International Journal of Theoretical Physics, Vol. 46, No. 6, June 2007 (© 2007)
DOI: 10.1007/s10773-006-9284-z

Characteristics of Quantum Radiation as Tunneling from a Cylindrically Symmetric Black Hole

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Received August 5, 2006; accepted September 26, 2006
Published Online: December 12, 2006

Applying the semi-classical quantum tunneling model, we have studied the Hawking radiation via tunneling from a cylindrically symmetric black hole. The derived results show that the tunneling rate of at the event horizon of the black hole is related to Bekenstein–Hawking entropy and the factual radiation spectrum is not strictly pure thermal, but is consistent with the underlying unitary theory.

KEY WORDS: cylindrically symmetric black hole; Bekenstein–Hawking entropy; tunneling rate; underlying unitary theory.

PACS numbers: 04.20.−s, 97.60.Lf.

1. INTRODUCTION

At the beginning of the 1970s, Hawking theoretically proved the thermal radiation of black hole (Hawking, 1975), which is greatly meaningful to research the evolution of fixed stars. Since then, black hole's thermodynamic characteristics have been studied by lots of researchers (Jiang *et al.*, 2005; Li *et al.*, 2005; Yang and Lin, 2004). There are two methods to research Hawking radiation, namely quantum field theory method and Damour–Ruffini method (Damour and Ruffini, 1976; Sannan, 1988). And the common point between these two methods is that the background space-time of black hole is fixed and that the obtained radiation spectrum is precisely thermal. In fact, the mass of black hole will be changed after considering self-gravitation interaction and energy conservation during the

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evaporation, and accordingly the position of the event horizon of black hole also varies before and after the particle emit out.

Recently, Parikh and Wilczek put forward a semi-classical tunneling model that was developed by Kraus and Wilczek, and carried out research on spherically symmetric black holes like Schwarzschild and Reissner-Nordström (Parikh, 2004; Parikh and Wilczek, 2000). Following that, Hemming and Keski-Vakkuri have investigated the Hawking radiation via tunneling from Anti-de Sitter black hole and Medved has researched those from a de Sitter cosmological horizon (Hemming and Keski-Vakkuri, 2000; Medved, 2002). The results both support the Parikh's opinion. After then, Zhang and Jiang have extended the Parikh's quantum tunneling method to discuss the tunneling radiation characteristics of the axi-symmetric black hole and the new forms of Kerr and Kerr-Newman black holes (Jiang *et al.*, 2006; Jiang and Wu, 2006; Zhang and Zhao, 2005a,b,c), the derived results have supported the Parikh's opinion, namely the true Hawking radiation spectrum of black holes are not pure thermal when the self-gravitation are taken into account, which extends our understanding on quantum tunnel effect of black holes.

However, up to now the tunneling radiation characteristics of a cylindrically symmetric black hole has not been studied. Lemos, Zanchin and Andrew have derived the cylindrically symmetric black hole solutions of Einstein's field equations with a negative cosmological constant in recent years. Cylindrical symmetric black hole plays an important role in the Anti-de Sitter conformal field theory(CFT) conjecture and the context of brane-word scenarios based on the set up of Randall and Sundrum because of the Anti-de Sitter space- time.

In this paper, we extend Parikh's semi-classical quantum tunneling method to research the cylindrically symmetric black hole in Anti-de Sitter space-time. Different from de Sitter space-time, the Anti-de Sitter-Painleve coordinates should not be the asymptotically flat space-time. The results show that the Hawking radiation spectrum via tunneling from the cylindrically symmetric black hole in Anti-de Sitter space-time is not pure thermal, and is related to Bekenstein-Hawking entropy. That supports the Parikh's opinion and gives a correct amendment to the Hawking pure thermal spectrum.

2. THE EVENT HORIZON AND THE PAINLEVÉ COORDINATE TRANSFORMATION

The space-time line element of a cylindrically symmetric black hole can be written as (Andrew, 1999)

$$ds^2 = -\left(\alpha^2\rho^2 - \frac{2\,(M+\Omega)}{\alpha\rho} + \frac{4Q^2}{\alpha^2\rho^2}\right) dt_c^2 + \left[\alpha^2\rho^2 - \frac{2\,(3\Omega - M)}{\alpha\rho} + \frac{4Q^2\,(3\Omega - M)}{(M+\Omega)\,\alpha^2\rho^2}\right]^{-1} d\rho^2$$

$$
\begin{aligned}
&-\frac{16J}{3\alpha\rho}\left(1-\frac{2Q^2}{(M+\Omega)\,\alpha\rho}\right)dt_c d\varphi \\
&+\left[\rho^2+\frac{4\,(M-\Omega)}{\alpha^3\rho}\left(1-\frac{2Q^2}{(M+\Omega)\,\alpha\rho}\right)\right]d\varphi^2+\alpha^2\rho^2dz^2, \qquad (1)
\end{aligned}
$$

in which t_c represents the time coordinate, and

$$
\Omega=\sqrt{M^2-\frac{8J^2\alpha^2}{9}}, \quad \alpha^2=-\frac{1}{3}\Lambda, \qquad (2)
$$

where M, Q and J are the mass, charge and angular momentum per unit length of the black hole, respectively. Λ is a negative cosmological constant, and it describes asymptotically anti-de Sitter behavior. When the charge and angular momentum are zero, the line element (1) can be written as following

$$
\begin{aligned}
ds^2=&-\left(\alpha^2\rho^2-\frac{4M}{\alpha\rho}\right)dt_c^2 \\
&+\left[\alpha^2\rho^2-\frac{4M}{\alpha\rho}\right]^{-1}d\rho^2+\rho^2d\varphi^2+\alpha^2\rho^2dz^2. \qquad (3)
\end{aligned}
$$

According to the null surface equation $g^{\mu\upsilon}\partial_\mu f\partial_\nu f=0$, we can get the event horizon, namely,

$$
\rho_h=\frac{\sqrt[3]{4M}}{\alpha}, \qquad (4)
$$

and correspondingly the area of the per unit length for the black hole is

$$
A_h=2\pi\alpha\rho^2. \qquad (5)
$$

Obviously, the infinite red-shift surface decided by $g_{00}=0$ coincides with the event horizon, however, there still exists a coordinate singularity that gives us inconvenience to study the tunneling behavior from the event horizon. So it is necessary to eliminate coordinate singularity. Accordingly further making the Painleve coordinate transformation as follows (Zhang and Zhao, 2005a,b,c)

$$
t_c=t+f\,(\rho)\,, \qquad (6)
$$

to Eq. (3) can we get

$$
\begin{aligned}
ds^2=&-\left(\alpha^2\rho^2-\frac{4M}{\alpha\rho}\right)dt^2-2f'(\rho)\left(\alpha^2\rho^2-\frac{4M}{\alpha\rho}\right)dtd\rho+\rho^2d\varphi^2 \\
&-\left[\left(\alpha^2\rho^2-\frac{4M}{\alpha\rho}\right)(f'(\rho))^2-\left(\alpha^2\rho^2-\frac{4M}{\alpha\rho}\right)^{-1}\right]d\rho^2+\alpha^2\rho^2dz^2. \qquad (7)
\end{aligned}
$$

In Anti-de Sitter space-time, according to reference (Hemming and Keski-Vakkuri, 2000), we can order

$$-\left[\left(\alpha^2\rho^2 - \frac{4M}{\alpha\rho}\right)(f'(\rho))^2 - \left(\alpha^2\rho^2 - \frac{4M}{\alpha\rho}\right)^{-1}\right] = \frac{1}{\alpha^2\rho^2}, \tag{8}$$

hence, the Painlevé-cylindrically-symmetric line element can be written as

$$ds^2 = -\left(\alpha^2\rho^2 - \frac{4M}{\alpha\rho}\right)dt^2 + 2\sqrt{\frac{4M}{\alpha^3\rho^3}}dt d\rho$$
$$+ \frac{1}{\alpha^2\rho^2}d\rho^2 + \rho^2 d\varphi^2 + \alpha^2\rho^2 dz^2. \tag{9}$$

Obviously, the Painlevé coordinate system has many attractive features. First, there is no singularity at event horizon. Secondly, the event horizon is in coincidence with the infinite red-shift surface. Finally, the new line element is stationary. All these characteristics provide superior conditions to study the quantum tunneling radiation of the black hole.

3. THE RADIATION CHARACTERISTICS AS TUNNELING

In the following section, we will discuss the Hawking radiation via tunneling from the event horizon of a cylindrically symmetric black hole in Anti-de Sitter space-time. From Eq. (9), the radial null geodesics is

$$\dot{\rho} = \alpha^2\rho^2\left(-\sqrt{\frac{4M}{\alpha^3\rho^3}} \pm 1\right), \tag{10}$$

where the $\pm$ signs correspond to the outgoing and ingoing geodesics at the event horizon. Considering the energy conservation, we fix the total ADM mass of the whole space-time and allow the mass of the black hole to vary. When a particle with the energy ω tunnels out, the line element of the black hole is expressed as

$$ds^2 = -\left(\alpha^2\rho^2 - \frac{4(M-\omega)}{\alpha\rho}\right)dt^2 + 2\sqrt{\frac{4(M-\omega)}{\alpha^3\rho^3}}dt d\rho$$
$$+ \frac{1}{\alpha^2\rho^2}d\rho^2 + \rho^2 d\varphi^2 + \alpha^2\rho^2 dz^2, \tag{11}$$

and correspondingly, Eqs. (4), (5), and (10) should also replace M with $(M-\omega)$. During the tunneling process, we consider the tunneling particle as a shell (a cylinder shell) of energy ω. Since the event horizon coincides with the infinite red-shift surface, the geometrical optics limit is reliable, which permits describing the cylinder-wave at the event horizon by using the language of particles, and rather than having to use the second-quantized method. In the semi-classical limit, applying the WKB approximation, we can obtain the relation between the tunneling rate and the imaginary part of the particle action (Kraus and Keski-Vakkuri, 1997)

$$\Gamma \sim e^{-2\mathrm{Im}S}, \tag{12}$$

where the imaginary part of the particle action is

$$\mathrm{Im}S = \mathrm{Im}\int_{\rho_i}^{\rho_f} P_\rho d\rho = \mathrm{Im}\int_{\rho_i}^{\rho_f}\int_0^{P_\rho} dP'_\rho d\rho, \tag{13}$$

in which, ρ_i and ρ_f are the locations of the event horizon before and after the particle emission respectively, which can be regarded as the two turning points of the potential barrier, and the width between the two points is determined by the energy of the outgoing particle. Applying the Hamilton canonical equation

$$\dot{\rho} = \left.\frac{dH}{dP_\rho}\right|_\rho = \frac{d(M-\omega)}{dP_\rho} = -\frac{d\omega}{dP_\rho}, \tag{14}$$

to Eq. (13), we have

$$\begin{aligned}\mathrm{Im}S &= \mathrm{Im}\int_M^{M-\omega}\int_{\rho_i}^{\rho_f}\frac{d\rho}{\dot{\rho}}dH \\ &= \mathrm{Im}\int_0^{\omega}\int_{\rho_i}^{\rho_f}\frac{d\rho}{\alpha^2\rho^2\left(1-\sqrt{\frac{4(M-\omega')}{\alpha^3\rho^3}}\right)}d(-\omega'),\end{aligned} \tag{15}$$

where $\rho = \rho'_h = \frac{\sqrt[3]{4(M-\omega')}}{\alpha}$ is a pole, the integral can be evaluated by deforming the contour around the pole. Then carrying on integrality on r first and then ω' in turn, we obtain.

$$\mathrm{Im}S = \mathrm{Im}\int_0^{\omega}\frac{-2\pi i}{3\alpha\sqrt[3]{4(M-\omega')}}d(-\omega') = \frac{\pi}{2\alpha}\left[\sqrt[3]{2M^2} - \sqrt[3]{2(M-\omega)^2}\right]. \tag{16}$$

So the tunneling rate from the event horizon is

$$\begin{aligned}\Gamma e^{-2\,\mathrm{Im}\,S} &= \exp\left[\frac{\pi}{\alpha}\left(\sqrt[3]{2(M-\omega)^2} - \sqrt[3]{2M^2}\right)\right] \\ &= e^{S_{BH}(M-\omega)-S_{BH}(M)} = e^{\Delta S_{BH}},\end{aligned} \tag{17}$$

where $\Delta S_{\mathrm{BH}} = S_{\mathrm{BH}}(M-\omega) - S_{\mathrm{BH}}(M)$ is the change of Bekenstein–Hawking entropy. Obviously, the tunneling rate is consistent with the underlying unitary theory in Quantum Mechanics, and the tunneling radiate spectrum is not strictly thermal, but gives a correct modification to the Hawking thermal spectrum.

4. CONCLUSION

We have studied the tunneling radiation characteristics of a cylindrically symmetric black hole in anti-de Sitter space-time. From Eq. (17) we can see clearly that the tunneling rate is relevant to the change of Bekenstein-Hawking entropy at the event horizon of a cylindrically symmetric black hole, which supports the Parikh's opinion. Expending Eq. (17) in ω, namely

$$\Gamma \sim e^{-2\,\mathrm{Im}\,S} \approx \exp\left[-\frac{4\pi}{3\alpha}\left(\frac{\omega}{\sqrt[3]{4M}} + \frac{\omega^2}{6M\sqrt[3]{4M}} + \frac{\omega^3}{27M^2\sqrt[3]{4M}} + \cdots\right)\right], \quad (18)$$

and neglecting the higher-order term of ω, we get

$$\Gamma \approx \exp\left[-\frac{4\pi}{3\alpha\sqrt[3]{4M}}\omega\right] = e^{-\beta\omega}. \quad (19)$$

where $\beta = T_h^{-1}$, and $T_h = \frac{\alpha}{2\pi}\frac{3}{2}(4M)^{1/3}$ is the temperature of the black hole. So only neglecting the higher order terms of ω in Eq. (18) can we obtain the Hawking pure thermal spectrum. The higher order terms caused by the self-gravitation and energy conservation provide a correction to the strictly thermal spectrum.

ACKNOWLEDGMENTS

Great thanks are due to Prof. Jing-Yi Zhang for his helpful discussions. The work is supported by the National Foundation of China under Grant No. 10347008.

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