Phantom Wormholes in $(2 + 1)$-Dimensions

Mubasher Jamil · M. Umar Farooq

Abstract In this paper, we have constructed a $(2 + 1)$-dimensional wormhole using inhomogeneous and anisotropic distribution of phantom energy. We have determined the exact form of the equation of state of phantom energy that supports the wormhole structure. Interestingly, this equation of state is linear but variable one and is dependent only on the radial parameter of the model.

A typical wormhole is characterized by a tunnel in spacetime connecting two arbitrary spacetime sections. These sections could either belong to the same spacetime or to two different spacetimes. The wormhole geometry arises naturally as a solution of the Einstein field equations [1–3]. Interest in wormhole physics was initiated when Morris and Thorne investigated the wormhole structure and proposed that the material required to construct it has to be exotic, i.e. its (negative) radial pressure and energy density must satisfy the inequality $|p| > \rho$ [4]. They also concluded that this structure could also serve as a time travel machine if it is horizon-free.

From the cosmological perspective, a candidate for exotic matter exists namely the phantom energy. Presently it is well-motivated from the observational data that the observable universe is pervaded with the phantom energy, which is characterized by $\omega = p/\rho < -1$ [5, 6]. In recent years, several studies are performed regarding construction of wormholes with the use of phantom energy as an exotic matter [7–11]. The phantom energy is exotic due to its weird and esoteric properties: its energy density increases as the universe expands; its accretion onto all gravitationally bound objects results in disassociating them; it can rip apart the spacetime itself in a finite time which is called the Big Rip.

M. Jamil (✉) · M.U. Farooq
Center for Advanced Mathematics and Physics, National University of Sciences and Technology, Rawalpindi 46000, Pakistan
e-mail: mjamil@camp.nust.edu.pk

M.U. Farooq
e-mail: m_ufarooq@yahoo.com

Understanding the nature of gravity is one of the hardest and most challenging problems in theoretical physics. Interest in $(2 + 1)$-dimensional theories of gravity—especially general relativity—dates back to early sixties. Since then several toy models have been built up in $(2 + 1)$ gravity which help in understanding the corresponding $(3 + 1)$ dimensional problems [12]. Gravity in $(2 + 1)$ dimensions behaves very differently compared to the usual $(3 + 1)$ dimensional gravity, for example, the gravity does not exist outside the matter source and remains confined locally. Since gravity does not propagate outside the gravitating source, gravitational waves don't arise in this case. In recent years, models of wormholes in $(2 + 1)$ dimensional gravity are presented [13–16]. In these models, coordinate θ is fixed so that $d\theta = 0$. Consequently, this reduces the complexity of the field equations.

The metric of a $(2 + 1)$-dimensional Morris-Thorne (MT) wormhole is given by [13–15]

$$ds^2 = -e^{2f(r)}dt^2 + \frac{1}{1 - \frac{b(r)}{r}}dr^2 + r^2 d\phi^2, \tag{1}$$

where $f(r)$ is called potential function while $b(r)$ is the shape function. These functions are arbitrary functions of radial coordinate r and will be determined below for a specific choice of matter distribution. The radial coordinate has a range that increases from a minimum value at r_0, corresponding to the wormhole throat, and extends to infinity. For the wormhole to be traversable, conditions commonly termed stability and traversability, are imposed on these two functions, namely: $f(r)$ must be bounded for all values of r; $b'(r_0) < 1$ at $r = r_0$; $b(r) < r$ for all $r > r_0$ and $b/r \to 0$ as $|r| \to \infty$. The stress energy components in an orthonormal frame of reference are $T_{00} = \rho$, $T_{11} = p$ and $T_{22} = p_t$. Here ρ is the energy density, p is the radial pressure while p_t is the transverse pressure.

The Einstein field equations become (units are $c = 1 = G$)

$$\rho(r) = \frac{b'r - b}{16\pi r^3}, \tag{2}$$

$$p(r) = \frac{(1 - b/r)f'}{8\pi r}, \tag{3}$$

$$p_t(r) = \frac{(1 - b/r)}{8\pi}\left[f'' - \frac{(b'r - b)}{2r(r - b)}f' + f'^2\right]. \tag{4}$$

Here prime ($'$) denotes differentiation with respect to r. The energy conservation equation is obtained by evaluating $T^{AB}_{;A} = 0$, with $A, B = 0, 1, 2$. It gives

$$p' + f'\rho + \left(f' + \frac{1}{r}\right)p - \frac{p_t}{r} = 0. \tag{5}$$

Note that in the above equations (3) to (5), the function f must not be a constant (i.e. $f' \neq 0$) otherwise the field equations become identically zero. Below we shall determine explicit form of f for a specific choice of two parameters. To solve the field equations, we choose the following ansatz for the shape function and pressures

$$b(r) = \frac{b_0}{r^m}, \quad m = 0, 1, 2, \ldots, \tag{6}$$

$$p_t = \alpha p, \tag{7}$$

where α and b_0 are constants. Notice that α is dimensionless while b_0 possesses dimensions of $(length)^{m+1}$. It is easy to check that (6) satisfies the stability conditions for the wormhole.

The second ansatz (7) says that the ratio of transverse to radial pressure will remain constant although both can vary differently. Our task is to find ρ, f, p and p_t using (2) to (7).

Inserting (6) in (2), we have

$$\rho(r) = -\frac{b_0(m+1)}{16\pi r^{m+3}}. \tag{8}$$

Since ρ is always positive so we require $b_0 < 0$. Making use of (2) to (7) and after simplification, we arrive at

$$f'' + \left[\frac{b_0(m+1)}{2r(r^{m+1}-b_0)} - \frac{\alpha}{r}\right]f' + f'^2 = 0. \tag{9}$$

To solve this equation, we rewrite it as

$$\frac{f''}{f'} - \frac{m+1}{2}\left(\frac{1}{r} - \frac{r^m}{r^{m+1}-b_0}\right) - \frac{\alpha}{r} = -f'. \tag{10}$$

Integrating it we get

$$f'e^f = \frac{Cr^{\alpha+\frac{1}{2}+\frac{m}{2}}}{\sqrt{r^{m+1}-b_0}}, \tag{11}$$

where C is a constant of integration and for the sake of convenience we fix $C = 1$. Integration once more leads to

$$f(r) = \ln\left[\frac{2r^{\frac{1}{2}(3+m+2\alpha)}\sqrt{1-\frac{r^{1+m}}{b_0}}\,_2F_1\left(\frac{3+m+2\alpha}{2+2m}, \frac{1}{2}, \frac{5+3m+2\alpha}{2+2m}, \frac{r^{1+m}}{b_0}\right)}{\sqrt{r^{1+m}-b_0}(3+m+2\alpha)}\right]. \tag{12}$$

Here $_2F_1$ is a hypergeometric function representing a series expression. It should be noted that logarithmic form for $f(r)$ have been obtained for a MT wormhole in $(3+1)$-dimensions as well [10]. Also notice that $f(r)$ does not give a finite value as $r \to \infty$, so the solution is not asymptotically flat. Hence we may match this interior solution to an exterior vacuum spacetime at a junction radius R [17]. Notice that in $(2+1)$ dimensions, the only exterior vacuum solution is the stationary BTZ spacetime [18, 19], given by

$$ds^2 = -\left(-M + \frac{r^2}{l^2}\right)dt^2 + \left(-M + \frac{r^2}{l^2}\right)^{-1}dr^2 + r^2d\phi^2. \tag{13}$$

Here M corresponds to mass of the spherically symmetrical object while $l = 1/\sqrt{-\Lambda} > 0$ and $\Lambda < 0$ is the cosmological constant. In order to match the interior wormhole solution with the exterior BTZ solution, we impose the continuity of the metric coefficients, g_{AB}, across a surface S, i.e.

$$g_{AB}^{\text{int}}|_S = g_{AB}^{\text{ext}}|_S. \tag{14}$$

The wormhole metric is continuous from the throat radius $r = r_0$ to a finite distance $r = R$. Explicitly (14) can be written as

$$g_{00}^{\text{int}}|_S = g_{00}^{\text{ext}}|_S, \tag{15}$$

$$g_{11}^{\text{int}}|_S = g_{11}^{\text{ext}}|_S. \tag{16}$$

Notice that g_{22} is already continuous, so we don't need any matching equation for it. The last two equations yield respectively

$$\frac{2R^{\frac{1}{2}(3+m+2\alpha)}\sqrt{1-\frac{R^{1+m}}{b_0}}\,_2F_1(\frac{3+m+2\alpha}{2+2m},\frac{1}{2},\frac{5+3m+2\alpha}{2+2m},\frac{R^{1+m}}{b_0})}{\sqrt{R^{1+m}-b_0}(3+m+2\alpha)}=-M+\frac{R^2}{l^2}, \qquad (17)$$

$$1-\frac{b_0}{R^{1+m}}=-M+\frac{R^2}{l^2}. \qquad (18)$$

Here M now refers to the mass of wormhole. Using (6) and (12) in (3), we get

$$p=-\frac{b_0 r^{-(3+m)}\sqrt{1-\frac{r^{1+m}}{b_0}}(3+m+2\alpha)}{16\pi\,_2F_1(\frac{3+m+2\alpha}{2+2m},\frac{1}{2},\frac{5+3m+2\alpha}{2+2m},\frac{r^{1+m}}{b_0})}. \qquad (19)$$

Putting (19) in (7), we obtain

$$p_t=-\frac{b_0 r^{-(3+m)}\sqrt{1-\frac{r^{1+m}}{b_0}}\,\alpha(3+m+2\alpha)}{16\pi\,_2F_1(\frac{3+m+2\alpha}{2+2m},\frac{1}{2},\frac{5+3m+2\alpha}{2+2m},\frac{r^{1+m}}{b_0})}. \qquad (20)$$

Alternatively, (20) can be obtained by inserting (6) and (12) in (4). In Figs. 1 and 2, we have plotted the magnitudes of the radial and the transverse pressures against the radial coordinate. These show that both the pressures have arbitrary large values near the throat while these vanish in the asymptotic limit of r. This shows that the matter distribution also satisfies the condition of asymptotic flatness, consistently with the wormhole geometry. The difference of radial and transverse pressures represents surface tension which plays very crucial role in compact stars [20]. Figure 3 shows that the behavior of surface tension is analogous to the two pressures.

Comparison of (8) and (19) yields a relationship between pressure and energy density, given by

$$p=\left[\frac{\sqrt{1-\frac{r^{1+m}}{b_0}}(3+m+2\alpha)}{(m+1)\,_2F_1(\frac{3+m+2\alpha}{2+2m},\frac{1}{2},\frac{5+3m+2\alpha}{2+2m},\frac{r^{1+m}}{b_0})}\right]\rho. \qquad (21)$$

Fig. 1 The radial pressure is plotted against radial coordinate while other parameters are fixed at $b_0=-1$, $m=2$ and $\alpha=1,2,3$ (*top to bottom*)
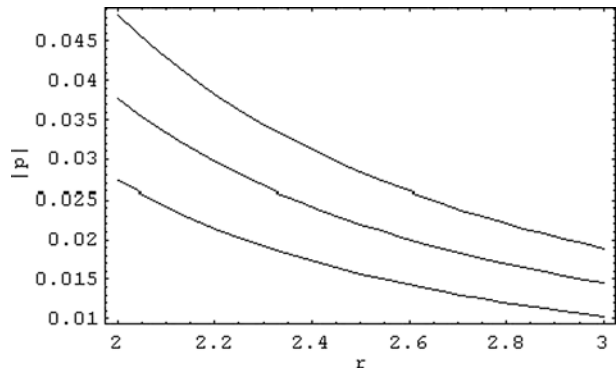
Fig. 2 The transverse pressure is plotted against radial coordinate while other parameters are fixed at $b_0 = -1$, $m = 2$ and $\alpha = 1, 2, 3$ (*top to bottom*)
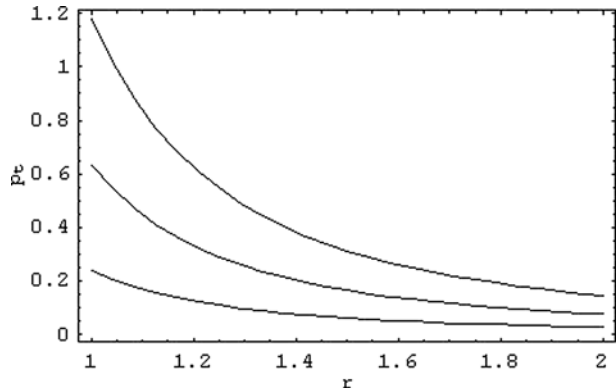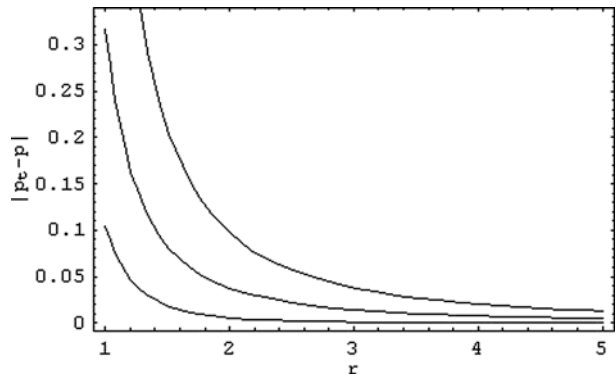

Fig. 3 The surface tension is plotted against radial coordinate while other parameters are fixed at $b_0 = -1$, $m = 2$ and $\alpha = -3, 2, 3$ (*top to bottom*)

On comparing (21) with $p = \omega\rho$, we get

$$\omega(r) = \frac{\sqrt{1 - \frac{r^{1+m}}{b_0}}(3 + m + 2\alpha)}{(m+1)_2F_1(\frac{3+m+2\alpha}{2+2m}, \frac{1}{2}, \frac{5+3m+2\alpha}{2+2m}, \frac{r^{1+m}}{b_0})}. \tag{22}$$

It shows that the wormhole under consideration satisfies a variable equation of state. The variable EoS arises naturally while solving the field equations for the wormhole. Interestingly a variable EoS unifies various forms of dark energy including phantom energy and Chaplygin gas, both of which support the wormhole spacetime [21]. The behavior of $\omega(r)$ is given in Fig. 4 and it shows that the EoS parameter ω has to be negative to model a wormhole. It naturally yields negative radial pressure and positive energy density.

The case of isotropic pressure $p = p_t$ is obtained by fixing $\alpha = 1$. We have

$$p = p_t = \left[\frac{\sqrt{1 - \frac{r^{1+m}}{b_0}}(5 + m)}{(m+1)_2F_1(\frac{5+m}{2+2m}, \frac{1}{2}, \frac{7+3m}{2+2m}, \frac{r^{1+m}}{b_0})}\right]\rho. \tag{23}$$

Similarly, the EoS parameter ω becomes

$$\omega(r) = \frac{\sqrt{1 - \frac{r^{1+m}}{b_0}}(5 + m)}{(m+1)_2F_1(\frac{5+m}{2+2m}, \frac{1}{2}, \frac{7+3m}{2+2m}, \frac{r^{1+m}}{b_0})}. \tag{24}$$

Fig. 4 The equation of state parameter ω is plotted against radial coordinate while other parameters are fixed at $b_0 = -1$, $m = 2$ and $\alpha = 1, 2, 3$ (*top to bottom*)
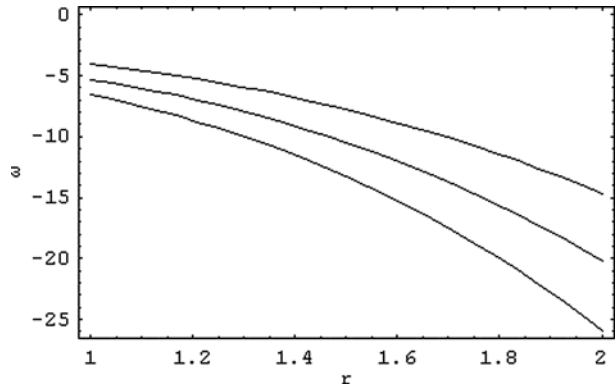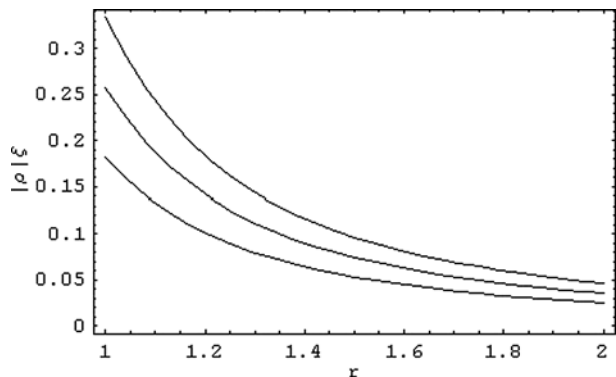

Fig. 5 The exoticity $|\rho|\xi$ is plotted against radial coordinate while other parameters are fixed at $b_0 = -1$, $m = 2$ and $\alpha = 1, 2, 3$ (*top to bottom*)

We would also comment that the case of vanishing pressure $p = 0$ (dust) is not allowed in the present formalism since it will make $f(r)$ unbounded. The dust cases in the framework of braneworld wormholes are investigated in [22, 23].

The specific dimensionless parameter ξ, defined by $\xi = (p - \rho)/|\rho|$, characterizes how the exotic or normal matters are distributed around the wormhole's throat [4, 15]. The *exoticity* at or near the throat of the wormhole is required to be non-negative, $\xi > 0$. The positivity of the exoticity ensures that wormhole will satisfy the flare-out condition as well. From (15), the exoticity becomes

$$|\rho|\xi = -\frac{b_0(m+1)}{16\pi r^{m+3}} \left[\frac{\sqrt{1 - \frac{r^{1+m}}{b_0}}(3 + m + 2\alpha)}{(m+1)_2 F_1(\frac{3+m+2\alpha}{2+2m}, \frac{1}{2}, \frac{5+3m+2\alpha}{2+2m}, \frac{r^{1+m}}{b_0})} - 1 \right]. \qquad (25)$$

In Fig. 5, we have plotted the exoticity against the radial coordinate. It shows that the exoticity remains positive while it converges to zero for large r. Hence the wormhole is surrounded by the exotic phantom energy, right from its throat to a sufficiently large distance. This also suggests that we can construct a wormhole with a sufficiently large radius that could be traversable for two dimensional beings.

In summary, our objective in this article has been to present a mathematical prescription for obtaining a wormhole in low dimensional spacetime. The wormhole is supported by an external source of phantom energy which is anistropically distributed. The wormhole spacetime satisfies a variable equation of state, which is in good agreement with earlier

models available in the literature. It is also shown that the otherwise asymptotically non-flat wormhole could be converted to an asymptotically flat one by matching the various components of its metric with the exterior BTZ spacetime.

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