Int J Thermophys (2012) 33:1288–1301
DOI 10.1007/s10765-012-1248-3

ORIGINAL PAPER

Piezoelectric Behavior of an Inhomogeneous Hollow Cylinder with Thermal Gradient

Ashraf M. Zenkour

Received: 31 March 2011 / Accepted: 11 July 2012 / Published online: 25 July 2012

Abstract An analytical solution to the axisymmetric problem of a radially polarized, radially orthotropic piezoelectric hollow cylinder with a thermal gradient and subjected to various boundary conditions is developed. The elastic coefficients, piezoelectric coefficients, stress-temperature moduli, dielectric coefficient, pyroelectric coefficients, thermal conductivity coefficient, and thermal expansion coefficients of the hollow cylinder are assumed to be graded in the radial direction according to a simple power-law distribution. The governing second-order differential equations are derived from the equilibrium equation, the charge equation of electrostatics, and steady state heat transfer equation through the radial direction of the inhomogeneous hollow cylinder. The displacement, stresses, and potential field distributions in the cylinder are examined. The influence of the inhomogeneity parameter on the numerical results is investigated.

Keywords Electric potential · Inhomogeneous hollow cylinder · Piezoelectric · Thermal gradient

1 Introduction

In recent years, the use of piezoelectric materials in intelligent structures has attracted extensive attention. Due to the intrinsic direct and converse piezoelectric effects,

A. M. Zenkour (✉)
Department of Mathematics, Faculty of Science, King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia
e-mail: zenkour@gmail.com

A. M. Zenkour
Department of Mathematics, Faculty of Science, Kafrelsheikh University,
Kafr El-Sheikh 33516, Egypt

piezoelectric materials can be effectively used as sensors or actuators for the active shape or vibration control of structures. The study of the coupling effect between elastic and piezoelectric materials has become an important topic in modern science and technology. An inhomogeneous piezoelectric material has a composition and properties varying continuously along certain radial or thickness directions. This material is an intentionally designed composite so that it possesses desirable properties for some specific applications. An advantage of this new kind of material is improvement in the reliability of the lifespan of piezoelectric devices. Recently, there has been growing interest in materials deliberately fabricated so that their electric, magnetic, thermal, and mechanical properties vary continuously in space on the macroscopic scale. This research subject is so new that only a few results can be found in the literature that used exact solutions.

Many achievements have been obtained in responses for elastic and piezoelectric hollow cylinders [1–6]. For piezoelectric media, Adelman and Stavsky [7,8] studied the axisymmetric free vibrations of radially and axially polarized piezoelectric ceramic hollow cylinders. Shul'ga et al. [9] and Paul and Venkatesan [10] investigated the axisymmetric and three-dimensional (3D) electroelastic waves in a hollow piezoelectric ceramic cylinder. Wang and Gong [11] obtained the elastodynamic solution for multilayered cylinders subjected to axisymmetric dynamic loads. Yin and Yue [12] studied the transient plane-strain response of multilayered cylinders due to an axisymmetric impulse. Han and Liu [13] studied elastic waves in a functionally graded piezoelectric cylinder. Utilizing the Fourier transform technique, Ueda [14] investigated the thermally induced fracture of a functionally graded piezoelectric layer. Lu et al. [15] derived exact solutions of a simply supported functionally graded piezoelectric plate/laminate under cylindrical bending. Wang et al. [16] obtained the dynamic solution of a multilayered orthotropic piezoelectric hollow cylinder for axisymmetric plane strain problems. Wang and Dong [17] showed that the characteristics of wave propagation in piezoelectric cylindrically laminated shells are related to the large deformation, rotator inertia, and thermal environment of the shells.

Some approximate 3D numerical modeling has also been developed for the static and dynamic analyses of multilayered composites and functionally graded material (FGM) circular hollow cylinders. For example, Wu and Yang [18] developed both the Reissner mixed variational theorem based on meshless collocation and an element-free Galerkin method for the approximate 3D analysis of multilayered composites and FGM hollow cylinders under mechanical loads. Chen et al. [19] developed an analytical solution of transversely isotropic FGM rotating discs, which can degenerate into the solution for the corresponding isotropic FGM rotating discs. Asghari and Ghafoori [20] presented a semi-analytical 3D elasticity solution for rotating functionally graded hollow and solid disks. Chen et al. [21] used the state-space approach coupled with the general linear spring-layer model to analyze 3D bending and free vibration of simply supported, cross-ply laminated cylindrical panels with weak interfaces. In fact, the formulation and analytical solution of the hollow cylinder may be easily obtained by neglecting the dependence of all parameters to the axial coordinate z, i.e., assuming planar conditions.

There are also many studies that have been carried out for inhomogeneous materials. Among them, the special case that Young's modulus has a power-law dependence on the radial coordinate, while the linear thermal expansion coefficient and Poisson's ratio are constants, has been considered by many scientists and engineers [22–24]. In the present investigation, thermo-electro-elastic equations for linear piezoelectric inhomogeneous hollow cylinders are given. The governing equilibrium equations in a radially polarized form are shown to reduce to a coupled system of second-order ordinary differential equations for the radial displacement and electric potential field. The analytical solution for these equations is developed for various boundary conditions. The exact expressions for responses of the radial displacement, radial stress, hoop stress, and electrostatic potential in inhomogeneous piezoelectric hollow cylinders are obtained.

2 Formulation of the Problem

Let us consider a long cylinder of outer radius b and inner radius a and made of an inhomogeneous orthotropic material. The cylindrical coordinate system (r, θ, z) is used with the z-axis coinciding with the axis of the cylinder. The cylinder is in a state of generalized planar strain, i.e., temperature, displacements, and stresses may be only functions of the coordinates r and θ. In addition, the strain tensor is considered to be symmetric about the z-axis. So, we have only the radial displacement u_r which is independent of θ and z. The material properties of the inhomogeneous hollow cylinder are assumed to be functions of the radial direction. The relationship between the physical properties and the radial direction r for the present cylinder is given by

$$P(r) = P^0 \left(\frac{r}{b}\right)^{2k}, \tag{1}$$

where P^0 is the material property of the homogeneous hollow cylinder and k is a geometric parameter. The value of k equal to zero represents a fully homogeneous hollow cylinder. The above power-law assumption reflects variable properties applied only in the radial direction. The power-law exponent k may be varied to obtain different distributions of the component materials through the radial direction of the cylinder.

The constitutive relations for an inhomogeneous long piezoelectric hollow cylinder are given by [3,4]

$$\left.\begin{aligned}
\sigma_{11} &= c_{11}(r)\frac{\mathrm{d}u_r}{\mathrm{d}r} + c_{12}(r)\frac{u_r}{r} + e_{11}(r)\frac{\mathrm{d}\varphi}{\mathrm{d}r} - \gamma_1(r)T(r),\\
\sigma_{22} &= c_{12}(r)\frac{\mathrm{d}u_r}{\mathrm{d}r} + c_{22}(r)\frac{u_r}{r} + e_{12}(r)\frac{\mathrm{d}\varphi}{\mathrm{d}r} - \gamma_2(r)T(r),\\
D_{11} &= e_{11}(r)\frac{\mathrm{d}u_r}{\mathrm{d}r} + e_{12}(r)\frac{u_r}{r} + \beta_{11}(r)\frac{\mathrm{d}\varphi}{\mathrm{d}r} + p_{11}(r)T(r),
\end{aligned}\right\} \tag{2}$$

where $c_{ij}, e_{ij}, \gamma_i, \beta_{11}$, and p_{11} are the elastic coefficients, piezoelectric coefficients, stress-temperature moduli, dielectric coefficient, and pyroelectric coefficients, respectively. In addition, σ_{ij}'s are the stress components and D_{11} is the radial electric displacement. Note that the stress-temperature moduli γ_i are given in terms of the elastic coefficients c_{ij} and the thermal expansion coefficients α_i by the relations,

$$\begin{aligned}
\gamma_1 &= c_{11}\alpha_1 + c_{12}\alpha_2 + c_{13}\alpha_3,\\
\gamma_2 &= c_{12}\alpha_1 + c_{22}\alpha_2 + c_{23}\alpha_3.
\end{aligned} \tag{3}$$

The equilibrium equation in the absence of a body force in the radial direction is given by

$$\frac{\mathrm{d}\sigma_{11}}{\mathrm{d}r}+\frac{\sigma_{11}-\sigma_{22}}{r}=0. \tag{4}$$

In the absence of free charge density, the charge equation of electrostatics is

$$\frac{\mathrm{d}D_{11}}{\mathrm{d}r}+\frac{D_{11}}{r}=0. \tag{5}$$

Substituting from Eq. 2 into Eq. 4, we get

$$c_{11}\frac{\mathrm{d}^2u_r}{\mathrm{d}r^2}+\left(\frac{\mathrm{d}c_{11}}{\mathrm{d}r}+\frac{c_{11}}{r}\right)\frac{\mathrm{d}u_r}{\mathrm{d}r}+\frac{1}{r}\left(\frac{\mathrm{d}c_{12}}{\mathrm{d}r}-\frac{c_{22}}{r}\right)u_r+e_{11}\frac{\mathrm{d}^2\varphi}{\mathrm{d}r^2}$$
$$+\left(\frac{\mathrm{d}e_{11}}{\mathrm{d}r}+\frac{e_{11}-e_{12}}{r}\right)\frac{\mathrm{d}\varphi}{\mathrm{d}r}-\gamma_1\frac{\mathrm{d}T}{\mathrm{d}r}-\left(\frac{\mathrm{d}\gamma_1}{\mathrm{d}r}+\frac{\gamma_1-\gamma_2}{r}\right)T=0. \tag{6}$$

The elastic coefficients, piezoelectric coefficients, thermal expansion coefficients, dielectric coefficient, and pyroelectric coefficients of the inhomogeneous hollow cylinder change continuously through the radial direction of the cylinder and obey the gradation relation given in Eq. 1. Inserting the material properties c_{1i}, e_{1i}, and $\gamma_i\,(i=1,2)$ from Eq. 1 into the above ordinary differential equation, we obtain

$$c_{11}^0\frac{\mathrm{d}^2u_r}{\mathrm{d}r^2}+\frac{c_{11}^0}{r}(1+2k)\frac{\mathrm{d}u_r}{\mathrm{d}r}-\frac{1}{r^2}\left[c_{22}^0-2kc_{12}^0\right]u_r$$
$$-\left\{\gamma_1^0\frac{\mathrm{d}}{\mathrm{d}r}+\frac{1}{r}\left[\gamma_1^0(1+2k)-\gamma_2^0\right]\right\}T$$
$$+e_{11}^0\frac{\mathrm{d}^2\varphi}{\mathrm{d}r^2}+\frac{1}{r}\left[e_{11}^0(1+2k)-e_{12}^0\right]\frac{\mathrm{d}\varphi}{\mathrm{d}r}=0, \tag{7}$$

where c_{1i}^0, e_{1i}^0, and γ_i^0 represent the elastic constants, piezoelectric constants, and stress-temperature moduli, respectively, for the homogeneous hollow cylinder.

3 Solution of the Problem

Introducing the following dimensionless forms:

$$R=\frac{r}{b},\quad S=\frac{a}{b},\quad u=\frac{u_r}{b},\quad \Phi=\sqrt{\frac{\beta_{11}^0}{c_{11}^0}}\frac{\varphi}{b},$$
$$c_i=\frac{c_{i2}^0}{c_{11}^0},\quad e_i=\frac{e_{i2}^0}{\sqrt{c_{11}^0\beta_{11}^0}},\quad \sigma_i=\frac{\sigma_{ii}}{c_{11}^0},\quad (i=1,2), \tag{8}$$
$$D_1=\frac{D_{11}}{\sqrt{c_{11}^0\beta_{11}^0}},\quad T_i(R)=\frac{\gamma_i^0T(r)}{c_{11}^0},\quad T_p(R)=\frac{p_{11}^0T(r)}{\sqrt{c_{11}^0\beta_{11}^0}}.$$

Then, Eq. 2 may be written in the following simple form:

$$\sigma_1 = R^{2k}\left(\frac{\mathrm{d}u}{\mathrm{d}R} + c_1\frac{u}{R} + e_1\frac{\mathrm{d}\Phi}{\mathrm{d}R} - T_1(R)\right), \quad (9)$$

$$\sigma_2 = R^{2k}\left(c_1\frac{\mathrm{d}u}{\mathrm{d}R} + c_2\frac{u}{R} + e_2\frac{\mathrm{d}\Phi}{\mathrm{d}R} - T_2(R)\right), \quad (10)$$

$$D_1 = R^{2k}\left(e_1\frac{\mathrm{d}u}{\mathrm{d}R} + e_2\frac{u}{R} - \frac{\mathrm{d}\Phi}{\mathrm{d}R} + T_p(R)\right). \quad (11)$$

Also, the charge equation should be

$$\frac{\mathrm{d}D_1}{\mathrm{d}R} + \frac{D_1}{R} = 0. \quad (12)$$

One can easily get the radial electric displacement from the above equation as

$$D_1(R) = \frac{C}{R}, \quad (13)$$

where C is a constant. Substituting the above equation into Eq. 11 yields

$$\frac{\mathrm{d}\Phi}{\mathrm{d}R} = e_1\frac{\mathrm{d}u}{\mathrm{d}R} + e_2\frac{u}{R} + T_p(R) - \frac{C}{R^{1+2k}}. \quad (14)$$

So, one can get the dimensionless stresses in the form

$$\sigma_1 = R^{2k}\left[(1+e_1^2)\frac{\mathrm{d}u}{\mathrm{d}R} + (c_1 + e_1e_2)\frac{u}{R} - T_1(R) + e_1T_p(R)\right] - e_1\frac{C}{R}, \quad (15)$$

$$\sigma_2 = R^{2k}\left[(c_1 + e_1e_2)\frac{\mathrm{d}u}{\mathrm{d}R} + (c_2 + e_2^2)\frac{u}{R} - T_2(R) + e_2T_p(R)\right] - e_2\frac{C}{R}. \quad (16)$$

In addition, the equilibrium differential equation given in Eq. 7 may be rewritten in the following form:

$$R^2\frac{\mathrm{d}^2u}{\mathrm{d}R^2} + (1+2k)R\frac{\mathrm{d}u}{\mathrm{d}R} + (k^2 - \mu^2)u + \Psi(R) = 0, \quad (17)$$

where

$$\mu^2 = k^2 - \frac{2k(c_1 + e_1e_2) - (c_2 + e_2^2)}{1 + e_1^2}, \quad (18)$$

and the function $\Psi(R)$ is given by

$$\Psi(R) = \frac{1}{1+e_1^2}\left[R^2\frac{\mathrm{d}T_{p1}}{\mathrm{d}R} + R(1+2k)T_{p1} - RT_{p2} + \frac{e_2C}{R^{2k}}\right], \quad (19)$$

in which

$$T_{p1} = e_1 T_p - T_1, \quad T_{p2} = e_2 T_p - T_2. \tag{20}$$

The general solution of Eq. 17 may be obtained as follows:

$$u(R) = R^{-k-\mu} \left[A + F_1(R)\right] + R^{-k+\mu} \left[B - F_2(R)\right], \tag{21}$$

where A and B are additional arbitrary constants. The functions F_1 and F_2 are given by

$$F_1 = \frac{1}{2\mu} \int R^{k+\mu-1} \Psi(R) \mathrm{d}R, \quad F_2 = \frac{1}{2\mu} \int R^{k-\mu-1} \Psi(R) \mathrm{d}R. \tag{22}$$

The solution of the present problem is completed when the temperature is determined. Since the variation of the temperature field is assumed to occur in the radial direction only, then the steady state heat transfer equation through the radial direction of the inhomogeneous hollow cylinder obeys the following equation:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \left(\frac{1}{\kappa(r)} \frac{\mathrm{d}\kappa(r)}{\mathrm{d}r} + \frac{1}{r}\right) \frac{\mathrm{d}}{\mathrm{d}r}\right] T(r) = 0, \tag{23}$$

where the thermal conductivity coefficient κ varies in the radial direction according to Eq. 1. Then

$$\left[R^2 \frac{\mathrm{d}^2}{\mathrm{d}r^2} + R(1 + 2k) \frac{\mathrm{d}}{\mathrm{d}R}\right] T(R) = 0, \tag{24}$$

and its solution is given by

$$T = \frac{1}{1 - S^{2k}} \left[T_b - T_a S^{2k} + \frac{(T_a - T_b) S^{2k}}{R^{2k}}\right], \tag{25}$$

in which T_a and T_b are the inner and outer temperatures, respectively. Substituting Eq. 21 with the aid of Eq. 25 into Eqs. 15 and 16 gives the radial and circumferential stresses in the following forms:

$$\begin{aligned}\sigma_1 = R^{2k} \Bigg\{ &(1 + e_1^2) \Bigg[R^{-k-\mu-1} \left(\frac{e_1 e_2 + c_1}{1 + e_1^2} - k - \mu\right) (A + F_1) + R^{-k-\mu} \frac{\mathrm{d}F_1}{\mathrm{d}R} \\ &- R^{-k+\mu} \frac{\mathrm{d}F_2}{\mathrm{d}R} + R^{-k+\mu-1} \left(\frac{e_1 e_2 + c_1}{1 + e_1^2} - k + \mu\right) (B - F_2) \Bigg] \\ &+ e_1 k_{p1} - k_{11} + (e_1 k\text{p}2 - k_{12}) R^{-2k} \Bigg\} - e_1 \frac{C}{R},\end{aligned} \tag{26}$$

$$\sigma_2 = R^{2k}\Bigg\{(c_1+e_1e_2)\Bigg[R^{-k-\mu-1}\left(\frac{c_2+e_2^2}{c_1+e_1e_2}-k-\mu\right)(A+F_1)+R^{-k-\mu}\frac{\mathrm{d}F_1}{\mathrm{d}R}$$
$$-R^{-k+\mu}\frac{\mathrm{d}F_2}{\mathrm{d}R}+R^{-k+\mu-1}\left(\frac{e_1e_2+c_1}{1+e_1^2}-k+\mu\right)(B-F_2)\Bigg]$$
$$+e_2k_{p1}-k_{21}+(e_2k_{p2}-k_{22})R^{-2k}\Bigg\}-e_2\frac{C}{R}, \tag{27}$$

where

$$(k_{i1},k_{i2}) = \frac{\gamma_i^0}{c_{11}^0(1-S^{2k})}\left(T_b-T_aS^{2k},(T_a-T_b)S^{2k}\right),$$
$$\left(k_{p1},k_{p2}\right) = \frac{p_{11}^0}{\sqrt{c_{11}^0\beta_{11}^0}(1-S^{2k})}\left(T_b-T_aS^{2k},(T_a-T_b)S^{2k}\right). \tag{28}$$

In addition, the first derivative of the electrostatic potential is given, using Eq. 11, by

$$\frac{\mathrm{d}\Phi}{\mathrm{d}R} = R^{-k-\mu-1}[e_2-e_1(k+\mu)](A+F_1)+R^{-k+\mu-1}[e_2-e_1(k-\mu)](B-F_2)$$
$$+e_1\left(R^{-k-\mu}\frac{\mathrm{d}F_1}{\mathrm{d}R}-R^{-k+\mu}\frac{\mathrm{d}F_2}{\mathrm{d}R}\right)+k_{p1}+k_{p2}R^{-2k}-CR^{-(1+2k)}. \tag{29}$$

The electrostatic potential is given by integrating the above equation which yields a fourth constant D.

Finally, Eq. 19 yields

$$\Psi(R) = \frac{1}{1+e_1^2}\Big\{\big[k_{21}-e_2k_{p1}+(e_1k_{p1}-k_{11})(1+2k)\big]R+e_2CR^{-2k}$$
$$+\big[k_{p2}(e_1-e_2)-k_{12}+k_{22}\big]R^{1-2k}\Big\}. \tag{30}$$

4 Boundary Conditions

The elastic solution for the inhomogeneous hollow cylinder with variable piezo-electro-elastic properties is completed by the application of the boundary conditions. For the present inhomogeneous hollow cylinder, the solution requires that one boundary condition be satisfied at each radius. These may be a given radial displacement and/or radial stress or some combination. In addition, the solution requires also two conditions applied on the value of the electric potential on the inner and outer radii of the cylinder. We can now formulate a variety of commonly encountered situations. The boundary conditions at the radii of the hollow cylinder of the inner radius $r=a(R=S)$ and outer radius $r=b(R=1)$ may readily be formulated as discussed below. The system of linear algebraic equations for the constants A, B, C, and D can be solved for the cases of the following boundary conditions.

4.1 Case 1

The cylinder is subjected to free inner and outer radii and a uniform potential difference across the cylindrical annulus. For convenience, it is assumed that the inner radius potential is zero and the potential on the outer radius is a nonzero constant.

$$\sigma_1(S) = 0, \quad \sigma_1(1) = 0, \quad \Phi(S) = 0, \quad \Phi(1) = 1. \tag{31}$$

4.2 Case 2

The cylinder is subjected to clamped inner and outer radii, and the inner radius potential is nonzero while the outer radius is

$$u(S) = 0, \quad u(1) = 0, \quad \Phi(S) = 1, \quad \Phi(1) = 0. \tag{32}$$

4.3 Case 3

The cylinder is subjected to an internal uniform pressure, free outer radius, and a uniform potential difference prescribed across the annulus (the inner radius potential is zero while the outer radius is nonzero).

$$\sigma_1(S) = 1, \quad \sigma_1(1) = 0, \quad \Phi(S) = 0, \quad \Phi(1) = 1. \tag{33}$$

4.4 Case 4

In this case, the cylinder has mixed boundary condition, i.e., clamped at its inner radius and free at its outer radius. In addition, it is assumed that the inner radius potential is zero and the potential on the outer radius is a nonzero constant.

$$u(S) = 0, \quad \sigma_1(1) = 0, \quad \Phi(S) = 0, \quad \Phi(1) = 1. \tag{34}$$

4.5 Case 5

Finally, the cylinder is subjected to an external uniform pressure, free inner radius, and zero electric potential difference across the cylindrical annulus.

$$\sigma_1(S) = 0, \quad \sigma_1(1) = 1, \quad \Phi(S) = 0, \quad \Phi(1) = 0. \tag{35}$$

5 Numerical Results

Numerical results have been obtained graphically to show the distribution of the radial displacement, radial stress, hoop stress, and electric potential through the radial direction of the inhomogeneous hollow cylinders. The results of the present investigations

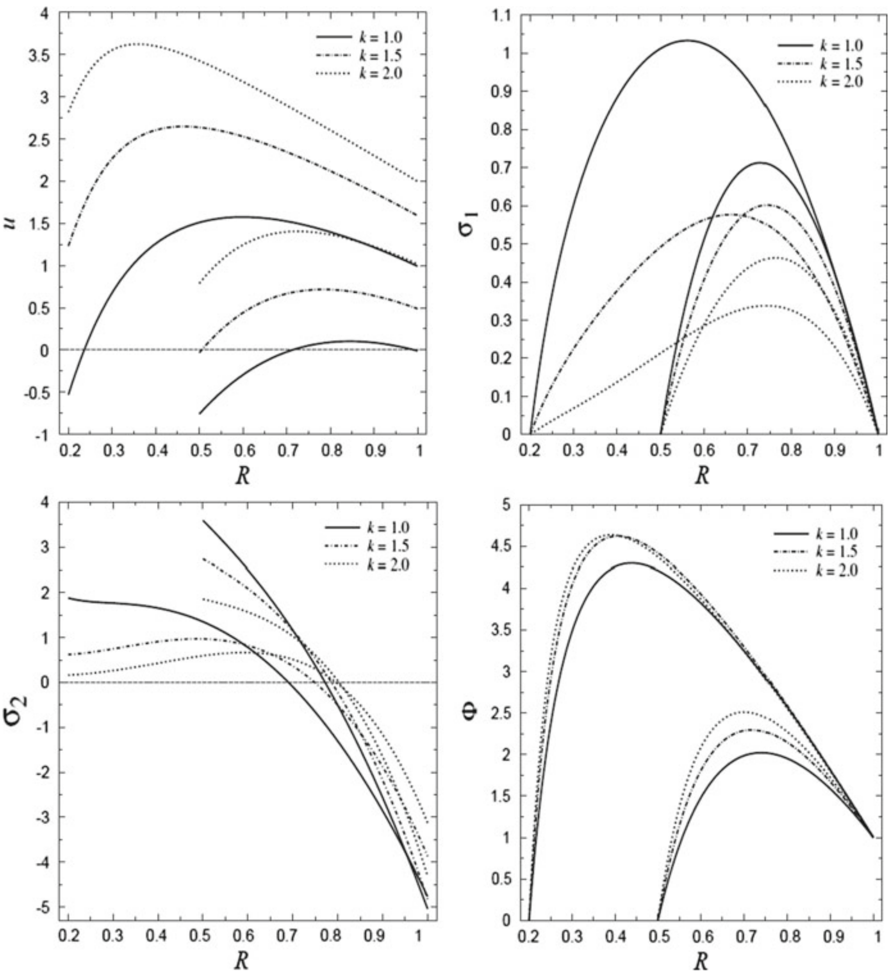

Fig. 1 Distribution of displacement, stresses, and electrostatic potential through the radial direction of the hollow cylinder according to Case 1

are displayed in Figs. 1, 2, 3, 4, and 5 corresponding to the five cases studied. Two hollow cylinders with $S = 0.2$ and 0.5 are considered. The plots depict results for $T_a = 0\,^\circ C$ and $T_b = 100\,^\circ C$. The basic material properties are taken as [3]

$$
\begin{aligned}
&c_{11}^0 = c_{33}^0 = 111\ \text{GPa}, \quad c_{13}^0 = c_{23}^0 = 115\ \text{GPa}, \quad c_{12}^0 = 77.8\ \text{GPa},\\
&c_{22}^0 = 220\ \text{GPa}, \quad e_{11}^0 = e_{13}^0 = -5.2\ \text{C}\cdot\text{m}^{-2}, \quad e_{12}^0 = 15.1\ \text{C}\cdot\text{m}^{-2},\\
&\beta_{11}^0(\times 10^9) = 5.62\ \text{C}^2\cdot\text{N}^{-1}\cdot\text{m}^{-2}, \quad p_{11}^0(\times 10^5) = -2.5\ \text{N}\cdot\text{m}^{-2}\cdot\ \text{K}^{-1},\\
&\alpha_1^0 = \alpha_3^0 = 0.0001\ \text{K}^{-1}, \quad \alpha_2^0 = 0.00001\ \text{K}^{-1}.
\end{aligned} \tag{36}
$$

Figures 1, 2, 3, 4, and 5 depict the variation of the displacement, stresses, and potential distributions along the radial direction of the inhomogeneous hollow cylinder with

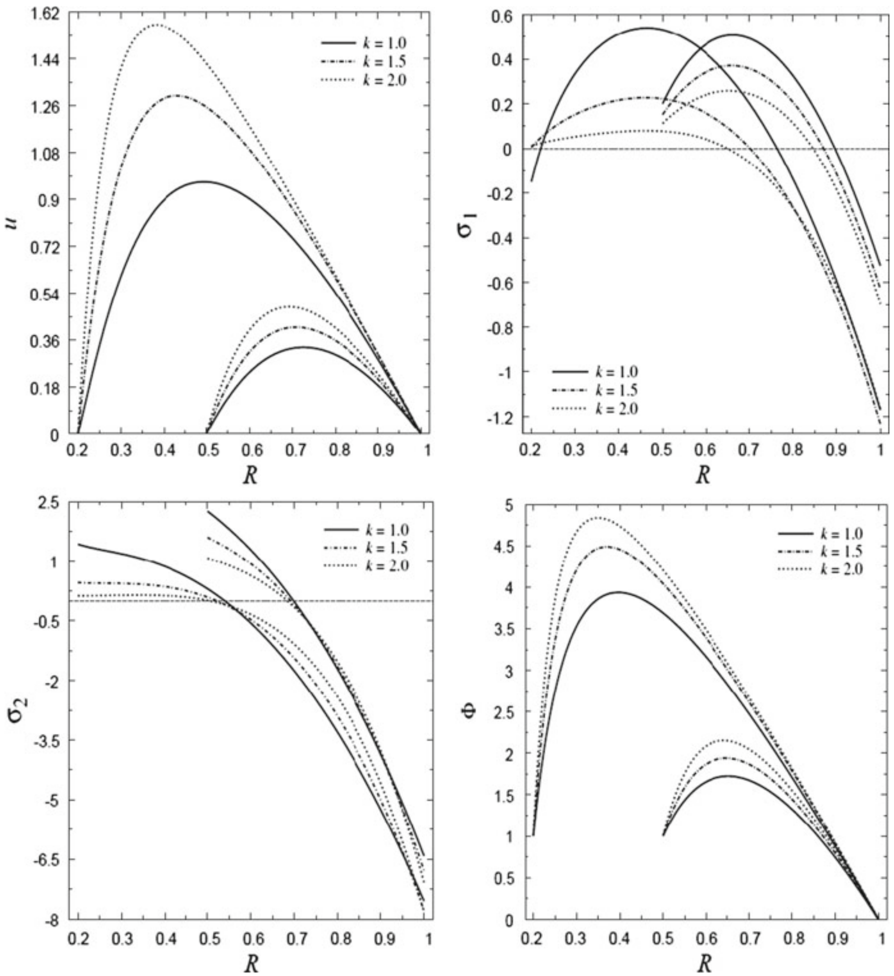

Fig. 2 Distribution of displacement, stresses, and electrostatic potential through the radial direction of the hollow cylinder according to Case 2

different values of the inhomogeneity exponent k. It is seen easily from all figures that the radial displacement and radial stress satisfy the mechanical boundary conditions while the electrostatic potential satisfies fully the electric boundary conditions.

Figure 1 shows that when k increases, the radial displacement and potential are increasing while the radial stress is decreasing. The hoop stress starts its curves with the behavior of the radial stress, then reflects the behavior to be as the radial displacement and potential. This is done at $R \cong 0.6$ for $S = 0.2$ and at $R \cong 0.73$ for $S = 0.5$. The radial stress according to this case (purely electrical) shows an interesting maximum at the interior surface. The maximum radial stress shifts to the outer radius as k increases, independent of the value of S. Also, the interior displacement of the same radial point is decreasing as S increases for $k = 1$ while it decreases for other values of k.

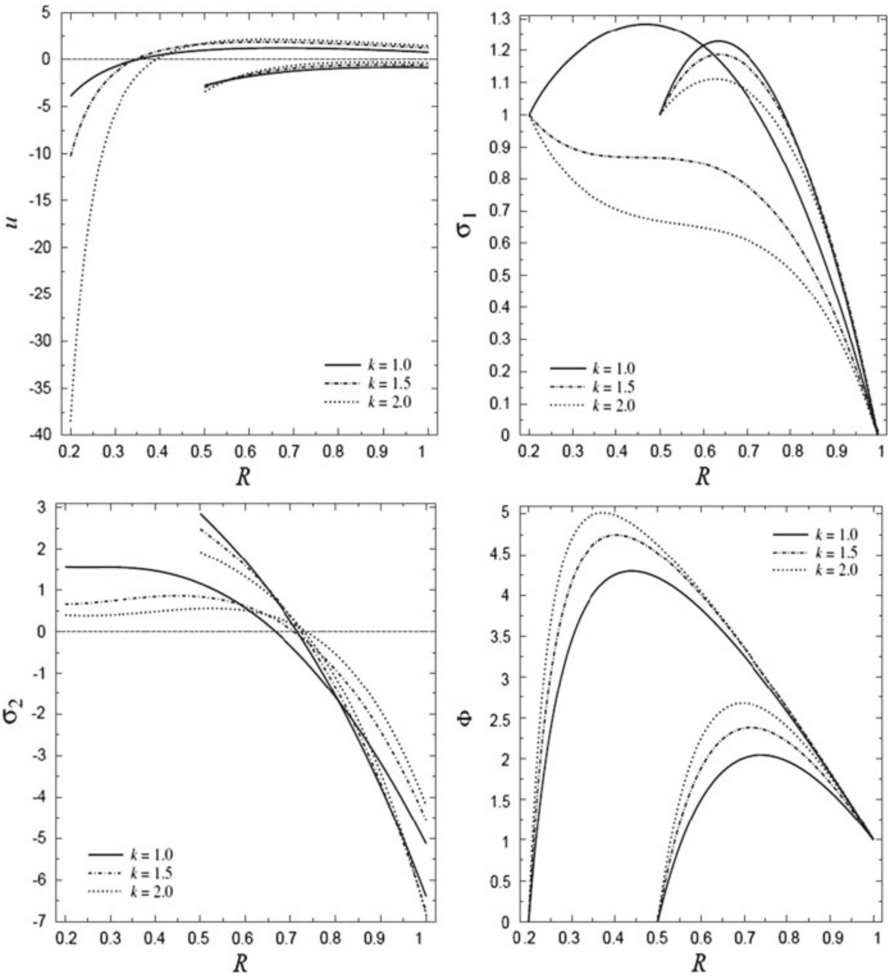

Fig. 3 Distribution of displacement, stresses, and electrostatic potential through the radial direction of the hollow cylinder according to Case 3

Figure 2 shows the results of a clamped cylinder with a uniform potential difference across the cylindrical annulus. The radial displacement is interestingly a maximum at the interior surface. The maximum displacement shifts to the inner radius as the inhomogeneity coefficient k increases, independent of the value of S. Also, the interior displacement decreases as S increases.

Figure 3 shows the results of an internal uniform pressure with a uniform potential difference across the cylindrical annulus. The radial displacement is significantly decreasing at the inner radius, especially for higher values of k. This is done of course from the radial stress at the inner radius. For $k > 1$, the radial stress is directly decreasing along the radial direction for $S = 0.2$. For $k = 1$ ($S = 0.2$) and all values of $k(S = 0.5)$, the stress has a maximum shift to the inner radius. In Fig. 4, the electrostatic potential has a changing trend similar to that of Fig 1. In these two cases, the

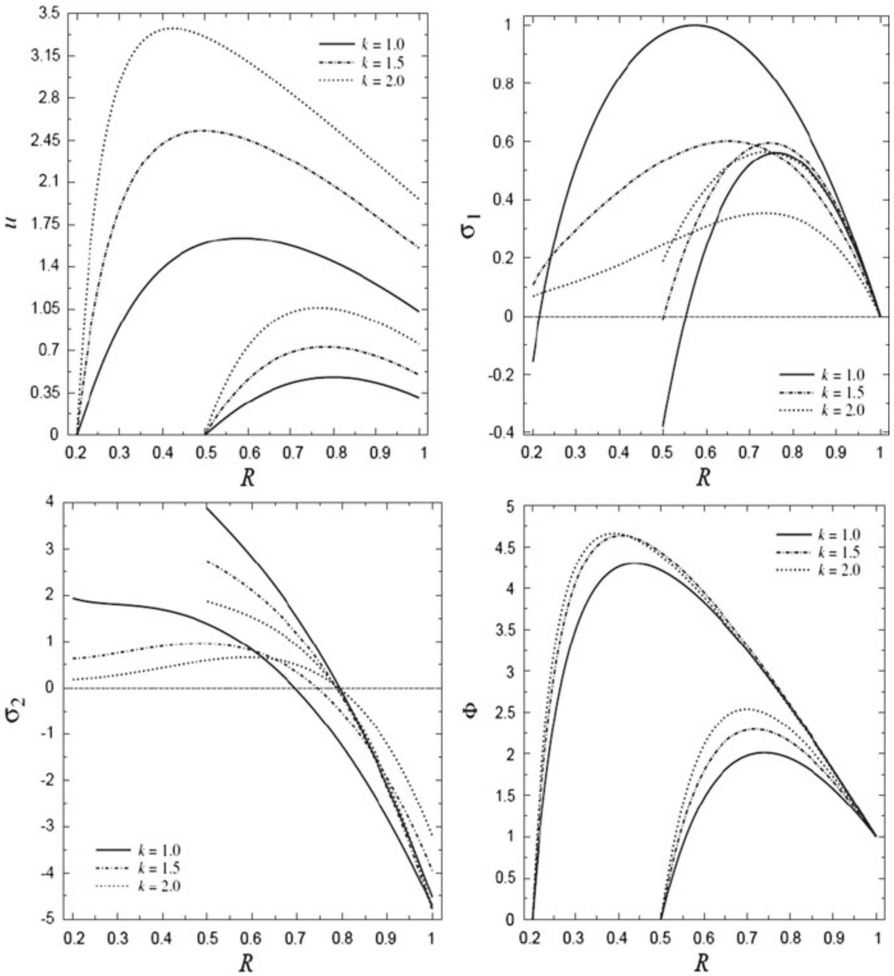

Fig. 4 Distribution of displacement, stresses, and electrostatic potential through the radial direction of the hollow cylinder according to Case 4

outer radius was free from radial stress. The inner radial stress may be negative for $k = 1$ only.

Figure 5 shows the results of an external uniform pressure with a zero electrostatic potential. The radial displacement shows similar behavior in comparisons with that of Fig. 3. Here, the inner radius is free of the radial stress. For $k = 2$, the radial stress is directly increasing along the radial direction to reach its maximum at the outer radius. While for $k < 1$, its maximum shifts to the outer radius.

It should be noted that the radial displacement u experiences a change of sign in Cases 4.1 and 4.3 for free inner and outer radii (Fig. 1) or for a free outer radius (Fig. 3). Also, the radial stress σ_1 experiences a change of sign in Cases 4.2 and 4.4 for clamped inner and outer radii (Fig. 2) or for a clamped inner radius (Fig. 4). However, the hoop

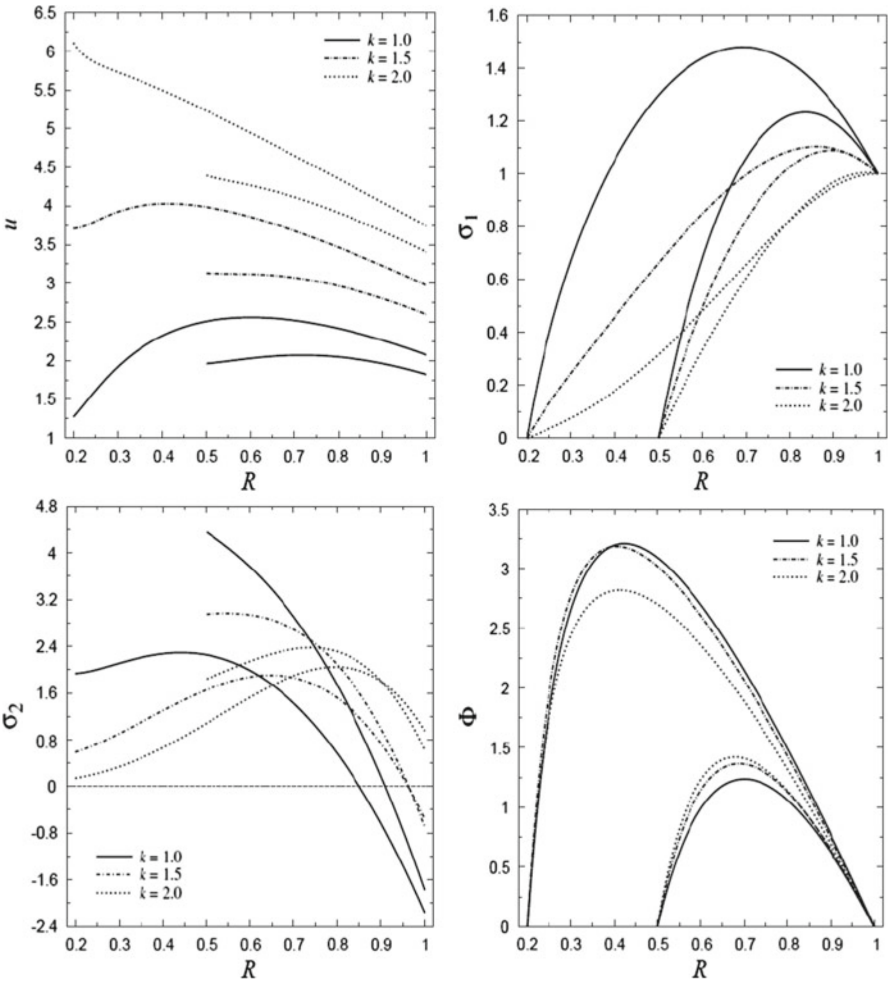

Fig. 5 Distribution of displacement, stresses, and electrostatic potential through the radial direction of the hollow cylinder according to Case 5

stress experiences a change of sign in all cases studied. In fact, all the cylindrical cross sections are in circumferential compression at the inner radii and in circumferential tension at the outer radii. This phenomenon does not occur in purely mechanical cylindrical orthotropic problems. Finally, all cases show the resulting induced electric effect. Although the boundary conditions require that the electric potential be zero or unity at the inner and outer radii, an electric potential has developed in the interior of the cross section.

6 Conclusion

An exact solution for inhomogeneous hollow cylinders subjected to electric, thermal, and mechanical load is obtained. All material coefficients are assumed to have the same exponent-law dependence on the radial direction of the hollow cylinder. The distributions of the dimensionless displacement and stresses as well as dimensionless electric potential curves are drawn and discussed in detail for various boundary conditions. The obtained solution is valid for arbitrary electric, thermal, and mechanical loads applied on the hollow cylinder. The results show that the inhomogeneity exponent has a great effect on the radial displacement, stresses, and electric potential. By selecting a proper value of k and suitable mechanical and electrical loads, it is possible for engineers to design such a cylinder that can meet some special requirements. Finally, this article considers the case in which the material coefficients follow a power function in the radial variable and then the present technique is applicable to other material inhomogeneities. In addition, the analysis approach presented in this study can be applied on all radially polarized piezoelectric cylinders.

References

1. N. Kharouf, P.R. Heyliger, J. Sound Vib. **174**, 539 (1994)
2. D. Galic, C.O. Horgan, J. Elast. **66**, 257 (2000)
3. H.L. Dai, X. Wang, Int. J. Solids Struct. **42**, 1151 (2005)
4. M. Saadatfar, A.S. Razavi, Int. J. Sci. Technol. **23**, 45 (2009)
5. H.M. Wang, H.J. Ding, W. Ge, Compos. Struct. **79**, 192 (2007)
6. H.L. Dai, L. Hong, Y.M. Fu, X. Xiao, Appl. Math. Model. **34**, 343 (2010)
7. N.T. Adelman, Y. Y. Stavsky, J. Sound Vib. **38**, 245 (1975)
8. N.T. Adelman, Y. Stavsky, J. Acoust. Soc. Am. **57**, 356 (1975)
9. N.A. Shul'ga, A.Y. Grigorenko, I.A. Loza, Sov. Appl. Mech. **20**, 26 (1984)
10. H.S. Paul, M. Venkatesan, J. Acoust. Soc. Am. **82**, 952 (1987)
11. X. Wang, Y.N. Gong, Int. J. Eng. Sci. **30**, 25 (1992)
12. X.C. Yin, Z.Q. Yue, ASME J. Appl. Mech. **69**, 825 (2002)
13. X. Han, G.R. Liu, Smart Mater. Struct. **12**, 962 (2003)
14. S. Ueda, J. Therm. Stress. **27**, 291 (2004)
15. P. Lu, H.P. Lee, C. Lu, Int. J. Mech. Sci. **47**, 437 (2005)
16. H.M. Wang, H.J. Ding, Y.M. Chen, Int. J. Solids Struct. **42**, 85 (2005)
17. X. Wang, K. Dong, Int. J. Solids Struct. **43**, 1710 (2006)
18. C-P. Wu, S-W. Yang, Composites B **42**, 1683 (2011)
19. J. Chen, H. Ding, W. Chen, Arch. Appl. Mech. **77**, 241 (2007)
20. M. Asghari, E. Ghafoori, Compos. Struct. **92**, 1092 (2010)
21. W.Q. Chen, Y.F. Wang, J.B. Cai, G.R. Ye, Int. J. Solids Struct. **41**, 2429 (2004)
22. A.M. Zenkour, Arch. Appl. Mech. **76**, 89 (2006)
23. A.M. Zenkour, Acta Mech. **187**, 85 (2006)
24. A.M. Zenkour, Appl. Math. Mech. Engl. Ed. **32**, 507 (2011)