY partner (*black curve*) of the Rosen–Morse potential (*gray curve*) for $a = 0.1,\ b = 1$, generated using the ground state eigenfunction of H_0

The corresponding eigenfunctions of H_0 become

$$\psi_n(x) \propto \mathrm{e}^{-[\frac{b}{n+a+1} - \mathrm{i}n]x} \sin^{1+a}(x) \\ \times {}_2F_1\left(-n, a+1-\frac{ib}{n+a+1}; 2a+2; 2\mathrm{i}\mathrm{e}^{-\mathrm{i}x}\sin(x)\right). \tag{47}$$

4.1 First-order SUSY partner of $V_0(x)$

Let us take the ground state eigenfunction,

$$\psi_0(x) \propto \mathrm{e}^{-\frac{bx}{a+1}} \sin^{1+a}(x),$$

as a seed for implementing the first-order SUSY transformation [18]. Thus, the SUSY partner potential of $V_0(x)$ becomes (see (10))

$$V_1(x) = \frac{1}{2}(a+1)(a+2)\csc^2(x) - b\cot(x), \tag{48}$$

which can be obtained from the Rosen–Morse potential by the change $a \to a+1$; this is the property of shape invariance [3]. An illustration of the supersymmetric partners $V_0(x)$ and $V_1(x)$ is given in Fig. 1.

4.2 Second-Order SUSY Partners of $V_0(x)$

Let us now take $\psi_j(x)$ and $\psi_{j+1}(x)$ as the two seeds to implement the second-order SUSY transformation. Since both solutions vanish at $x = 0$ and $x = \pi$, $W(\psi_j, \psi_{j+1})$ has the same null behavior at both ends, which is separated through the factorization

$$W(\psi_j, \psi_{j+1}) = \sin^{3+2a}(x)\, \mathcal{W}(x), \tag{49}$$

where $\mathcal{W}(x)$ is a nodeless function in $[0, \pi]$. Thus, the second-order SUSY partner potential of $V_0(x)$ becomes (see (27))

$$V_2(x) = \frac{1}{2}(a+2)(a+3)\csc^2(x) - b\cot(x) - \left[\ln \mathcal{W}(x)\right]''. \tag{50}$$

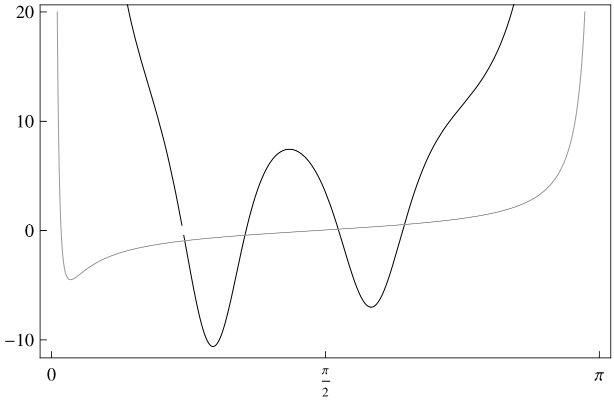

Fig. 2 Second-order SUSY partner (*black curve*) of the Rosen–Morse potential (*gray curve*) for $a=0.1,\ b=1$, generated by using as seeds $\psi_2(x)$ and $\psi_3(x)$

Notice that, in particular, for $j=0$ we get

$$V_2(x)=\frac{1}{2}(a+2)(a+3)\csc^2(x)-b\cot(x), \tag{51}$$

which is once again the shape invariance of the Rosen–Morse potentials. However, for $j>0$ it turns out that $[\ln\mathcal{W}(x)]''\neq 0$, which implies that in the second-order case the shape invariance will arise just when we delete simultaneously both the ground and the first excited states of H_0 for generating $V_2(x)$. This property will no longer arise if we delete instead the first and second excited states, or the second and third excited states, etc. An illustration of these properties is shown in Fig. 2, where we have deleted the second and third excited states of H_0 to produce $V_2(x)$.

5 Conclusions

In this paper the SUSY transformations of first and second order were used to generate new solvable Hamiltonians departing from the Rosen–Morse potentials. In particular, it was shown that their second-order SUSY partners do not belong in general to the initial family of potentials. The algebraic structure of the initial Hamiltonian and how this is inherited to the SUSY generated potentials remain to be studied in detail. Consequently, some related subjects, like the associated coherent states, could be studied once this algebraic structure is analyzed (see, e.g., [29]). This paper is just the starting point for addressing some of these topics for the Rosen–Morse potentials, which we hope to do in the near future.

Acknowledgement The authors acknowledge the support of Conacyt.

References

1. Miller, W.: Lie Theory and Special Functions. Academic Press, New York (1968)
2. de Lange, O.L., Raab, R.E.: Operator Methods in Quantum Mechanics. Clarendon, Oxford (1991)
3. Cooper, F., Khare, A., Sukhatme, U.: Supersymmetry and quantum mechanics. Phys. Rep. **251**, 267–385 (1995)
4. Mielnik, B., Rosas-Ortiz, O.: Factorization: little or great algorithm? J. Phys. A **37**, 10007–10035 (2004)
5. Fernández, D.J., Fernández-García, N.: Higher-order supersymmetric quantum mechanics. AIP Conf. Proc. **744**, 236–273 (2005)

6. Fernández, D.J.: Supersymmetric quantum mechanics. arXiv:0910.0192v1 [quant-ph] (2009)
7. Andrianov, A.A., Borisov, N.V., Ioffe, M.V.: Factorization method and Darboux transformation for multidimensional Hamiltonians. Theor. Math. Phys. **61**, 1078–1088 (1984)
8. Bagrov, V.G., Samsonov, B.F.: Darboux transformation of the Schrödinger equation. Phys. Part. Nucl. **28**, 374–397 (1997)
9. Fernández, D.J., Rosu, H.C.: Quantum mechanical spectral engineering by scaling intertwining. Phys. Scr. **64**, 177–183 (2001)
10. Andrianov, A.A., Cannata, F.: Nonlinear supersymmetry for spectral design in quantum mechanics. J. Phys. A **37**, 10297–10321 (2004)
11. Fernández, D.J., Glasser, M.L., Nieto, L.M.: New isospectral oscillator potentials. Phys. Lett. A **240**, 15–20 (1998)
12. Fernández, D.J., Hussin, V., Mielnik, B.: A simple generation of exactly solvable anharmonic oscillators. Phys. Lett. A **244**, 309–316 (1998)
13. Fernández, D.J., Hussin, V.: Higher order SUSY, linearized non-linear Heisenberg algebras and coherent states. J. Phys. A **32**, 3603–3619 (1999)
14. Díaz, J.I., Negro, J., Nieto, L.M., Rosas-Ortiz, O.: The supersymmetric modified Pöschl–Teller and delta-well potentials. J. Phys. A **32**, 8447–8460 (1999)
15. Negro, J., Nieto, L.M., Rosas-Ortiz, O.: Confluent hypergeometric equations and related solvable potentials in quantum mechanics. J. Math. Phys. **41**, 7964–7996 (2000)
16. Fernández, D.J., Hussin, V., Rosas-Ortiz, O.: Coherent states for Hamiltonians generated by supersymmetry. J. Phys. A **40**, 6491–6511 (2007)
17. Contreras-Astorga, A., Fernández, D.J.: Supersymmetric partners of the trigonometric Pöschl–Teller potentials. J. Phys. A **41**, 475303 (2008)
18. Compean, C.B., Kirchbach, M.: The trigonometric Rosen–Morse potential in the supersymmetric quantum mechanics and its exact solutions. J. Phys. A **39**, 547–557 (2006)
19. Nieto, L.M., Santander, M., Rosu, H.C.: Hydrogen atom as an eigenvalue problem in 3-D spaces of constant curvature and minimal length. Mod. Phys. Lett. A **14**, 2463–2469 (1999)
20. Andrianov, A.A., Ioffe, M.V., Spiridonov, V.: Higher-derivative supersymmetry and the Witten index. Phys. Lett. A **174**, 273–279 (1993)
21. Andrianov, A.A., Ioffe, M.V., Cannata, F., Dedonder, J.P.: 2nd-order derivative supersymmetry, q-deformations and the scattering problem. Int. J. Mod. Phys. A **10**, 2683–2702 (1995)
22. Fernández, D.J.: SUSUSY quantum mechanics. Int. J. Mod. Phys. A **12**, 171–176 (1997)
23. Samsonov, B.F.: New possibilities for supersymmetry breakdown in quantum mechanics and second-order irreducible Darboux transformations. Phys. Lett. A **263**, 274–280 (1999)
24. Plyushchay, M.: Hidden nonlinear supersymmetries in pure parabosonic systems. Int. J. Mod. Phys. A **15**, 3679–3698 (2000)
25. Aoyama, H., Sato, M., Tanaka, T.: N-fold supersymmetry in quantum mechanics: general formalism. Nucl. Phys. B **619**, 105–127 (2001)
26. Plyushchay, M.: Nonlinear supersymmetry: from classical to quantum mechanics. J. Phys. A **37**, 10375–10384 (2004)
27. Compean, C.B., Kirchbach, M.: Trigonometric quark confinement potential of QCD traits. Eur. Phys. J. A **33**, 1–4 (2007)
28. Bateman, H.: Higher Transcendental Functions, vol. 1. McGraw-Hill, New York (1953)
29. Chenaghlou, A., Faizy, O.: Gazeau–Klauder coherent states for trigonometric Rosen–Morse potential. J. Math. Phys. **49**, 022104 (2008)