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# Plane Waves in Generalized Thermo-microstretch Elastic Solid with Thermal Relaxation Using Finite Element Method

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**Abstract** The propagation of plane waves in a thermo-microstretch elastic solid half-space as proposed by Lord–Shulman as well as the classical dynamical coupled theory are discussed. The problem has been solved numerically using a finite element method. Numerical results for the displacement components, force stresses, temperature, couple stresses, and microstress distribution are obtained. The variations of the considered variables through the horizontal distance are given and illustrated graphically. Comparisons are made with the results predicted by the theory of generalized thermoelasticity for different values of the relaxation time.

**Keywords** Dynamic coupled theory · Finite element method · Lord–Shulman theory · Thermo-microstretch elastic solid

## 1 Introduction

In the classical theory of elasticity, the points of the material have translational degrees of freedom and the transmission of the load across a differential element of the surface

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is described by a force vector only. However, in the theory of micropolar elasticity, there is an additional degree of freedom characterized by rotation of material points, and there is an additional kind of stress called couple stress. Thus, in the classical theory of elasticity, the effect of couple stress is neglected. Eringen [1] introduced the theory of microstretch elastic solids. This theory is a generalization of the theory of micropolar elasticity [2]. The material points of microstretch solids can stretch and contract independently of their translations and rotations. Thus, in these solids, the motion is characterized by seven degrees of freedom, namely, three for translation, three for rotation, and one for stretch.

The transmission of the load across a differential element of the surface of a microstretch elastic solid is described by a force vector, a couple stress vector, and a microstretch vector. The theory of a microstretch elastic solid differs from the theory of micropolar elasticity in the sense that there is an additional degree of freedom called stretch and there is an additional kind of stress called the microstretch vector. Materials such as a porous elastic material filled with gas or inviscid fluid, asphalt, composite fibers, etc., lie in the category of microstretch elastic solids. Eringen [3] extended the theory of microstretch elastic solids to include heat conduction. In the framework of the theory of thermo-microstretch elastic solids, Eringen established a uniqueness theorem for the mixed boundary-initial value problem. The theory was illustrated with the solution of a one-dimensional wave and compared with lattice dynamical results. The asymptotic behavior of solutions and an existence result were presented by Bofill and Quintanilla [4]. Iesan and Quintanilla [5] investigated thermal stresses in microstretch elastic plates. Svanadze and De Cicco [6] analyzed a fundamental solution in the theory of thermo-microstretch elastic solids. Iesan and Scalia [7] discussed propagation of singular surfaces in thermo-microstretch continua with memory. Othman and Lotfy [8] studied the effect of rotation on plane waves in a generalized thermo-microstretch elastic solid with one relaxation time. The basic results in the theory of microstretch elastic solids were obtained in the literatures [9–12].

The theory of thermoelasticity (THE) deals with the effect of mechanical and thermal disturbances on an elastic body. The theory of uncoupled THE consists of the heat equation, which is independent of mechanical effects, and the equation of motion, which contains the temperature as a known function. There are two defects in this theory. First is that the mechanical state of the body has no effect on the temperature. Second, the heat equation, which is parabolic, implies that the speed of propagation of the temperature is infinite, which contradicts physical experiments. Biot [13] introduced the theory of coupled THE to overcome the first shortcoming. The governing equations for this theory are coupled, eliminating the first paradox of the classical theory. However, both theories share the second shortcoming as the heat equation for the coupled theory is also parabolic. To overcome this drawback, two generalizations of the coupled theory were introduced. The first is due to Lord and Shulman [14], who obtained a wave-type heat equation by postulating a new law of heat conduction to replace the classical Fourier's law. This new law contains the heat flux vector as well as its time derivative. It also contains a new constant that acts as a relaxation time. As the heat equation of this theory is of the wave-type, it automatically ensures finite speeds of propagation of heat and elastic waves. The remaining governing equations for this theory, namely, the equations of motion and constitutive relations, remain the same as those for the coupled and uncoupled theories.

The second generalization of the coupled theory of elasticity is what is known as the theory of THE with two relaxation times or the theory of temperature-rate-dependent THE. Mullar [15], in a review of the thermodynamics of thermoelastic solids, proposed an entropy production inequality, with the help of which he considered restrictions on a class of constitutive equations. A generalization of this inequality was proposed by Green and Laws [16]. Green and Lindsay [17] obtained another version of the constitutive equations. These equations were also obtained independently and more explicitly by Suhubi [18]. This theory contains two constants that act as relaxation times and modify all the equations of the coupled theory, not only the heat equation. The classical Fourier's law of heat conduction is not violated if the medium under consideration has a center of symmetry. Kumar and Singh [19,20] discussed the problems of wave propagation in a micropolar generalized thermoplastic body with stretch and in a generalized thermo-microstretch elastic solid. Othman [21] studied the Lord–Shulman theory under the dependence of the modulus of elasticity on the reference temperature in two-dimensional generalized THE. Othman and Singh [22] investigated the effect of rotation on generalized micropolar THE for a half-space under five theories. Abbas [23], Youssef and Abbas [24], Abbas and Abd-Alla [25], Abbas [26], and Abbas and Othman [27] applied the finite element method in different generalized thermoelastic problems.

The purpose of this paper is to obtain the normal displacement, temperature, normal force stress, and tangential couple stress in a microstretch elastic solid concerned with the LS and CD theories. The finite element method is used to obtain the expressions for the considered variables. The distributions of the considered variables are represented graphically. A comparison is carried out between the temperature, stresses, and displacements as calculated from the generalized THE Lord-Shulman (LS) and classical dynamical coupled (CD) theories for the propagation of waves in a semi-infinite microstretch elastic solid for different values of the relaxation time.

## 2 Formulation of the Problem

Following Eringen [3] and Lord and Shulman [14], the constitutive equations and field equations for a linear isotropic generalized thermo-microstretch elastic solid in the absence of body forces are obtained. We consider a rectangular coordinate system $(x, y, z)$ originating on the surface $y = 0$ and the $z$-axis pointing vertically into the medium (Fig. 1). The basic governing equations of linear generalized THE with rotation in the absence of body forces and heat sources are

$$(\lambda + \mu)\nabla(\nabla \cdot \vec{u}) + (\mu + k)\nabla^2\vec{u} + k(\nabla \times \vec{\phi}) + \lambda_0\nabla\phi^* - \hat{\gamma}\nabla T = \rho\ddot{\vec{u}}, \quad (1)$$

$$K\nabla^2 T = \left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)\left(\rho C_E T + \hat{\gamma}T_0 e + \hat{\gamma}_1 T_0\phi^*\right), \quad (2)$$

$$(\alpha + \beta + \gamma)\nabla(\nabla \cdot \vec{\phi}) - \gamma\nabla \times (\nabla \times \vec{\phi}) + k(\nabla \times \vec{u}) - 2k\vec{\phi} = j\rho\frac{\partial^2\vec{\phi}}{\partial t^2}, \quad (3)$$

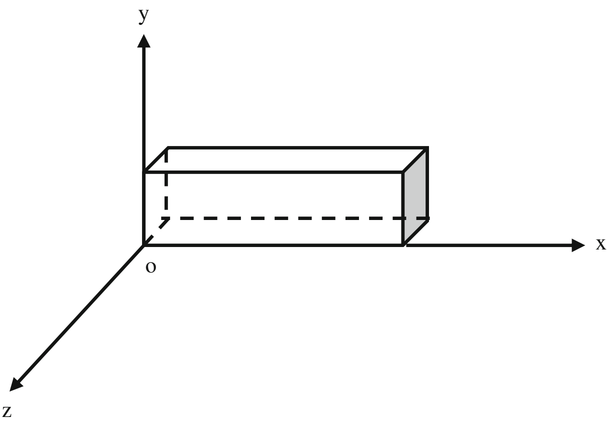


**Fig. 1** Geometry of the problem

$$\alpha_0\nabla^2\phi^* - \lambda_1\phi^* - \lambda_0(\nabla\cdot\vec{u}) + \hat{\gamma}_1 T = \frac{\rho j_0}{2}\frac{\partial^2\phi^*}{\partial t^2}, \tag{4}$$

$$\sigma_{il} = (\lambda_0\phi^* + \lambda u_{r,r})\delta_{il} + (\mu + k)u_{l,i} + \mu u_{i,l} - k\varepsilon_{ilr}\phi_r - \hat{\gamma} T\delta_{il}, \tag{5}$$

$$m_{il} = \alpha\phi_{r,r}\delta_{il} + \beta\phi_{i,l} + \gamma\phi_{l,i}, \tag{6}$$

$$\lambda_i = \alpha_0\phi^*_{,i} + b_0\varepsilon_{ilr}\phi_{l,r}, \tag{7}$$

$$e = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}. \tag{8}$$

The state of plane strain parallel to the *xz*-plane is defined by

$$\begin{aligned} &u_1 = u\,(x,z,t), \quad u_2 = 0, \quad u_3 = w(x,z,t), \quad \phi_1 = \phi_3 = 0,\\ &\phi_2 = \phi_2(x,z,t), \quad \phi^* = \phi^*(x,z,t). \end{aligned} \tag{9}$$

The field Eqs. 1–4 reduce to

$$\begin{aligned} &(\lambda + \mu)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 w}{\partial x\,\partial z}\right) + (\mu + k)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right) - k\frac{\partial\phi_2}{\partial z}\\ &\quad + \lambda_0\frac{\partial\phi^*}{\partial x} - \hat{\gamma}\frac{\partial T}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2}, \end{aligned} \tag{10}$$

$$\begin{aligned} &(\lambda + \mu)\left(\frac{\partial^2 u}{\partial x\,\partial z} + \frac{\partial^2 w}{\partial z^2}\right) + (\mu + k)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) + k\frac{\partial\phi_2}{\partial x}\\ &\quad + \lambda_0\frac{\partial\phi^*}{\partial z} - \hat{\gamma}\frac{\partial T}{\partial z} = \rho\frac{\partial^2 w}{\partial t^2}, \end{aligned} \tag{11}$$

$$K\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right) = \left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)\left(\rho C_E T + \hat{\gamma} T_0 e + \hat{\gamma}_1 T_0\phi^*\right), \tag{12}$$

$$\gamma\left(\frac{\partial^2\phi_2}{\partial x^2} + \frac{\partial^2\phi_2}{\partial z^2}\right) - 2k\phi_2 + k\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = j\rho\frac{\partial^2\phi_2}{\partial t^2}, \tag{13}$$

$$\alpha_0\left(\frac{\partial^2\phi^*}{\partial x^2}+\frac{\partial^2\phi^*}{\partial z^2}\right)-\lambda_1\phi^*-\lambda_0\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)+\hat{\gamma}_1 T=\frac{\rho j_0}{2}\frac{\partial^2\phi^*}{\partial t^2}, \quad (14)$$

where $\hat{\gamma}=(3\lambda+2\mu+k)\alpha_{t_1}$, $\hat{\gamma}_1=(3\lambda+2\mu+k)\alpha_{t_2}$, and $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}$.

The constants $\hat{\gamma}$ and $\hat{\gamma}_1$ depend on mechanical as well as the thermal properties of the body. The constitutive relations can be written as

$$\sigma_{xx}=\lambda_0\phi^*+(\lambda+2\mu+k)\frac{\partial u}{\partial x}+\lambda\frac{\partial w}{\partial z}-\hat{\gamma}T, \quad (15)$$

$$\sigma_{zz}=\lambda_0\phi^*+(\lambda+2\mu+k)\frac{\partial w}{\partial z}+\lambda\frac{\partial u}{\partial x}-\hat{\gamma}T, \quad (16)$$

$$\sigma_{xz}=\mu\frac{\partial u}{\partial z}+(\mu+k)\frac{\partial w}{\partial x}+k\phi_2, \quad (17)$$

$$\sigma_{zx}=\mu\frac{\partial w}{\partial x}+(\mu+k)\frac{\partial u}{\partial z}-k\phi_2, \quad (18)$$

$$m_{xy}=\gamma\frac{\partial\phi_2}{\partial x}, \quad (19)$$

$$m_{zy}=\gamma\frac{\partial\phi_2}{\partial z}, \quad (20)$$

$$\lambda_x=\alpha_0\frac{\partial\phi^*}{\partial x}+b_0\frac{\partial\phi_2}{\partial z}, \quad (21)$$

$$\lambda_z=\alpha_0\frac{\partial\phi^*}{\partial z}-b_0\frac{\partial\phi_2}{\partial x}. \quad (22)$$

For convenience, the following non-dimensional variables are used:

$$\begin{aligned}
&x_i'=\frac{\omega^*}{c_2}x_i,\quad u_i'=\frac{\rho c_2\omega^*}{\hat{\gamma}T_0}u_i,\quad t'=\omega^* t,\quad \tau_0'=\omega^*\tau_0,\quad T'=\frac{T}{T_0},\\
&\sigma_{ij}'=\frac{\sigma_{ij}}{\hat{\gamma}T_0},\quad \phi_2'=\frac{\rho c_2^2}{\hat{\gamma}T_0}\phi_2, \phi^{*'}=\frac{\rho c_2^2}{\hat{\gamma}T_0}\phi^*,\quad m_{ij}'=\frac{\omega^*}{c_2\hat{\gamma}T_0}m_{ij},\\
&\lambda_i'=\frac{\omega^*}{c_2\hat{\gamma}T_0}\lambda_i,\quad \omega^*=\frac{\rho C_E c_2^2}{K},\quad c_2^2=\frac{\lambda+2\mu+k}{\rho}. \qquad (23)
\end{aligned}$$

Using Eq. 23 in Eqs. 10–22, we obtain the equations in non-dimensional form after suppressing the primes as

$$\frac{\partial^2 u}{\partial x^2}+(a_1+a_2)\frac{\partial^2 w}{\partial x\partial z}+(a_2+a_3)\frac{\partial^2 u}{\partial z^2}+a_0\frac{\partial\phi^*}{\partial x}-a_3\frac{\partial\phi_2}{\partial z}-\frac{\partial T}{\partial x}=\frac{\partial^2 u}{\partial t^2}, \quad (24)$$

$$\frac{\partial^2 w}{\partial z^2}+(a_1+a_2)\frac{\partial^2 u}{\partial x\partial z}+(a_2+a_3)\frac{\partial^2 w}{\partial x^2}+a_0\frac{\partial\phi^*}{\partial z}+a_3\frac{\partial\phi_2}{\partial x}-\frac{\partial T}{\partial z}=\frac{\partial^2 w}{\partial t^2}, \quad (25)$$

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial z^2}=\left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right)\left[T+\varepsilon_1\phi^*+\varepsilon_2\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)\right], \quad (26)$$

$$\frac{\partial^2\phi_2}{\partial x^2}+\frac{\partial^2\phi_2}{\partial z^2}+\zeta_1\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)-2\zeta_1\phi_2=\zeta_2\frac{\partial^2\phi_2}{\partial t^2}, \tag{27}$$

$$\frac{\partial^2\phi^*}{\partial x^2}+\frac{\partial^2\phi^*}{\partial z^2}-b_1\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)+b_2T-b_3\phi^*=b_4\frac{\partial^2\phi^*}{\partial t^2}, \tag{28}$$

$$\sigma_{xx}=a_0\phi^*+\frac{\partial u}{\partial x}+a_1\frac{\partial w}{\partial z}-T, \tag{29}$$

$$\sigma_{zz}=a_0\phi^*+\frac{\partial w}{\partial z}+a_1\frac{\partial u}{\partial x}-T, \tag{30}$$

$$\sigma_{xz}=a_2\frac{\partial u}{\partial z}+(a_2+a_3)\frac{\partial w}{\partial x}+a_3\phi_2, \tag{31}$$

$$\sigma_{zx}=a_2\frac{\partial w}{\partial x}+(a_2+a_3)\frac{\partial u}{\partial z}-a_3\phi_2, \tag{32}$$

$$m_{xy}=a_4\frac{\partial\phi_2}{\partial x}, \tag{33}$$

$$m_{zy}=a_4\frac{\partial\phi_2}{\partial z}, \tag{34}$$

$$\lambda_x=a_5\frac{\partial\phi^*}{\partial x}+a_6\frac{\partial\phi_2}{\partial z}, \tag{35}$$

$$\lambda_z=a_5\frac{\partial\phi^*}{\partial z}-a_6\frac{\partial\phi_2}{\partial x}, \tag{36}$$

where $(a_0,a_1,a_2,a_3)=\frac{1}{\rho c_2^2}(\lambda_0,\lambda,\mu,k)$, $(a_4,a_5,a_6)=\frac{\omega^{*2}}{\rho c_2^4}(\gamma,\alpha_0,b_0)$, $(\varepsilon_1,\varepsilon_2)=\frac{T_0\hat{\gamma}}{\rho K\omega^*}\left(\hat{\gamma}_1,\hat{\gamma}\right)$, $(\zeta_1,\zeta_2)=\frac{c_2^2}{\gamma}\left(\frac{k}{\omega^{*2}},\rho j\right)$, $(b_1,b_2,b_3)=\frac{c_2^2}{\alpha_0\omega^{*2}}\left(\lambda_0,\frac{\rho c_2^2\hat{\gamma}_1}{\hat{\gamma}},\lambda_1\right)$, $b_4=\frac{\rho c_2^2 j_0}{2\alpha_0}$.

## 3 Initial and Boundary Conditions

The above equations are solved subject to initial conditions,

$$u=w=T=\phi_2=\phi^*=0,\quad \dot{u}=\dot{w}=\dot{T}=\dot{\phi}_2=\dot{\phi}^*=0,\quad t=0. \tag{37}$$

The boundary conditions for the problem may be taken as

$$T(0,z,t)=T_1H(2l-|z|),\quad \sigma_{xx}(0,z,t)=0,\quad \sigma_{xz}(0,z,t)=0,\quad m_{xy}(0,z,t)=0,\quad \lambda_x(0,z,t)=0, \tag{38}$$

where $H$ is the Heaviside unit step.

## 4 Finite Element Formulation

In this section, the governing equations of a microstretch-generalized thermoelastic half-space are summarized, followed by the corresponding finite element equations. In the finite element method, the eight-node isoparametric, quadrilateral element is used for displacement components, temperature, and micro-rotation calculations. The displacement components $u$, $w$, temperature $T$, micro-rotation $\phi_2$, and microstretch $\phi^*$ are related to the corresponding nodal values by

$$\begin{gathered} u=\sum_{i=1}^{m} N_i u_i(t), \quad w=\sum_{i=1}^{m} N_i w_i(t), \quad T=\sum_{i=1}^{m} N_i T_i(t), \\ \phi_2=\sum_{i=1}^{m} N_i \phi_{2i}(t), \quad \phi^*=\sum_{i=1}^{m} N_i \phi_i^*(t), \end{gathered} \tag{39}$$

where $m$ denotes the number of nodes per element and $N_i$'s are the shape functions. The weighting functions and the shape functions coincide. Thus,

$$\begin{gathered} \delta u=\sum_{i=1}^{m} N_i \delta u_i(t), \quad \delta w=\sum_{i=1}^{m} N_i \delta w_i(t), \quad \delta T=\sum_{i=1}^{m} N_i \delta T_i(t), \\ \delta\phi_2=\sum_{i=1}^{m} N_i \delta\phi_{2i}(t), \quad \delta\phi^*=\sum_{i=1}^{m} N_i \delta\phi_i^*(t). \end{gathered} \tag{40}$$

It should be noted that appropriate boundary conditions associated with the governing Eqs. 24–28 must be adopted in order to properly formulate a problem. Essential conditions are prescribed displacements $u$, $w$, temperature $T$, micro-rotation $\phi_2$, and microstretch $\phi^*$, while the natural boundary conditions are prescribed tractions, heat flux, and mass flux which are expressed as

$$\begin{gathered} \sigma_{xx}n_x+\sigma_{xz}n_z=\bar{\tau}_x, \quad \sigma_{xz}n_x+\sigma_{zz}n_z=\bar{\tau}_z, \\ q_x n_x+q_z n_z=\bar{q}, \quad m_{xy}n_x=\bar{m}, \lambda_x n_x=\bar{\lambda}, \end{gathered} \tag{41}$$

where $n_x$ and $n_z$ are direction cosines of the outward unit normal vector at the boundary; $\bar{\tau}_x$, $\bar{\tau}_z$ are the given tractions values; $\bar{q}$ is the given surface heat flux; and $\bar{m}$ and $\bar{\lambda}$ are the given couple traction and microstress components, respectively. In the absence of body forces, the governing equations are multiplied by weighting functions and then are integrated over the spatial domain $\Psi$ with the boundary $\Gamma$. Applying integration by parts and making use of the divergence theorem reduce the order of the spatial derivatives and allows for the application of the boundary conditions. Thus, the finite element equations corresponding to Eqs. 24–28 can be obtained as

$$\sum_{e=1}^{me}\left(\begin{array}{l}\begin{bmatrix} M_{11}^e & 0 & 0 & 0 & 0 \\ 0 & M_{22}^e & 0 & 0 & 0 \\ M_{31}^e & M_{32}^e & M_{33}^e & 0 & M_{35}^e \\ 0 & 0 & 0 & M_{44}^e & 0 \\ 0 & 0 & 0 & 0 & M_{55}^e \end{bmatrix}\begin{Bmatrix} \ddot{u}^e \\ \ddot{w}^e \\ \ddot{T}^e \\ \ddot{\phi}_2^e \\ \ddot{\phi}^{*e} \end{Bmatrix}+\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ C_{31}^e & C_{32}^e & C_{33}^e & 0 & C_{35}^e \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{u}^e \\ \dot{w}^e \\ \dot{T}^e \\ \dot{\phi}_2^e \\ \dot{\phi}^{*e} \end{Bmatrix} \\ +\begin{bmatrix} k_{11}^e & k_{12}^e & k_{13}^e & k_{14}^e & k_{15}^e \\ k_{21}^e & k_{22}^e & k_{23}^e & k_{24}^e & k_{25}^e \\ 0 & 0 & k_{33}^e & 0 & 0 \\ k_{41}^e & k_{42}^e & 0 & k_{44}^e & k_{45}^e \\ k_{51}^e & k_{52}^e & k_{53}^e & 0 & k_{55}^e \end{bmatrix}\begin{Bmatrix} u^e \\ w^e \\ T^e \\ \phi_2^e \\ \phi^{*e} \end{Bmatrix}=\begin{Bmatrix} F_1^e \\ F_2^e \\ F_3^e \\ F_4^e \\ F_5^e \end{Bmatrix}\end{array}\right) \tag{42}$$

where the coefficients in Eq. 42 are given below:

$$M_{11}^e=\int_\Psi [N]^T[N]\,\mathrm{d}\Psi,\quad M_{22}^e=\int_\Psi [N]^T[N]\,\mathrm{d}\Psi,\quad M_{31}^e=\int_\Psi \tau_0\varepsilon_2[N]^T\left[\frac{\partial N}{\partial x}\right]\mathrm{d}\Psi,$$

$$M_{32}^e=\int_\Psi \tau_0\varepsilon_2[N]^T\left[\frac{\partial N}{\partial z}\right]\mathrm{d}\Psi,\quad M_{33}^e=\int_\Psi \tau_0[N]^T[N]\,\mathrm{d}\Psi,$$

$$M_{35}^e=\int_\Psi \tau_0\varepsilon_1[N]^T[N]\,\mathrm{d}\Psi,\quad M_{44}^e=\int_\Psi \zeta_2[N]^T[N]\,\mathrm{d}\Psi,$$

$$M_{55}^e=\int_\Psi b_4[N]^T[N]\,\mathrm{d}\Psi,\quad C_{31}^e=\int_\Psi \varepsilon_2[N]^T\left[\frac{\partial N}{\partial x}\right]\mathrm{d}\Psi,$$

$$C_{32}^e=\int_\Psi \varepsilon_2[N]^T\left[\frac{\partial N}{\partial z}\right]\mathrm{d}\Psi,\quad C_{33}^e=\int_\Psi [N]^T[N]\,\mathrm{d}\Psi,\quad C_{35}^e=\int_\Psi \varepsilon_1[N]^T[N]\,\mathrm{d}\Psi,$$

$$K_{11}^e=\int_\Psi\left(\left[\frac{\partial N}{\partial x}\right]^T\left[\frac{\partial N}{\partial x}\right]+(a_2+a_3)\left[\frac{\partial N}{\partial z}\right]^T\left[\frac{\partial N}{\partial z}\right]\right)\mathrm{d}\Psi\ ,$$

$$K_{12}^e=\int_\Psi (a_1+a_2)\left[\frac{\partial N}{\partial z}\right]^T\left[\frac{\partial N}{\partial x}\right]\mathrm{d}\Psi,\quad K_{13}^e=-\int_\Psi\left[\frac{\partial N}{\partial x}\right]^T[N]\,\mathrm{d}\Psi,$$

$$K_{14}^e=-\int_\Psi a_3\left[\frac{\partial N}{\partial z}\right]^T[N]\,\mathrm{d}\Psi,\quad K_{15}^e=\int_\Psi a_0\left[\frac{\partial N}{\partial z}\right]^T[N]\,\mathrm{d}\Psi,$$

$$K_{21}^e=\int_\Psi (a_1+a_2)\left[\frac{\partial N}{\partial x}\right]^T\left[\frac{\partial N}{\partial z}\right]\mathrm{d}\Psi,$$

$$K_{22}^e=\int_\Psi\left(\left[\frac{\partial N}{\partial z}\right]^T\left[\frac{\partial N}{\partial z}\right]+(a_2+a_3)\left[\frac{\partial N}{\partial x}\right]^T\left[\frac{\partial N}{\partial x}\right]\right)\mathrm{d}\Psi,$$

$$K_{23}^e=-\int_\Psi\left[\frac{\partial N}{\partial z}\right]^T[N]\,\mathrm{d}\Psi,\quad K_{24}^e=\int_\Psi a_3\left[\frac{\partial N}{\partial x}\right]^T[N]\,\mathrm{d}\Psi,$$

$$K_{25}^{e}=\int\limits_{\Psi} a_0\left[\frac{\partial N}{\partial z}\right]^T[N]\,\mathrm{d}\Psi,$$

$$K_{33}^{e}=\int\limits_{\Psi}\left(\left[\frac{\partial N}{\partial x}\right]^T\left[\frac{\partial N}{\partial x}\right]+\left[\frac{\partial N}{\partial z}\right]^T\left[\frac{\partial N}{\partial z}\right]\right)\mathrm{d}\Psi,$$

$$K_{41}^{e}=-\int\limits_{\Psi}\zeta_1[N]^T\left[\frac{\partial N}{\partial z}\right]\mathrm{d}\Psi,\quad K_{42}^{e}=\int\limits_{\Psi}\zeta_1[N]^T\left[\frac{\partial N}{\partial x}\right]\mathrm{d}\Psi,$$

$$K_{44}^{e}=\int\limits_{\Psi}\left(\left[\frac{\partial N}{\partial x}\right]^T\left[\frac{\partial N}{\partial x}\right]+\left[\frac{\partial N}{\partial z}\right]^T\left[\frac{\partial N}{\partial z}\right]+2[N]^T[N]\right)\mathrm{d}\Psi,$$

$$K_{51}^{e}=\int\limits_{\Psi} b_1[N]^T\left[\frac{\partial N}{\partial x}\right]\mathrm{d}\Psi,\quad K_{52}^{e}=\int\limits_{\Psi} b_1[N]^T\left[\frac{\partial N}{\partial z}\right]\mathrm{d}\Psi,$$

$$K_{53}^{e}=-\int\limits_{\Psi} b_2[N]^T[N]\,\mathrm{d}\Psi,$$

$$K_{55}^{e}=\int\limits_{\Psi}\left(\left[\frac{\partial N}{\partial x}\right]^T\left[\frac{\partial N}{\partial x}\right]+\left[\frac{\partial N}{\partial z}\right]^T\left[\frac{\partial N}{\partial z}\right]+b_3[N]^T[N]\right)\mathrm{d}\Psi,$$

$$F_1^e=\int\limits_{\Gamma}[N]^T\bar{\tau}_x\mathrm{d}\Gamma,\quad F_2^e=\int\limits_{\Gamma}[N]^T\bar{\tau}_z\mathrm{d}\Gamma,\quad F_3^e=\int\limits_{\Gamma}[N]^T\bar{q}\mathrm{d}\Gamma,$$

$$F_4^e=\int\limits_{\Gamma}[N]^T\bar{m}\mathrm{d}\Gamma,\quad F_5^e=\int\limits_{\Gamma}[N]^T\bar{\lambda}\mathrm{d}\Gamma, \tag{43}$$

Symbolically, the discretized equations of Eqs. 42 can be written as

$$M\ddot{d}+C\dot{d}+Kd=F^{\mathrm{ext}}, \tag{44}$$

where $M$, $C$, $K$, and $F^{\mathrm{ext}}$ represent the mass, damping, stiffness matrices, and external force vectors, respectively, $d=[u,w,T,\phi_2,\phi^*]^T$. On the other hand, the time derivatives of the unknown variables have to be determined by the Newmark time integration method (see Wriggers [28]).

## 5 Numerical Results and Discussion

With the view of illustrating the theoretical results obtained in the preceding sections and concerned with Lord–Shulman theory, we present some numerical results. The material chosen for this purpose is magnesium crystal (a microstretch thermoelastic solid). The micropolar parameters are the following [20]:

$$\rho=1.74\times10^3\,\mathrm{kg\cdot m^{-3}},\quad j=0.2\times10^{-19}\,\mathrm{m}^2,\quad \lambda=9.4\times10^{10}\,\mathrm{N\cdot m^{-2}},$$
$$\gamma=0.779\times10^{-9}\,\mathrm{N},\quad k=10^{10}\,\mathrm{N\cdot m^{-2}},\ \mu=4.0\times10^{10}\,\mathrm{N\cdot m^{-2}}.$$

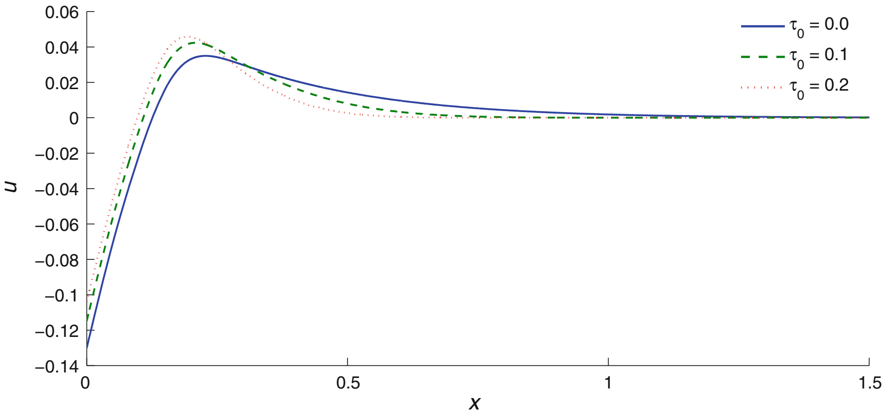


**Fig. 2** Horizontal displacement distribution $u$ for different values of $\tau_0$

The thermal characteristics were taken from Ref. [21]:

$$\tau_0 = 0.1, \quad T_0 = 298\,\mathrm{K}, \quad \alpha_{t_1} = 0.05 \times 10^{-3}\,\mathrm{K}^{-1}, \qquad \alpha_{t_2} = 0.04 \times 10^{-3}\,\mathrm{K}^{-1},$$
$$K = 1.7 \times 10^{2}\,\mathrm{J \cdot m^{-1} \cdot s^{-1} \cdot K^{-1}}, \quad C_E = 1.04 \times 10^{3}\,\mathrm{J \cdot kg^{-1} \cdot K^{-1}}$$

The stretch parameters from Ref. [22] are as follows:

$$\lambda_0 = 2.1 \times 10^{10}\,\mathrm{N \cdot m^{-2}}, \quad \lambda_1 = 0.7 \times 10^{10}\,\mathrm{N \cdot m^{-2}}, \quad j_0 = 0.19 \times 10^{-19}\,\mathrm{m}^2,$$
$$\alpha_0 = 0.779 \times 10^{-9}\,\mathrm{N}, \quad b_0 = 0.9 \times 10^{-9}\,\mathrm{N}$$

The results for the temperature distribution $T$; displacement components $u$, $v$, stress components $\sigma_{xx}$, $\sigma_{xz}$, $\sigma_{zz}$; the micro-rotation $\phi_2$; the microstretch $\phi^*$, $\lambda_x$, and $m_{xy}$ under the LS theory for different values of the thermal relaxation time ($\tau_0 = 0,\ 0.1,\ 0.2$) are shown in Figs. 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11 for a thermoelastic solid with different values of the relaxation time: $\tau_0 = 0$ for the solid line, $\tau_0 = 0.1$ for the dashed line, and $\tau_0 = 0.2$ for the dotted line.

Figure 2 depicts that the values of displacement components $u$ are increasing in the initial range $0 < x < 0.25$ as $\tau_0$ increases; in the further range $0.25 < x < 1.0$, the values decrease as $\tau_0$ increases, then converge to zero. Figure 3 shows that the values of displacement component $w$ decrease as $\tau_0$ increases in the range $0 < x < 1.4$, then converge to zero. Figure 4 exhibits that the temperature under the influence of thermal relaxation decreases sharply in the initial range $0 < x < 0.5$ as $\tau_0$ increases; in the further range $x > 0.5$, the values decrease as $\tau_0$ decreases. Figure 5 explains that the values of micro-rotation $\phi_2$ decrease in the initial range $0 < x < 1.0$, then increase in the range $1.0 < x < 3.0$ as $\tau_0$ increases; in the further range $x > 3.0$, the values decrease as $\tau_0$ increases. Figure 6 depicts that the values of microstretch $\phi^*$ increase as $\tau_0$ increases. Figure 7 shows that the values of the stress components $\sigma_{xx}$ are decreasing in the initial range $0 < x < 0.38$ as $\tau_0$ increases, in the further

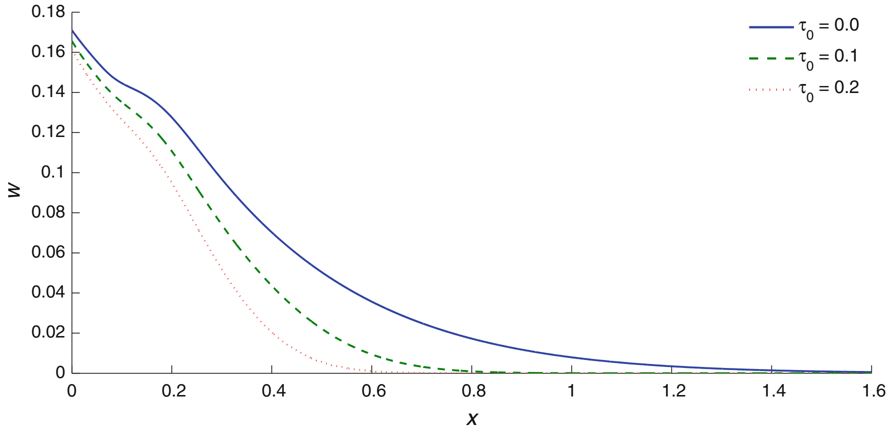


**Fig. 3** Vertical displacement distribution $w$ for different values of $\tau_0$

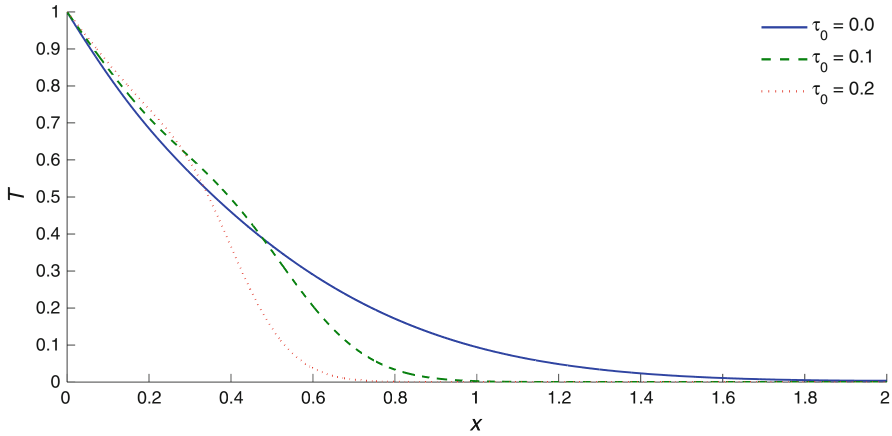


**Fig. 4** Temperature distribution $T$ for different values of $\tau_0$

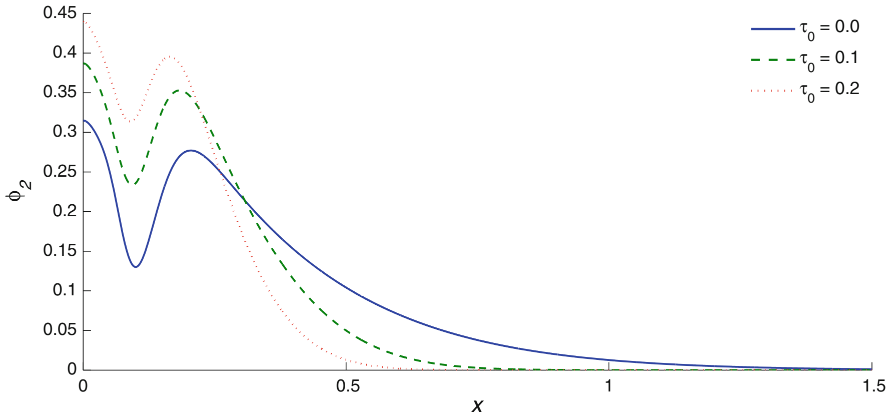


**Fig. 5** The micro-rotation $\phi_2$ distribution for different values of $\tau_0$

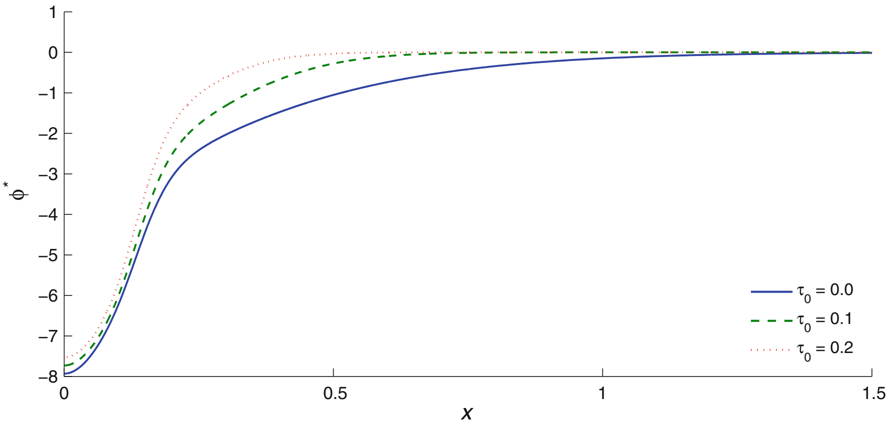


**Fig. 6** The microstretch $\phi^*$ distribution for different values of $\tau_0$

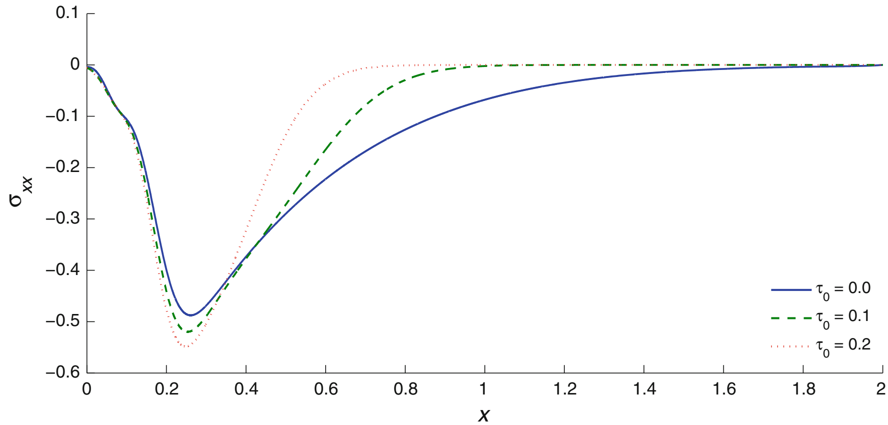


**Fig. 7** Stress $\sigma_{xx}$ distribution for different values of $\tau_0$

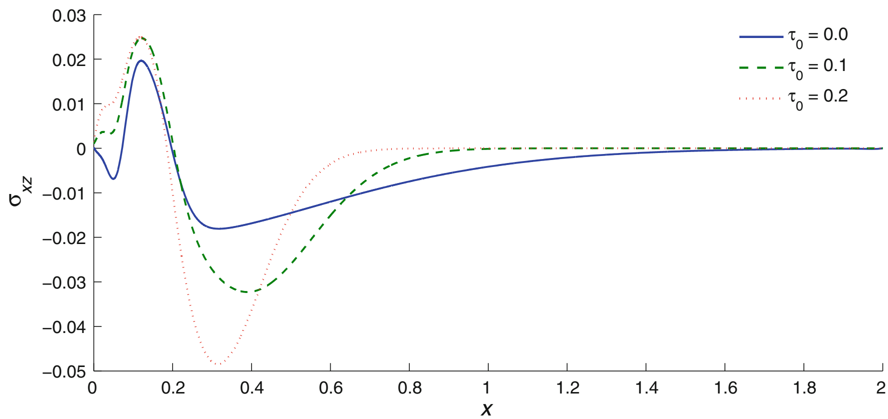


**Fig. 8** Stress $\sigma_{xz}$ distribution for different values of $\tau_0$

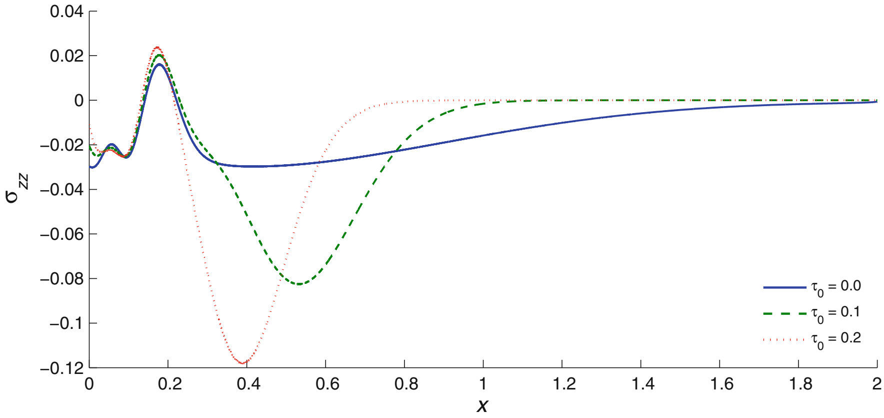


**Fig. 9** Stress $\sigma_{zz}$ distribution for different values of $\tau_0$

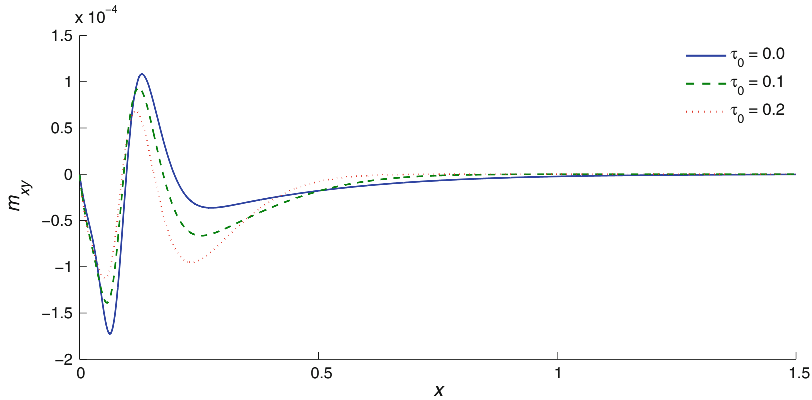


**Fig. 10** $m_{xy}$ distribution for different values of $\tau_0$

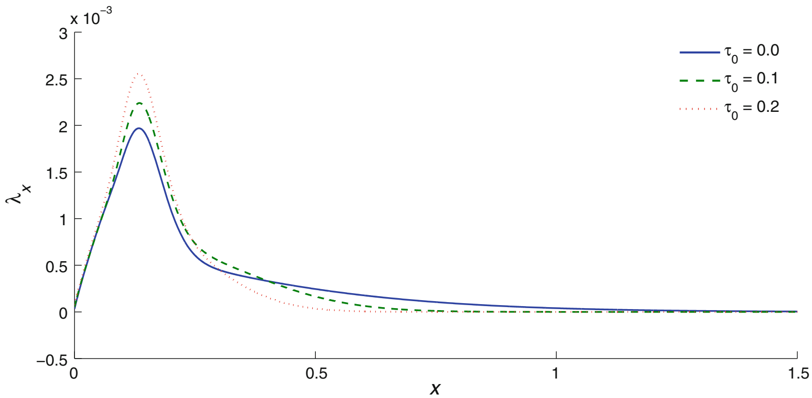


**Fig. 11** Microstress $\lambda_x$ distribution for different values of $\tau_0$

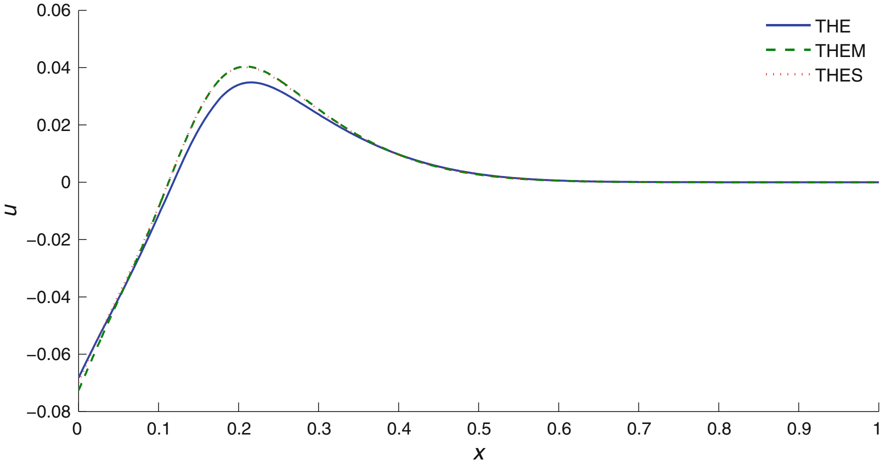


**Fig. 12** Horizontal displacement $u$ distribution

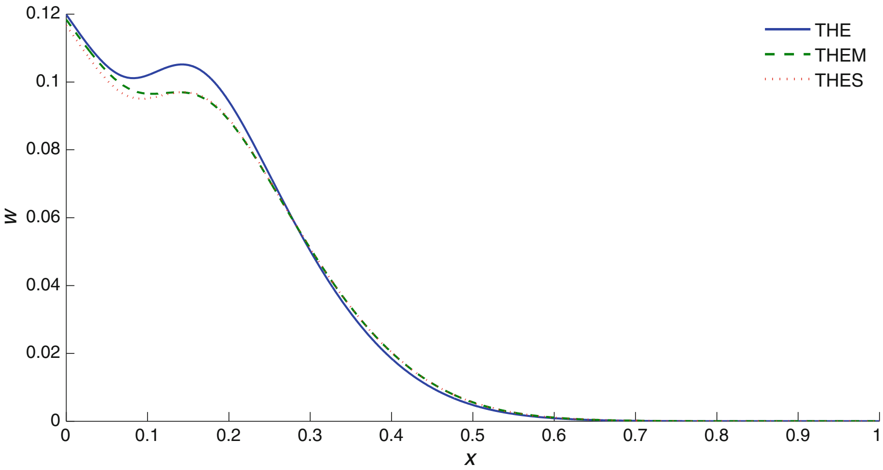


**Fig. 13** Vertical displacement $w$ distribution

range $0.38 < x < 1.6$, the values increase as $\tau_0$ increases. Figure 8 shows that the values of the stress component $\sigma_{xz}$ are increasing in the initial range ss as $\tau_0$ increases, while they decrease in the range $0.2 < x < 0.5$ as $\tau_0$ increases; in the further range $0.5 < x < 1.6$, the values increase as $\tau_0$ increases and finally all curves converge to zero. Figure 9 explains that the values of the stress component $\sigma_{zz}$ initially coincide within the range $0 < x < 0.2$, while they decrease in the range $0.2 < x < 0.8$ as $\tau_0$ increases; in a further range $0.8 < x < 1.9$, the values increase as $\tau_0$ increases and finally all curves converge to zero. Figure 10 shows the distribution of $m_{xy}$ are decreasing in the initial range $0 < x < 0.2$ as $\tau_0$ increases; in the further range $0.2 < x < 0.5$, the values decrease as $\tau_0$ increases, while they increase in the range $0.5 < x < 1.1$ as $\tau_0$ increases; finally, all curves converge to zero. Figure 11 depicts that the distribution

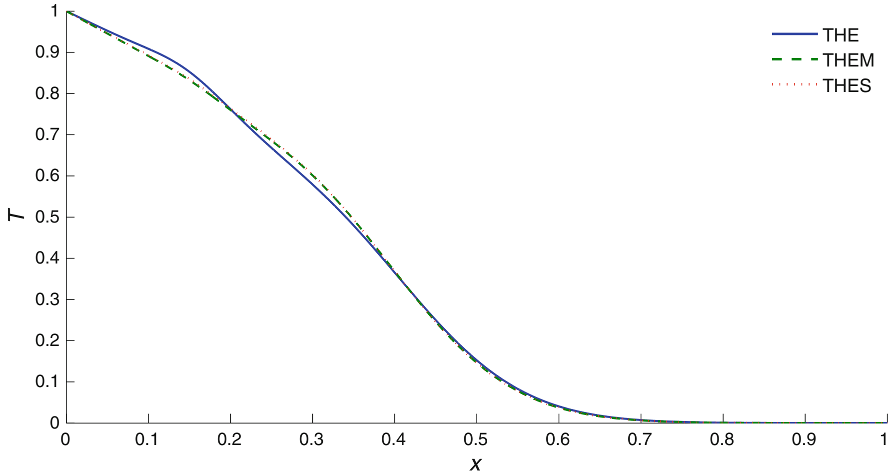


**Fig. 14** Temperature distribution

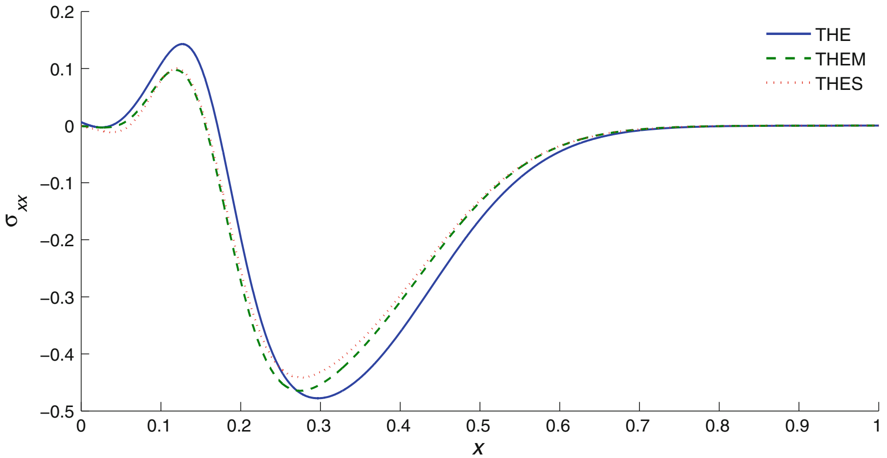


**Fig. 15** Stress $\sigma_{xx}$ distribution

of microstress $\lambda_x$ are increasing in the initial range $0 < x < 0.35$ as $\tau_0$ increases; in the further range $0.35 < x < 1.2$, the values decrease as $\tau_0$ increases and finally all curves converge to zero.

Figures 12, 13, 14, 15, 16, and 17 exhibit comparisons between the temperature distributions $T$; the displacement components $u$, $w$; and the stress components $\sigma_{xx}$, $\sigma_{xz}$, $\sigma_{zz}$ versus the distance $x$ for the LS theory in the case of generalized THE, micropolar thermoelasticity (THEM), and thermoelasticity with stretch (THES).

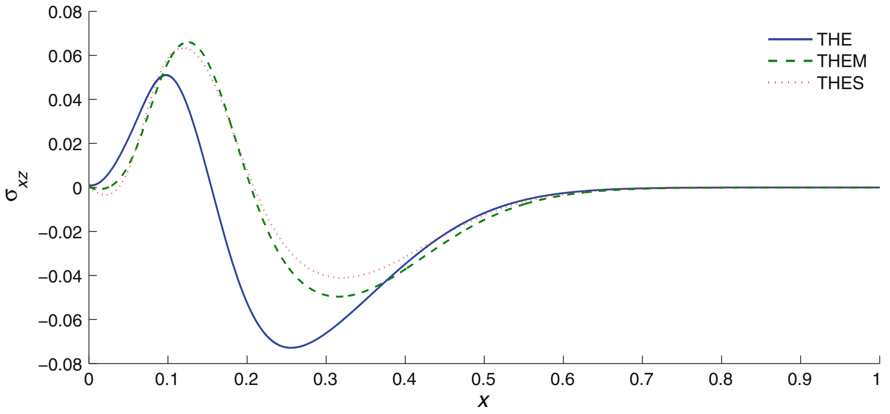


**Fig. 16** Stress $\sigma_{xz}$ distribution

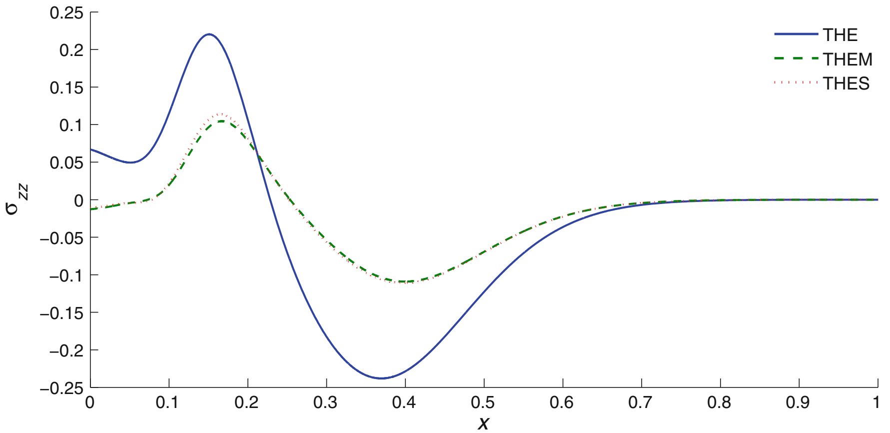


**Fig. 17** Stress $\sigma_{zz}$ distribution

## 6 Concluding Remarks

In this study, the finite element method is used to study the problem of the effect of the thermal relaxation time on a microstretch thermoelastic solid. We can obtain the following conclusions based on the above analysis:

1. The thermal relaxation time has a significant effect on the field quantities.
2. The presence of the microstretch plays a significant role in all the physical quantities.
3. Numerical solutions based upon the finite element method on the THE problem in solids have been developed and utilized.
4. The values of all the physical quantities converge to zero with an increase in the distance $x$ and all functions are continuous.

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