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Quantum Randomness as a Result of Random Fluctuations at the Planck Time Scale?

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Abstract We show that the mathematical formalism of the quantum statistical model can be interpreted as a method for approximation of classical (measure-theoretic) averages on the infinite-dimensional phase space. The technique of approximation is based on the Taylor expansion of functionals of classical fields. To find the order of the deviation of quantum statistical predictions from the classical predictions, we use the time-scaling arguments. We show that quantum randomness might be considered as the result of random fluctuations at the Planck time-scale.

Keywords Quantum and classical statistical models · Quantum and classical averages · Approximation · Taylor expansion · Infinite-dimensional phase space · Planck time scale · Random fluctuations · Wiener process with values in Hilbert space

1 Introduction

In Khrennikov [12–15] it was shown that the mathematical formalism of the quantum statistical model can be interpreted as a method for approximation of averages given by classical statistical mechanics on the *infinite dimensional phase space*. Such an approximation is based on the asymptotic expansion of classical statistical averages with respect to some parameter κ. (One of the aims of this paper is to provide a reasonable physical interpretation of this parameter.)

By representing points of the phase space by classical vector fields, $\psi(x) = (q(x), p(x))$, we can interpret our prequantum theory as a field theory, *prequantum classical statistical field theory—PCSFT*. In the present paper we explore the probabilistic approach to the quantum approximation of PCSFT. In Sect. 3 we show that the standard method of approximation of averages by using the Taylor formula can be interpreted as approximating of classical measure-theoretic averages by quantum averages (given by the von Neumann trace

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formula) for a finite-dimensional state-space. In Sect. 5 we generalize this method to the infinite-dimensional case—to the Hilbert state-space.

As was already mentioned, the infinite-dimensional phase space can be represented as the space of classical fields. The PCSFT can be interpreted as a special model with hidden variables given by the classical random field, cf. with the well known models [1, 3, 4, 6]. The author was also strongly influenced by works of G. 't Hooft [19, 20] (as well as a series of conversations with him). The PCSFT might be considered as one of (possible) realizations of "Einstein's dream" about a purely classical field model of physical reality [7].

In the present paper we consider the parameter κ (the basis of the asymptotic expansion of classical averages) as the *time scaling parameter*. We remark that this paper is about *non-relativistic theory*. We consider only conventional non relativistic QM, see Dirac [5] or von Neumann [22] and, consequently, non-relativistic prequantum models.

2 Classical and Quantum Statistical Models

A *classical statistical model* is defined in the following way:

(a) States ψ are represented by points of some set Ω (state space).
(b) Variables are represented by functions $f : \Omega \to \mathbf{R}$ belonging to some functional space $V(\Omega)$.[1]
(c) Statistical states are represented by probability measures on Ω belonging to some class $S(\Omega)$.[2]
(d) The average of a variable which is represented by a function $f \in V(\Omega)$ with respect to a statistical state which is represented by a probability measure $\mu \in S(\Omega)$ is given by

$$\langle f \rangle_\rho \equiv \int_\Omega f(\psi) d\mu(\psi). \tag{1}$$

A *classical statistical model* is a pair $M = (S(\Omega), V(\Omega))$.

We recall that classical statistical mechanics on the phase space Ω_{2n} gives an example of a classical statistical model. But we shall not be interested in this example in our further considerations. To clarify coupling between the classical and quantum statistical models, we shall use a classical statistical model with *the infinite-dimensional phase-space*.

A *quantum statistical model*, see e.g. Holevo [8], in a complex Hilbert space Ω_c is described in the following way:

(a) Observables are represented by operators $A : \Omega_c \to \Omega_c$ belonging to the class of continuous[3] self-adjoint operators $\mathcal{L}_s \equiv \mathcal{L}_s(\Omega_c)$.
(b) Statistical states are represented by density operators, see von Neumann [22] or Holevo [8]. The class of such operators is denoted by $\mathcal{D} \equiv \mathcal{D}(\Omega_c)$.

[1] The choice of a concrete functional space $V(\Omega)$ depends on various physical and mathematical factors.

[2] The choice of a concrete space of probability measures $S(\Omega)$ depends on various physical and mathematical factors.

[3] To simplify considerations, we shall consider only observables represented by bounded operators. To obtain the general quantum statistical model with observables represented by unbounded operators, we should consider a prequantum classical statistical model based on the rigged Hilbert space: $\Omega_c^+ \subset \Omega_c \subset \Omega_c^-$.

(c) The average of an observable which is represented by the operator $A \in \mathcal{L}_s(\Omega_c)$ with respect to a statistical state which is represented by the density operator $D \in \mathcal{D}(\Omega_c)$ is given by von Neumann's formula:

$$\langle A \rangle_D \equiv \mathrm{Tr}\, DA. \tag{2}$$

The *quantum statistical model* is the pair $N_{\mathrm{quant}} = (\mathcal{D}(\Omega_c), \mathcal{L}_s(\Omega_c))$.

3 Mean Value of a Function of a Random Variable

Here we follow Chap. 11 of the book of Ventzel [21]. This book was the basic book for teaching probability theory in Soviet military colleges.[4] Elena Ventzel wrote her book in the form of precise instructions of what a student should follow to solve a problem:

"In practice we have very often situations in that, although investigated function of random arguments is not strictly linear, but it differs practically so negligibly from a linear function that it can be approximately considered as linear. This is a consequence of the fact that in many problems fluctuations of random variables play the role of small deviations from the basic law. Since such deviations are relatively small, functions which are not linear in the whole range of variation of their arguments are *almost linear* in a restricted range of their random changes" [21, p. 238].

Let $y = f(x)$. Here in general f is not linear, but it does not differ so much from linear on some interval $[m_x - \delta, m_x + \delta]$, where $x = x(\omega)$ is a random variable and

$$m_x \equiv Ex = \int x(\omega)\, d\mathbf{P}(\omega)$$

is its average. Here $\delta > 0$ is sufficiently small. A student of a military college should approximate f by using the first order Taylor expansion at the point m_x:

$$y(\omega) \approx f(m_x) + f'(m_x)(x(\omega) - m_x). \tag{3}$$

By taking the average of both sides the student would obtain:

$$m_y \approx f(m_x). \tag{4}$$

The crucial point is that the linear term $f'(m_x)(x(\omega) - m_x)$ does not give any contribution! We remark that the approximate formula (4) was first discovered by Gauss and in the probabilistic literature it is sometimes called the Gaussian formula for averages.

Further Elena Ventzel pointed out (see [21, p. 245]): "For some problems the above linearization procedure may be unjustified, because the method of finalization may be not produce a sufficiently good approximation. In such cases to test the applicability of the finalization method and to improve results there can be applied the method which is based on preserving not only the linear term in the expansion of function, but also some terms of higher orders."

[4] I am thankful to my farther-in-law, Alexander Choustov (marine artillery officer) who pointed out this chapter to me.

A student now should preserve the first three terms in the expansion of f in the Taylor series at the point m_x:

$$y(\omega) \approx f(m_x) + f'(m_x)(x(\omega) - m_x) + \frac{1}{2} f''(m_x)(x(\omega) - m_x)^2. \tag{5}$$

Hence

$$m_y \approx f(m_x) + \frac{\sigma_x^2}{2} f''(m_x), \tag{6}$$

where

$$\sigma_x^2 = E(x - m_x)^2 = \int (x(\omega) - m_x)^2 \, d\mathbf{P}(\omega)$$

is the dispersion of the random variable x.

Let us now consider the special case of symmetric fluctuations:

$$m_x = 0$$

and let us restrict considerations to functions f such that

$$f(0) = 0.$$

Then we obtain the following special form of (6):

$$m_y \approx \frac{\sigma_x^2}{2} f''(0). \tag{7}$$

We emphasize again that the first derivative does not give any contribution to the average.

Thus at the some level of approximation we can calculate averages not by using the Lebesgue integral (as we do in classical probability theory), but by finding the second derivative. Such a "calculus of probability" would match well with experiment. I hope that reader has already found analogy with the quantum calculus of probabilities. But for a better expression of this analogy we shall also study the multi-dimensional case. Consider a system of n random variables

$$x = (x_1, \ldots, x_n).$$

We consider vector average:

$$m_x = (m_{x_1}, \ldots, m_{x_n})$$

and the covariance matrix:

$$D_x = (D_x^{ij}), \; D_x^{ij} = E[(x_i - m_{x_i})(x_j - m_{x_j})].$$

We now consider the random variable $y(\omega) = f(x_1(\omega), \ldots, x_n(\omega))$. By using the Taylor expansion we would like to obtain an algorithm for approximation of the average m_y. We start directly from the second order Taylor expansion valid for any function f in $C^3(\mathbf{R})$:

$$y(\omega) \approx f(m_{x_1}, \ldots, m_{x_n}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(m_{x_1}, \ldots, m_{x_n})(x_i(\omega) - m_{x_i})$$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(m_{x_1}, \ldots, m_{x_n})(x_i(\omega) - m_{x_i})(x_j(\omega) - m_{x_j}), \tag{8}$$

and hence:

$$m_y \approx f(m_{x_1}, \ldots, m_{x_1}) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(m_{x_1}, \ldots, m_{x_1}) D_x^{ij}. \tag{9}$$

By using vector notation, we can rewrite the previous formulas as:

$$y(\omega) \approx f(m_x) + (f'(m_x), x(\omega) - m_x) + \frac{1}{2}(f''(m_x)(x(\omega) - m_x), x(\omega) - m_x). \tag{10}$$

and

$$m_y \approx f(m_x) + \frac{1}{2}\,\mathrm{Tr}\,\mathrm{D_x f''(m_x)}. \tag{11}$$

Let us again consider the special case: $m_x = 0$ and $f(0) = 0$. We have:

$$m_y \approx \frac{1}{2}\,\mathrm{Tr}\,\mathrm{D_x f''(0)}. \tag{12}$$

We now remark that the Hessian $f''(0)$ is *always a symmetric operator*. Let us now represent f by its second derivative at zero:

$$f \to A = \frac{1}{2} f''(0).$$

Then we see that, at some level of approximation, instead of operation with Lebesgue integrals, one can use linear algebra:

$$m_y \approx \mathrm{Tr}\,\mathrm{D_x A}. \tag{13}$$

4 Classical Statistical Model with Infinite-Dimensional Phase Space

We choose the space

$$\Omega = Q \times P,$$

where $Q = P = H$ and H is the infinite-dimensional real (separable) Hilbert space. We consider Ω as the real Hilbert space with the scalar product

$$(\psi_1, \psi_2) = (q_1, q_2) + (p_1, p_2)$$

for $\psi_i = (q_i, p_i)$. We denote by J the symplectic operator on Ω

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

making Ω a phase space. Let us consider the class $\mathcal{L}_{\mathrm{symp}}(\Omega)$ of bounded $\mathbf{R}$-linear operators $A : \Omega \to \Omega$ which commute with the symplectic operator: $\mathrm{A\,J = J\,A}$. This is a sub-algebra of the algebra of bounded linear operators $\mathcal{L}(\Omega)$ We also consider the space of $\mathcal{L}_{\mathrm{symp},s}(\Omega)$ consisting of self-adjoint operators.

By using the operator J we can introduce on the phase space Ω a complex structure. A real vector ψ with the coordinates q, p is represented by the complex vector $q + ip$. We shall denote it by the same symbol ψ. We denote Ω endowed with this complex structure

by $\Omega_c : \Omega_c \equiv Q \oplus iP$. This is a complex linear space. The operation of multiplication by the complex scalar i is well defined. The map $\psi \to i\psi$ is the complex image of the operator $-J$. We shall use the complex space Ω_c later.

At the moment consider Ω as a real linear space and consider its complexification $\Omega^{\mathbf{C}} = \Omega \oplus i\Omega$.

Let us consider the functional space $\mathcal{V}_{\mathrm{symp}}(\Omega)$ consisting of functions $f : \Omega \to \mathbf{R}$ such that:

(a) the zero vector is preserved:

$$f(0) = 0;$$

(b) f is J-invariant: $f(J\psi) = f(\psi)$;

(c) f can be extended to the analytic function $f : \Omega^{\mathbf{C}} \to \mathbf{C}$ having exponential growth: $|f(\psi)| \leq c_f e^{r_f \|\psi\|}$ for some $c_f, r_f \geq 0$ and for all $\psi \in \Omega^{\mathbf{C}}$.

The following mathematical result plays the fundamental role in establishing the correspondence between classical and quantum averages:

Proposition 1 *Let f be a smooth J-invariant function. Then*

$$f''(0) \in \mathcal{L}_{\mathrm{symp},s}(\Omega).$$

We now consider the complex realization Ω_c of the phase space and the corresponding complex scalar product $\langle \cdot, \cdot \rangle$. We remark that the class of operators $\mathcal{L}_{\mathrm{symp}}(\Omega)$ is mapped onto the class of $\mathbf{C}$-linear operators $\mathcal{L}(\Omega_c)$. Thus consideration of operators of the class $\mathcal{L}_{\mathrm{symp}}(\Omega)$ in the real model is equivalent to consideration of operators of the class $\mathcal{L}(\Omega_c)$ in the complex model. Our viewpoint of the complex structure of QM is very pragmatical. This is nothing else than the complex representation of the phase space structure.

We call classical statistical mechanics on the phase space Ω with variables of the class $\mathcal{V}_{\mathrm{symp}}(\Omega)$ and statistical states given by Gaussian measures with covariance operators belonging $\mathcal{L}_{\mathrm{symp},s}(\Omega)$ *prequantum classical statistical field theory*, PCSFT.

5 Time-Scaling of Random Fluctuations

Let $w_s^D, s \geq 0$, be the Ω-valued Wiener process[5] corresponding to the trace class (self-adjoint) operator $D \geq 0$ with $\mathrm{Tr}\, D = 1$. As usual, we interpret s as a time parameter. Trajectories $s \to w_s^D(\omega)$ of the Wiener process describes stochastic dynamics in the Hilbert space.

The ordinary Wiener process taking values in $\mathbf{R}^3$ describes the Brownian motion. As we know, Einstein and Smoluchowski observed that by the kinetic theory of fluids the molecules of water move at random. Therefore, a small particle in water would receive a random number of impacts of random strength and from random directions in any short period of time. This random bombardment by the molecules of the fluid would cause a sufficiently small particle to move in exactly the way described by Brown. In our model the role of a particle is played by a classical field $\psi = (q, p)$. It is assumed that this field moves in

[5]The reader who is not familiar with theory of infinite-dimensional stochastic processes can either study some literature, e.g. Daletski and Fomin [2] (see Khrennikov [9–11] for related results) or just consider the finite dimensional phase space $\Omega_{2n} = \mathbf{R}^{2n}$.

a random background field, cf. SED and stochastic mechanics models—De la Pena and Cetto [6], Nelson [16] or Davidson [3]. Hence a trajectory $s \to w_s^D(\omega)$ can be interpreted as the results of "collissions" of the field ψ with the random background field. The time scale s under consideration can be called a prequantum time scale. Here by "quantum time scale" we mean the time scale of quantum observations, see Sect. 6. Thus our meaning of the term "quantum time scale" should be distinguished from a rather common use the terminology "quantum time scale" for the Planck time scale. We shall consider the Planck time scale as one of possible candidates to be chosen as the prequantum time scale, see Sect. 6.

We also assume that the covariance operator D determining the Wiener process belongs to the class $\mathcal{L}_{\mathrm{symp}}(\Omega)$. Thus we have:

$$E[\langle \phi, w_s^D \rangle] = 0, \quad \phi \in \Omega, \tag{14}$$

$$E[\langle \phi_1, w_s^D \rangle \langle w_s^D, \phi_2 \rangle] = s\langle D\phi_1, \phi_2 \rangle, \quad \phi_1, \phi_2 \in \Omega. \tag{15}$$

The following time scaling law for the Wiener process is well known, see e.g. Shiryaev [18]:

$$\text{Prob. law } (w_{\kappa s}^D : s \geq 0) = \text{Prob. law } (\kappa^{1/2} w_s^D : s \geq 0) \tag{16}$$

for any $\kappa > 0$. We shall see that by (16) our $\kappa^{1/2}$-scaling of $\psi \in \Omega$ can be considered as the result of κ-scaling of time.

Our basic postulate is that *quantum formalism arises as the result of an approximation based on the time scaling*.

Let us consider a "*prequantum time scale*" that is essentially finer[6] than the quantum time scale. We suppose that these two time scales can be coupled through a scaling parameter κ. We denote the prequantum and quantum times by symbols s and t respectively. We suppose that:

$$t = \kappa s. \tag{17}$$

Here κ is a dimensionless parameter. It is assumed that

$$\kappa \ll 1. \tag{18}$$

Thus the unit interval of the prequantum time corresponds to the interval $t = \kappa$ of the quantum time. We can also say that the unit interval $t = 1$ of quantum time corresponds to a huge interval $s = \frac{1}{\kappa}$ of the prequantum time. Moreover, if $\kappa \to 0$, then $s = \frac{1}{\kappa} \to \infty$. At the prequantum time scale quantum processes have practically infinite duration.

In such a model a quantum measurement has a huge duration with respect to the prequantum time scale. In particular "collapse of the wave function" is a very long process in the prequantum world.

Let us consider the time scaling (17) for the Wiener processes w_s^D. We set

$$W_t^D = w_{\kappa s}^D.$$

The formula (16) implies that, for any continuous function $f : \Omega \to \mathcal{P}$ (which is integrable with respect to any Gaussian measure on Ω), we have:

$$Ef(W_\kappa^D) = Ef(\kappa^{1/2} w_1^D). \tag{19}$$

[6]The meaning of "essentially" would be discussed later.

This is nothing else than the basic "field-scaling" formula which was used in Khrennikov [12–15]. We interpret W_t^D as the Wiener process with respect to the quantum time t and w_s^D as the Wiener process with respect to the prequantum time s.

By our interpretation the quantum formalism does not provide a possibility to find exactly the average $Ef(W_t^D)$ with respect to the "quantum Wiener process" W_t^D. The main problem is that the interval $t = \kappa$ is negligible compare with natural quantum scales of time. The quantum formalism provides only an approximation of the classical average $Ef(W_\kappa^D)$ (it is again only our special interpretation of QM).

Moreover, to produce observable effects, the classical physical variable f should be strongly amplified:

$$f \to f_\kappa \equiv \frac{1}{\kappa} f.$$

We can proceed through expanding the right-hand side of (19) into the Taylor series with respect to the scaling parameter $\kappa^{1/2}$, cf. with Sect. 3. We obtain the following result on the asymptotic expansion:

Theorem 1 *Let $f \in \mathcal{V}_{\rm symp}(\Omega.)$ Then*:

$$Ef_\kappa(W_\kappa^D) = Ef_\kappa(\kappa^{1/2} w_1^D) = \frac{1}{2} E(f''(0) w_1^D, w_1^D) + O(\kappa), \quad \kappa \to 0. \tag{20}$$

Thus for nonquadratic maps $f : \Omega \to \mathbf{R}$, the quantum statistical model gives only an approximation $\langle f \rangle_D = \mathrm{Tr}\, Df''(0)$ of the classical average $Ef_\kappa(W_\kappa^D)$.

The difference between statistical predictions of the quantum statistical model and PCSFT is of the magnitude κ, where κ is the scaling parameter for the prequantum and quantum time scales, see (17). What is a magnitude of the time scaling factor κ? We shall come back to this problem in Sect. 6.

Thus by taking into account Brownian fluctuations at the prequantum time scale we can say that prequantum statistical states are given by Wiener measures P_D on the space $C_0([0,\kappa])$ of trajectories $\psi : [0,\kappa] \to \Omega$, $\psi(0) = 0$. Denote the space of such Wiener measures by the symbol $S_{G,\rm symp}(C_0([0,\kappa]), \Omega)$ (we recall that $[D, J] = 0$).

This is the space of statistical states of our prequantum classical statistical model. As the space of classical physical variables, we should choose some subspace of the space of continuous functionals $f : C_0([0,\kappa], \Omega) \to \mathbf{R}$.

Since all our considerations are coupled to the fixed moment of (quantum) time $t = \kappa$, we can restrict them to the class of functionals which depend only on $\psi(\kappa)$. So we can choose the space of classical physical variables consisting of functionals of trajectories, $\psi : [0,\kappa] \to \Omega$, of the form $\psi(\cdot) \to f(\psi(\kappa))$, $f \in \mathcal{V}_{\rm symp}(\Omega)$ We denote this class by the symbol $\mathcal{V}_{\rm symp}(C_0([0,\kappa], \Omega))$.

Thus, finally, we consider the following classical statistical model on phase space consisting of trajectories $\tilde{\Omega}^\kappa = C_0([0,\kappa], \Omega)$:

$$\tilde{M}^\kappa = (S_{G,\rm symp}(\tilde{\Omega}^\kappa), \mathcal{V}_{\rm symp}(\tilde{\Omega}^\kappa)).$$

We define the maps T mapping the classical model into the quantum model:

$$T : S_{G,\rm symp}(\tilde{\Omega}^\kappa) \to \mathcal{D}(\Omega_c), \qquad T(P_D) = D; \tag{21}$$

$$T : \mathcal{V}_{\rm symp}(\tilde{\Omega}^\kappa) \to \mathcal{L}_s(\Omega_c), \qquad T(f) = f''(0)/2. \tag{22}$$

Theorem 2 *Both maps* (21) *and* (22) *are surjections. The map* (21) *is even an injection. However, the map* (22) *is not one-to-one. The latter map is* **R***-linear.*

We can say that the family of classical statistical models $\tilde{M}^{\kappa}$, $\kappa > 0$, and the pair of maps (21), (22) provide "dequantization" of the conventional quantum statistical model $N_{\text{quant}} = (\mathcal{D}(\Omega_c), \mathcal{L}_s(\Omega_c))$, see Sect. 2.

6 The Magnitude of Time-Scaling

To get the basic parameter of our model κ, we should choose quantum and prequantum time scales. There are a few different possibilities and we shall discuss one of them. We choose the *atom time*-scale in QM and the *Planck time*-scale in the prequantum classical theory.

We recall that Max Planck first listed his set of units and gave values for them remarkably close to those used today, at the time, see Planck [17], when QM had not been invented. He had not yet discovered the theory of black-body radiation.

We assume that the prequantum time scale is based on the Planck time:

$$t_{\text{prq}} = t_P = \sqrt{\frac{\hbar G}{c^5}} \approx 5.391 \times 10^{-44} \text{ s}. \tag{23}$$

We remark that the Planck time t_P is expressed as

$$t_P = \frac{\hbar}{c^2 M_P}, \tag{24}$$

where the Planck mass is given by

$$m_P = \sqrt{\frac{\hbar c}{G}} \approx 2.176 \times 10^{-8} \text{ kg}.$$

We recall that, in contrast to the Planck time as well as the Planck length, the Planck mass is a *macroscopic quantity*. It is often considered as a pathological feature of the Planck systems of units. We shall see that in our approach the mass scale of a prequantum model should really be macroscopic. It would imply that the difference between statistical predictions of quantum and prequantum models would be very small.

To obtain the atom time-scale we choose the electron mass scale, $m_e \approx 9.109 \times 10^{-31}$ kg. The corresponding time scale can be set in the same way as it was done by Max Planck:

$$t_{\text{q}} = t_e = \frac{\hbar}{c^2 m_e} = 1.288 \times 10^{-21}. \tag{25}$$

Therefore our time scaling parameter

$$\kappa = \frac{t_{\text{prq}}}{t_q} = \frac{t_P}{t_e} \approx 4.185 \times 10^{-23}. \tag{26}$$

We also remark that

$$\kappa = \frac{m_e}{m_P}. \tag{27}$$

Thus our time-scaling parameter has the magnitude:

$$\kappa \sim 10^{-23}.$$

Under such a choice of the prequantum scale the difference between statistical predictions of PCSFT and the quantum statistical model (given by (20)) is of the order 10^{-23}. Thus, if, e.g., the classical physical variable

$$f(\psi) = \frac{1}{2}\langle A\psi, \psi\rangle + \frac{1}{4}\langle A\psi, \psi\rangle^2, \quad A \in \mathcal{L}_s,$$

then the difference between the quantum prediction ($\langle A\rangle_D = \mathrm{Tr}\,DA$) and the PCSFT-prediction should be of the order 10^{-23}.

Of course, all previous consideration have sense only under the assumption that the Planck time t_P really provides the correct prequantum time-scale. As was already mentioned, at the moment we cannot justify this assumption. The main problem induced by such a choice is that there is a huge gap between the atomic and Planck scales. It might be found another natural scale between the Planck and atomic scales. In such a case κ would be larger. This would simplify the experimental verification of PCSFT. On the other hand, using the Planck scale (and hence the macroscopic mass scale) and the corresponding parameter $\kappa \equiv \kappa_e \sim 10^{-23}$ clarify why predictions of the quantum statistical model have not yet been violated—the deviation is really negligibly small.

One of the reasons in favor of the Planck scale as the scale of prequantum fluctuations is that the Planck mass is of the macroscopic magnitude.

Let a system have the mass m. Then by choosing the corresponding time scale

$$t_m = \frac{\hbar}{c^2 m},$$

we obtain

$$\kappa = \frac{m}{m_P}.$$

Therefore predictions of the quantum statistical model should be violated for systems of macroscopic mass. In principle, one may expect that it would be easier to produce deviations from QM for heavy elementary particles, e.g., muons. Let take $m = m_{\mathrm{muon}}$. The corresponding time scale

$$\kappa_{\mathrm{muon}} = \kappa_e \frac{m_{\mathrm{muon}}}{m_e} \approx 207\kappa_e.$$

Thus statistical deviations for muons are essentially larger than for electrons, but they are still very small

$$\kappa_{\mathrm{muon}}\kappa_e \sim 10^{-21}.$$

On the other hand, decreasing of the mass increases strongly the precision of quantum approximation. For electron neutrino and antineutrino

$$\kappa_{\text{e-neutrino}} = \kappa_e \frac{m_{\text{e-neutrino}}}{m_e} < 4.31 \times 10^{-7}\kappa_e \sim 10^{-30}.$$

It is impossible to interpolate our theory directly to photons, since we considered nonrelativistic QM. The direct interpolation would give us

$$\kappa_{\mathrm{photon}} = 0.$$

Thus it would imply that the quantum statistical model is precise for photons. However, as was already mentioned, such an interpolation is too straightforward.

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