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Mis-Out and Mis-In Examples: The Case of Rational Numbers

Dina Tirosh[1] · Pessia Tsamir[1]

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Abstract

This paper focuses on the definitions and the mis-out and mis-in examples of rational numbers that four prospective elementary teachers presented while working on rational number assignments. The participants were first asked to respond, individually, to an Individual Rational Number Assignment, consisting of items aiming at detecting their personal concept definitions of rational numbers and identifying the entities that they regarded as rational numbers. Then, to share their work with another prospective teacher, to identify similarities and differences in their responses, and to list issues that were raised during the individual or pair work, that they would like to discuss in class. The data exposed a tendency to provide one definition of rational numbers, to identify the term "rational" with "natural", not to include a clarification that a rational number is a number, and a controversy regarding including (or not including) a statement that $b \neq 0$ in the definition. Other observations related to a tendency not to categorize negative numbers (and perhaps also zero) as rational numbers and an inconsistency between their responses to the question "what is a rational number?" and their classification of examples of rational numbers. Recommendations for topics for discussion with prospective teachers, in light of the responses to the assignments, are suggested and methodological issues for considerations are proposed.

Keywords Definitions · Mis-in examples · Mis-out examples · Rational numbers

Prolog

A class of elementary school prospective teachers were provided with a list of mathematical expressions. They were asked to circle all the rational numbers. Roni and Ben (pseudonyms) were arguing about the status of the number 2:

✉ Dina Tirosh
dina@tauex.tau.ac.il

1 Department of Mathematics, Science and Technology Education, Tel-Aviv University, Tel-Aviv, Israel

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Roni1: 2 is a rational number. All the integers are rational numbers.
Ben2: 2 is not a rational number.
Roni3: I can write 2 as $\frac{2}{1}$. So, it is a rational number.
Ben4: We had a mathematical course last year, and we talked about rational numbers … I don't remember the details, but I remember that integers are not rational numbers. It was bizarre, so I remember it. I am sure.
Roni5: I'm almost sure that 2 is a rational number. Do you agree that $\frac{2}{1}$ is a rational number?
Beni6: ummmm…
Roni7: but $\frac{2}{1}$ and 2 is the same number.
Beni8: Yes. But I remember that we learnt that $\frac{2}{1}$ is a rational number but 2 is not. I'll look in my notes.

Introduction

"Rational number" is a mathematical concept. As such, a common expectation is that rational numbers will have a definition that provides a means to determine, unequivocally, if a given entity is (or is not) a rational number. Knowing the definitions of mathematical concepts and the capability to identify examples (and non-examples) according to the definitions of mathematical concepts in general, and of rational numbers in particular, are regarded as essential parts of the mathematical knowledge needed for teaching (Ball et al., 2008; Blömeke et al., 2016; Campbell et al., 2014; Hill et al., 2008; Ottmar et al., 2015; Rowland et al., 2005).

For about 35 years, we have presented elementary school, middle school, and high school mathematics prospective teachers with various tasks related to rational numbers and to other mathematical concepts (e.g., infinity, quadrilaterals, extrema points). We aimed at identifying their related ideas about each concept, about the nature of mathematical definitions and about the role of examples and non-examples in mathematics instruction, as a starting point to teaching various courses. Recently, in several articles, we surveyed our collected data and consequently suggested to distinguish between two types of misidentifications of examples and non-examples of mathematical concepts: mis-out examples versus mis-in examples (e.g., Tirosh & Tsamir, 2022; Tsamir & Tirosh, 2023). Mis-out examples are examples of a concept that are erroneously categorized as non-examples of this concept. Mis-in examples are non-examples of a concept that are erroneously identified as examples of the concept. Evidently, mis-out and mis-in examples of rational numbers should be identified according to a definition of rational numbers.

In this paper, we describe the definitions and the images that were expressed in the responses of four elementary school prospective teachers to rational number assignments. We examine their suggested definitions of rational numbers and analyze the mis-out and mis-in examples of rational numbers that they provide. We share with the readers some insights that we arrive at while looking at the data with these new lenses.

Literature Review

Before detailing the theoretical framework of this article, we address the question: What is a rational number.

What is a Rational Number?

In response to this question, we surveyed mathematical definitions of rational numbers in various sources (e.g., dictionaries, encyclopedias, textbooks, websites). We focused on sources relating to rational numbers as a specific type of numbers (not as order pairs of numbers), as this is the way that rational numbers are commonly presented to elementary prospective teachers. The following are three definitions that various sources provided:

Definition 1 A rational number is often defined as a *number* that *can be expressed* in the form $\frac{a}{b}$, a and b are *integers*. This definition follows the essential, hierarchical criterion that was given by Aristotle for defining a concept, namely, that a new concept is described as a specific case of a more general concept (see for example, Heath, 1956). In the case of rational numbers, this definition addresses a specific case of numbers, listing a minimal set of necessary and sufficient conditions for the concept (Winicki-Landman & Leikin, 2000) and thereby creating two separate sets: A set of examples and a set of non-examples of the concept.

Definition 2 A rational number is *a number* that *can be written* in the form $\frac{a}{b}$, where a and b are *integers* and *b is not equal to 0*.

This definition provides a similar definition to Definition 1, adding that b should not be zero. Note that mathematically, there is no need to mention the $b \neq 0$ condition as it is stated, at the very beginning of the definition, that a rational number *is a number*, and this implies that b could not be zero. Including this unnecessary information in the definition defies the mathematical requirement that the set of conditions that are included in a definition should be minimal (e.g., Khinchin, 1968; Solow, 1984; Vinner, 1991).

Definition 3 Rational numbers are *numbers* of the form $\frac{a}{b}$ where a and b are *integers* and *b is not zero*. This definition, like the previous ones, follows the essential, hierarchical criterion that was given by Aristotle for defining a concept. It also lists a minimal set of necessary and sufficient conditions for the concept. According to this definition, expressions that are not written in the form $\frac{a}{b}$ where a and b are integers, but can be written in this form, are not included in the set of rational numbers.

Examining these definitions from the standpoint of the sets of examples and non-examples that are created by the criteria that each definition poses to determine if a given entity is a rational number reveals that the sets that are created by Definition 1 and Definition 2 are identical. The criteria are: (1) being a *number* (2) *can be written* in the form $\frac{a}{b}$ (3) *a* and *b* are integers. The fourth criterion that Definition 2 poses (i.e., $b \neq 0$) is derived from the first criterion. These two definitions are equivalent.

Definition 3, however, poses the following criteria: (1) being a *number* (2) *is written* in the form $\frac{a}{b}$ (3), *a* and *b* are *integers* (here, too, the fourth criterion, i.e., $b \neq 0$, is derived from the first criterion). Definition 3 is not equivalent to Definitions 1 and 2. The set of rational numbers that is created by Definition 3 is a proper subset of the set of rational numbers that is formed by Definitions 1 and 2. For example, the numbers 2, 0, -8 (and all the other integers written in this form) are examples of rational numbers according to Definitions 1 and 2, but they are non-examples according to Definition 3.

Theoretical Background

In their classical article on concept images and concept definitions, Tall and Vinner (1981, p. 152) suggested the notions concept image, evoked concept images, concept definition, formal concept definition, and personal concept definition. They described the construct concept image in the following manner: "We shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures." Evoked concept image is "the portion of the concept image which is activated at a particular time", Concept definition is "a form of words used to specify that concept." "A formal concept definition, … [is] a concept definition which is accepted by the mathematical community at large." And "Personal concept definition… is the form of words that the student uses for his own explanation of his (evoked) concept image." Vinner further defined the phenomenon of compartmentalization, that is, a situation in which individuals believe in the correctness of two (or more) incompatible images (Vinner, 1990).

A definition of a mathematical notion determines two mutually exclusive sets—a related set of examples and a set of non-examples. However, one's concept images might cause errors in this respect, as examples or non-examples might not be regarded as such. We distinguish between cases where students mistakenly take examples out of the set of examples to address them as non-examples, referred to as *mis-out examples*, and cases of mistakenly including non-examples in the set of examples, referred to as *mis-in examples* (Tirosh & Tsamir, 2022; Tsamir & Tirosh, 2023). We regard the distinctions between mis-out examples and mis-in examples that we recently suggested (and briefly described in the introduction) as part of the concept image construct that is defined in the concept image– concept definition framework. Both mis-out and mis-in examples are erroneously included in the concept images of a concept, and thus deserve specific attention.

In line with our attempts to examine mis-out and mis-in examples of various mathematical notions, we focus in this article on records that describe the responses of four elementary school prospective teachers to rational number assignments. The research questions that serve as the basis of our analysis of these records were:

How do elementary prospective teachers define rational numbers?

Are their example set of rational numbers consistent with their definitions?

What examples (if any) of rational numbers do elementary prospective teachers tend to mis-out?

What examples (if any) of rational numbers do elementary prospective teachers tend to mis-in?

The Study

Participants

Nancy, Ana, Ken and Peter (pseudonyms) were prospective elementary school teachers, in their second out of a four-year undergraduate program toward a certificate for teaching mathematics in elementary schools. They studied in an academic teacher college in Israel. When their class was asked to work in pairs on a given task, Nancy often worked with Ana, and Ken with Peter. Various elements of their records reflect issues that were commonly raised in other records, albeit not always in the same, transparent manner.

Tools

The research tools were two assignments: *The Individual Rational Number Assignment* and *The Pair Rational Number Assignment*.

The Individual Rational Number Assignment This assignment consisted of four items (Fig. 1). Item 1 aimed at detecting the personal concept definition of rational numbers by asking the participants to reply to the question: "What is a rational number?". The other three items attempted to identify the entities that the participants regarded as rational numbers and those that they viewed as not-rational numbers, by asking to list three examples of rational numbers (Item 2), three non-examples of rational numbers (Item 3) and to sort the rational numbers out of a list of 25 expressions (Item 4).

Item 1: What is a rational number?

Item 2: Write 3 examples of rational numbers.

Item 3: Write 3 non-examples of rational numbers.

Item 4: Which of the following are rational numbers?

4, -8, 0, 0.12122122212222…, $\frac{2}{3}, \frac{4}{0}, \frac{0}{2}, \frac{a}{6}, \frac{4}{3}, \frac{\pi}{2}, \frac{x+4}{8}, \frac{-3}{5}; \frac{-5}{3}, \frac{4}{8}, \frac{8}{1}, \frac{6}{2}, \frac{4.5}{6}, \frac{8}{3.5}, \frac{6}{6}, \frac{8}{-11}, \frac{\frac{4}{9}}{\frac{5}{7}}, \frac{\frac{-2}{5}}{\frac{6}{11}}$, 0.3,

0.3333…$\sqrt{7}$

Fig. 1 The Rational Number Assignment

According to Definitions 1 and 2, the list of numbers in Item 4 included 19 rational numbers $\left(\frac{2}{3}, \frac{4}{3}, \frac{4}{8}, \frac{-3}{5}, \frac{-5}{3}, 4, 0, -8, \frac{0}{2}, \frac{6}{6}, \frac{8}{1}, \frac{6}{2}, \frac{\frac{4}{9}}{\frac{5}{7}}, \frac{\frac{-2}{5}}{\frac{6}{11}}, \frac{8}{-11}, \frac{4.5}{6}, \frac{8}{3.5}, 0.3, 0.333\ldots\right)$. According to Definition 3, it included ten rational numbers $\left(\frac{2}{3}, \frac{4}{3}, \frac{4}{8}, \frac{-3}{5}, \frac{-5}{3}, \frac{0}{2}, \frac{6}{6}, \frac{8}{1}, \frac{6}{2}, \frac{8}{-11}\right)$. The list also included the irrational numbers $\sqrt{7}$, $\frac{\pi}{2}$, 0.12122122212222…, the algebraic fractions $\frac{a}{6}$, $\frac{x+4}{8}$ and the undefined expression $\frac{4}{0}$. The collection of Items 2,3, and 4 provided different, yet complementary lenses to study the expressions that each prospective teacher regarded as examples (and those that were regarded as non-examples) of rational numbers.

The Pair, Rational Number Assignment This assignment was based on the same four items that were included in the Individual Rational Number Assignment. Each prospective teacher was asked to share his/her work with another, prospective teacher and to fill in, together, the *Pair, Rational Number Form* (see Fig. 2) according to the following instructions:

Share your individual responses to each task with your pair. Identify similarities and differences in your responses to items 1–4. If you agree on a response to an item (a response that was provided by both of you, by one of you or another response) write it down in the "Pair Response" column, as your mutual response to this item. If you disagree – write your different responses to this item in the "Pair Response" column.

In the "Issues of Concerns" column, list issues that were raised during your individual or pair work, that you would like to discuss in class.

The pair-part of the assignment provided the participants with an opportunity to reflect on their own responses to the Individual Rational Number Assignment with another prospective teacher that they chose to work with, who responded to the same assignment. This mode of work created a situation that encouraged the members of each pair to re-think, in a trustable environment, their own work in the light of another, possibly different responses to the same items, to uncover similarities

Item	Pair Response	Issues of Concerns
1. What is a rational number?		
2. Examples of rational numbers		
3. Non-examples of rational numbers		
4. Identification of rational numbers		

Fig. 2 The Pair, Rational Number Form

and differences, and to raise hesitations and dilemmas about their own definitions, their suggestions of examples and non-examples of rational numbers and their decisions regarding the sorting of expressions. The Pair Rational Number Assignment provided us with an additional window to the prospective teachers' conceptions of rational numbers.

Procedure

Two major stages took place in this study: *The Individual Stage* and the *Pair Stage*.

Stage 1: The individual Stage. In this stage, the prospective teachers worked, individually, on the individual Rational Number Assignment. Each prospective teacher was asked to respond to Items 1,2, and 3 of the assignment. Upon submission, the prospective teachers received the second part of this assignment, consisting of Item 4 of the Individual Rational Number Assignment.
Stage 2: Pair work. At this stage, the prospective teachers were asked to share their work with another prospective teacher and to fill in, together, The Pair Rational Number Form. Upon finishing their work, the prospective teachers submitted their responses to Stage 2 to the lecturer of the course.

The two authors reviewed, together, the two individual responses of each member of each pair to the Individual Rational Number Assignment and their responses to the Pair, Rational Number Form that they had gathered over the years. We chose to analyze the records of Nancy and Ana, and of Ken and Peter because they reflected issues that were commonly raised by other prospective teachers that deserved attention and because Nancy, Ana, Ken and Peter expressed themselves, in writing, in a very clear manner.

Results

In this section, we first share with the readers the responses of Nancy, Ana, Ken and Peter to the Individual Rational Number Assignment, then we describe the responses of Nancy and Ana to the Pair, Rational Number Form and the responses of Ken and Peter to this form.

Individual Responses to the Individual, Rational Number Assignment

Table 1 provides the responses of Nancy, Ana, Ken and Peter to the four items that were included in Stage 1. A glance at the responses reveals that Ken is the only prospective teacher whose responses to all four items are in accordance with one of the definitions (Definition 2).

Table 2 specifies the attributes that were written by each prospective teacher in response to Item 1. Ken and Peter wrote that a rational number is *a number* – an attribute included in all definitions. Three prospective teachers noted

Table 1 Responses of Nancy, Ana, Ken and Peter to the Rational Number Assignment

Item	Nancy	Ana	Ken	Peter
1. What is a rational number?	It can be written as $\frac{a}{b}$ and a and b are integers	It can be written as $\frac{a}{b}$, b is not zero	It is a number. It can be expressed as $\frac{a}{b}$, a and b are integers, b is not zero	It is a number. It is written as $\frac{a}{b}$, a and b are normal numbers
2. Examples of rational numbers	$\frac{1}{2}, \frac{8}{3}, -\frac{7}{2}$	$\frac{3}{4}, \frac{73}{92}, \frac{125}{189}$	$\frac{1}{2}, \frac{7}{3}, \frac{-19}{5}$	$\frac{1}{2}, \frac{5}{6}, \frac{591}{978}$
3. Non-examples of rational numbers	$\frac{\pi}{2}, \sqrt{2}, \sqrt{5}$	$\frac{\pi}{2}, \frac{2}{0}, \sqrt{2}$	$\frac{\pi}{2}, \sqrt{2}, \frac{2}{0}$	$\frac{\pi}{2}, \sqrt{2}, \sqrt{3}$
4. Identification of rational numbers	$0.3, 4, -8, \frac{8}{1}, 0, \frac{6}{6}, \frac{8}{-11}, \frac{0}{2}, \frac{8}{3.5}, \frac{\frac{4}{9}}{\frac{5}{7}}, \frac{\frac{-2}{5}}{\frac{6}{11}}, \frac{4.5}{6}, \frac{4}{8}, \frac{4}{3}, \frac{-3}{5}, \frac{6}{2}, \frac{2}{3}, \frac{-5}{3}, 0.3333\ldots$	$0.3, 4, 0, \frac{2}{3}, \frac{0}{2}, \frac{4}{3}, \frac{4}{8}, \frac{6}{2}, \frac{8}{1}, \frac{8}{3.5}, \frac{4.5}{6}, \frac{6}{6}, \frac{\frac{4}{9}}{\frac{5}{7}}, 0.3333\ldots$	$0.3, 4, -8, \frac{8}{1}, 0, \frac{6}{6}, \frac{-8}{11}, \frac{0}{2}, \frac{8}{3.5}, \frac{4.5}{6}, \frac{4}{8}, \frac{4}{3}, \frac{-3}{5}, \frac{\frac{4}{9}}{\frac{5}{7}}, \frac{\frac{-2}{5}}{\frac{6}{11}}, \frac{6}{2}, \frac{2}{3}, \frac{-5}{3}, 0.333\ldots$	$\frac{8}{1}, \frac{6}{6}, \frac{0}{2}, \frac{4}{8}, \frac{4}{3}, \frac{6}{2}, \frac{2}{3}, \frac{4}{0}$

Table 2 Responses to "What is a rational number? (Item 1)

What is a rational number?	Nancy	Ana	Ken	Peter
a number—Definitions 1 2 3	-	-	+	+
can be written (expressed) as $\frac{a}{b}$- Definitions 1 2	+	+	+	
is written as $\frac{a}{b}$- Definition 3				+
a and b are integers—Definitions 1 2 3	+		+	
a and b are normal numbers				+
b is not zero—Definitions 2 3		+	+	

Table 3 Responses to "examples of rational numbers" (Item 2)

	Nancy	Ana	Ken	Peter
Numbers written in the form $\frac{a}{b}$, $a<b$, a and b are natural numbers	$\frac{1}{2}$	$\frac{3}{4}, \frac{73}{92}, \frac{125}{189}$	$\frac{1}{2}$	$\frac{1}{2}, \frac{73}{92}, \frac{591}{978}$
Numbers written in the form $\frac{a}{b}$, $a>b$, a and b are natural numbers	$\frac{8}{3}$		$\frac{7}{3}$	
Negative numbers	$-\frac{7}{2}$		$\frac{-19}{5}$	

that'it' (Nancy and Ana) or 'the number' (Ken) *can be written as* $\frac{a}{b}$ – an attribute included in Definitions 1 and 2 while Peter wrote that *it is written as* $\frac{a}{b}$ – an attribute included in Definition 3. Two (Nancy and Ken) stated that *a and b are integers*, another attribute included in all definitions. Peter argued that a and b are "normal numbers" (the responses to the Pair, Rational Number Form indicated that normal numbers, according to Peter, were either natural numbers or whole numbers). Finally, Ana and Ken claimed, in line with Definitions 2 and 3, that *b is not zero*.

Notably, all the examples (Table 3) of rational numbers that the four prospective teachers provided are *numbers*, written in the form $\frac{a}{b}$, a and b are integers (and $b \neq 0$). Thus, they were in accordance with Definitions 1, 2, and 3. Notice that most examples were of numbers between zero and one and that all the examples that Ana and Peter provided are such examples. Nancy and Ken wrote an example of a number between zero and one ($\frac{1}{2}$), a number bigger than 1, and a negative number. Similarly, the non-examples that the prospective teachers wrote were not examples of rational numbers according to the three definitions (Table 4). the four prospective teachers wrote two identical non-examples: the irrational numbers $\frac{\pi}{2}$ and $\sqrt{2}$. The third non-examples that Nancy and Peter wrote are irrational numbers ($\sqrt{5}$ and $\sqrt{3}$) while Ana and Ken wrote the undefined $\frac{2}{0}$. Markedly, the examples and non-examples that the prospective teacher presented were consistent with their own responses to the item: "What is a rational number?" (Item 1).

Item 4 called for identifying the rational numbers in a list of mathematical entities. We first examined the responses of the four prospective teachers to this item according to Definitions 1 and 2. Nancy and Ken depicted all the rational

Table 4 Responses to "non-examples of rational numbers" (Item 3)

Non-examples of rational numbers	Nancy	Ana	Ken	Peter
Irrational numbers written as square roots	$\sqrt{2}, \sqrt{5}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}, \sqrt{3}$
The irrational number $\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
The undefined number $\frac{2}{0}$		$\frac{2}{0}$	$\frac{2}{0}$	

numbers (19) that were provided in the list, and only the rational numbers. Ana included 14 rational numbers, missing out the five negative, rational numbers $\left(-8, \frac{-3}{5}; \frac{-5}{3}, \frac{8}{-11}, \frac{\frac{-2}{5}}{\frac{6}{11}}\right)$ and Peter listed seven rational numbers, missing- out the numbers $4, 0, \frac{-5}{3}, 0.3333\ldots, \frac{4.5}{6}, \frac{8}{3.5}, \frac{8}{-11}, \frac{-3}{5}, \frac{\frac{4}{9}}{\frac{5}{7}}, \frac{\frac{-2}{5}}{\frac{6}{11}}$, 0.3, -8 and missing-in the expression $\frac{4}{0.}$

Examining the identification of the rational numbers according to Definition 3 reveals that Nancy and Ken missed-in nine numbers $\left(0.3, 4, -8, 0, \frac{8}{3.5}, \frac{4.5}{6}, \frac{\frac{4}{9}}{\frac{5}{7}}, \frac{\frac{-2}{5}}{\frac{6}{11}} 0.3333\ldots\right)$, Ana missed-out three numbers $\left(\frac{-3}{5}, \frac{-5}{3}, \frac{8}{-11}\right)$ and missed-in six numbers $\left(0.3, 4, \frac{\frac{4}{9}}{\frac{5}{7}}, 0, \frac{8}{3.5}, \frac{4.5,}{6}\right)$, and Peter missed-out three numbers $\left(\frac{-8}{11}, \frac{-3}{5}, \frac{-5}{3}\right)$, and missed-in the expression $\frac{4}{0}$.

An examination of the consistency of the responses of the prospective teachers to the four items revealed that the responses of all of them to Item 1 (their definitions of rational numbers) were consistent with their responses to Item 2 and Item 3 (providing their own examples and non-examples of rational numbers). The responses to Item 4 of both Ken and Peter were in line with their own responses to Item 1. However, according to their own written definitions of rational numbers,

Table 5 Classifications of Expressions as Rational Numbers (Item 4)

Classifications of expressions	Nancy	Ana	Ken	Peter
Numbers written in the form $\frac{a}{b}$, a and b whole numbers $\frac{6}{2}, \frac{4}{8}, \frac{8}{1}, \frac{6}{6}, \frac{4}{3}, \frac{2}{3}, \frac{0}{2}$	+	+	+	+
Numbers written in the form $\frac{a}{b}$, a or b are positive decimals $\frac{4.5}{6}, \frac{8}{3.5}$	+	+	+	
Whole numbers -not written in the form $\frac{a}{b}$ 0, 4	+	+	+	
Numbers written in the form $\frac{\frac{a}{b}}{\frac{c}{d}}$, *a, b, c, d* are natural numbers $\frac{\frac{4}{9}}{\frac{5}{7}}$	+	+	+	
Decimals -terminating and non-terminating- 0.3, 0.3333…	+	+	+	
Negative numbers $\frac{-5}{3}, \frac{8}{-11}, -8, \frac{-3}{5}, \frac{\frac{-2}{5}}{\frac{6}{11}}$	+		+	
Irrational numbers $\frac{\pi}{2}, \sqrt{7}, 0.12122122212222\ldots$				
Algebraic expressions $\frac{a}{6}, \frac{x+4}{8}$				
Undefined expression $\frac{4}{0}$				+

Note. + = classify as rational numbers

Nancy missed-out the expression $\frac{4}{0}$ and Ana missed-out eight expressions $\left(\frac{-3}{5}, \frac{-5}{3}, -8, \frac{8}{-11}, \frac{\frac{-2}{5}}{\frac{6}{11}}, \frac{\pi}{2}, \frac{a}{6}, \frac{x+4}{8}\right)$.

Table 5 reveals that the seven numbers that were written in the form $\frac{a}{b}$ when both a and b were whole numbers, were regarded as rational numbers by the four prospective teachers. However, when one of these numbers (either a or b) was written as a positive decimal, or when the whole numbers were not written in the form $\frac{a}{b}$, or when the number was written as $\frac{\frac{a}{b}}{\frac{c}{d}}$ where $a,b,$ $c,$ d were natural numbers, or when the numbers were terminating and non-terminating decimals, one prospective teacher (Peter) did not classify them as rational numbers. The negative numbers were classified as rational numbers by Nancy and Ken, but not by Peter and Ana.

Table 5 also reveals that the irrational numbers and the algebraic expressions that were included in Item 4 were not classified as rational numbers by the prospective teachers. The expression $\frac{4}{0}$ was classified as a rational number by Peter.

Responses to The Pair, Rational Number Form

Nancy and Ana As can be seen from Table 6, Nancy and Ana agreed that each of them provided appropriate examples and non-examples of rational numbers (Item 2 and Item 3, respectively). Regarding the identification of rational numbers (Item 4), Ana accepted Nancy's position that not only 0.3, 4, 0, $\frac{2}{3}, \frac{0}{2}, \frac{4}{3}, \frac{4}{8}, \frac{6}{2}, \frac{8}{1}, \frac{8}{3.5}, \frac{4.5}{6}, \frac{6}{6}, \frac{\frac{4}{9}}{\frac{5}{7}}$, 0.3333…but also the negative number -8, $\frac{-3}{5}$; $\frac{-5}{3}$ and $\frac{8}{-11}, \frac{\frac{-2}{5}}{\frac{6}{11}}$ are rational numbers. The only item that caused some disagreements and hesitations was Item 1 ("What is a rational number?"). They both wrote that rational numbers can be written as $\frac{a}{b}$ in their individual responses and in their pair response to the assignment. Ana accepted Nancy's position that a and b should be integers. They also wrote, in their Pair Response, that rational numbers are numbers. Interestingly, Nancy claimed that "rational numbers are numbers (it is written in their names)" and Ana accepted her assertion. Thus, in their pair work they agreed on listing three critical attributes of rational numbers (all the attributes of Definition 1 and three of the four of Definition 2). They disagreed on writing (or not writing) that b should not be zero (the fourth attribute according to Definition 2). Nancy claimed that there is no need to mention that b should not be zero because the word "number" is included in "rational numbers" and since $\frac{a}{b}$ is a number, b could not be zero. She also noted that "it is *forbidden* to write that b is not zero because definitions are minimal". Ana, however, although she accepted Nancy's position that rational numbers are numbers, felt that "*it should be written* that b is not zero -to make it clear".

Notably, Item 1 is not only the sole item that Nancy and Ana had some opposing opinions about, but also the only item that they felt a need to discuss in class, as expressed in the Issues of Concerns column. Nancy and Ana stated, in that column,

Table 6 Individual and Mutual Responses of Nancy and Ana

Item	Nancy	Ana	Pair Response	Issues of Concerns
1. What is a rational number?	It can be written as $\frac{a}{b}$ and a and b are integers	It can be written as $\frac{a}{b}$, b is not zero	We agree that it can be written as $\frac{a}{b}$ and that *a* and *b* are integers. We disagree about writing that *b* is not zero (Nancy said -no need to write it because division by zero is not a number and rational numbers are numbers (it is written in their names). We agree that rational numbers are numbers but Ana thinks that it should be written that *b* is not zero – to make it clear.	We know that rational numbers can be written as $\frac{a}{b}$and that *a* and *b* are integers. We do not agree – and would like to know if it is correct to write that *b* is not zero. One of us (Nancy) argues that it is forbidden to write it because rational numbers are numbers (it is written in their names) and it is forbidden to write that b is not zero because definitions are minimal. We agree that rational numbers are numbers and that division by zero is not a number. But, one of us (Ana) thinks that it is important to make it clear that *b* is not zero
2. Examples of rational numbers	$\frac{1}{2}, \frac{8}{3}, -\frac{7}{2}$	$\frac{3}{4}, \frac{73}{92}, \frac{125}{189}$	We agreed that all these are examples of rational numbers (also the negative ones)	
3. Non-examples of rational numbers	$\frac{\pi}{2}, \sqrt{2}, \sqrt{5}$	$\frac{\pi}{2}, \frac{2}{0}, \sqrt{2}$	We agreed that all these are non-examples of rational numbers	
4. Identification of rational numbers	$0.3, 4, -8, \frac{8}{1}, 0, \frac{6}{6}, \frac{8}{-11}, \frac{0}{2}, \frac{8}{3.5}$ $\frac{4.5}{6}, \frac{4}{8}, -\frac{4}{3}, \frac{-3}{5}, \frac{\frac{-2}{5}}{\frac{6}{11}}$ $\frac{\frac{4}{9}}{\frac{5}{7}}, \frac{6}{2}, \frac{2}{3}, \frac{-5}{3}, 0.3333\ldots$	$0.3, 4, 0, \frac{2}{3},$ $\frac{0}{2}, \frac{4}{3}, \frac{4}{8}, \frac{6}{2},$ $\frac{8}{1}, \frac{8}{2.5}, \frac{4.5}{6}, \frac{6}{6},$ $\frac{\frac{4}{9}}{\frac{5}{7}}, 0.3333\ldots$	We agreed that all the 19 are rational numbers	

Table 7 Individual and Mutual Responses of Ken and Peter

Item	Ken	Peter	Pair Response	Issues of Concerns
1. What is a rational number?	It is a number. It can be expressed as $\frac{a}{b}$, a and b are integers, b is not zero	It is a number. It is written as $\frac{a}{b}$, a and b are normal numbers	A rational number is a number -It can be written as $\frac{a}{b}$, a and b are the normal numbers 1, 2, 3…. and maybe zero	We know that rational numbers are numbers, that they can be written as $\frac{a}{b}$, a and b are normal numbers, as they are the normal numbers that we use. We think that negative numbers are not rationals because they are not the regular numbers . We are sure that 1,2,3,… are normal numbers but – what about zero? Is zero a normal number?
2. Examples of rational numbers	$\frac{1}{2}, \frac{7}{3}, \frac{-19}{5}$	$\frac{1}{2}, \frac{5}{6}, \frac{591}{978}$	All but $\frac{-19}{5}$ are rational numbers	We are not sure about $\frac{2}{0}$. It is a rational number if zero is a normal number. If zero is not – it is not.
3. Non-examples of rational numbers	$\frac{\pi}{2}, \sqrt{2}, \frac{2}{0}$	$\frac{\pi}{2}, \sqrt{2}, \sqrt{3}$	all but $\frac{2}{0}$ are not examples of rational numbers	
4. Identification of rational numbers	$0.3, 4, -8, \frac{8}{1}, 0, \frac{6}{6}, \frac{-8}{11}, \frac{0}{2}, \frac{8}{3.5}, \frac{4.5}{6}, \frac{4}{8}, \frac{4}{3}, \frac{-3}{5}, \frac{\frac{4}{9}}{\frac{5}{7}} \frac{\frac{-2}{5}}{\frac{6}{11}}, \frac{6}{2}, \frac{2}{3}, \frac{-5}{3}, 0.333\ldots$	$\frac{8}{1}, \frac{6}{6}, \frac{0}{2}, \frac{4}{8}, \frac{4}{3}, \frac{6}{2}, \frac{2}{3}, \frac{4}{0}$	$\frac{6}{2}, 4, \frac{8}{1}, \frac{4}{3}, \frac{4}{8}, 0, \frac{6}{6}, \frac{2}{3}, \frac{0}{2}, \frac{4}{0}$, are rational numbers	We do not know about 0, $\frac{0}{2}$ and $\frac{4}{0}$ (yes if zero is a normal number)

their diverse positions on the issue of writing (or not writing) that b should not be zero in the definition, raising it as an issue of concern.

Ken and Peter Table 7 shows that Ken and Peter agreed, in line with Definitions 1 and 2, that a rational number is a number and that it can be written as $\frac{a}{b}$ (Peter accepted Ken's position that *numbers that can be written as* $\frac{a}{b}$ are also rational numbers). However, they also wrote that a and b are the numbers 1, 2, 3…. and "maybe zero". Thus, Ken gave up his position that a and b are integers (a position in line with Definitions 1, 2, and 3 of rational numbers). He accepted Peter's judgement that a and b should be "normal numbers" (clearly natural numbers and maybe also zero). The issues of concern that they raised (in each and every item) related to the status of zero, explaining that they knew that 1,2,3,4… are normal numbers but they were not sure if zero was also a normal number.

Discussion

In this section, we address each of the four research questions. We also put the findings in the context of relevant research.

Prospective Teachers' Definitions of Rational Numbers

The first research question was: How do elementary prospective teachers define rational numbers? A question that naturally comes to mind is: Are the personal concept definitions of rational numbers of the prospective teachers consistent with the formal concept definitions of this concept? In this regard, we remind the readers that in the literature review, we listed three definitions of rational numbers, stressing that according to two definitions (Definitions 1 and 2), a rational number is (1) a number (2) can be written in the form $\frac{a}{b}$ and (3) a and b are integers. The nuance among these two definitions, that does not change the two, distinct sets of examples and non-examples of rational numbers that are formed via these two definitions, is stating (in Definition 2), or not stating (in Definition 1) that $b \neq 0$. Two of the three critical attributes of rational numbers that are posed by Definition 3 are identical to those that are posed by Definitions 1 and 2 (being a number, a and b are integers). The third condition, however, is that a and b *are written* in the form $\frac{a}{b}$ (and not that they *can be written* in this form). Consequently, while Definitions 1 and 2 form the same set of examples of rational numbers, the set of rational numbers resulting from Definition 3 is a proper subset of that set.

The examinations of the personal concept definitions of the prospective teachers, according to the definitions of rational numbers, revealed that two prospective teachers (Nancy and Ana) mentioned, in their individual works, some of the critical attributes of Definition 1 and 2 but neither of them listed *all* the critical attributes of one of these definitions. However, in their mutual work, they listed *all* the critical attributes that constitute Definition 1. They raised the issue of the status of the claim that b should not be zero (thereby doubting the single difference between

Definition 1 and Definition 2, and listing it as an issue of concern). Another prospective teacher, Ken, wrote the four, critical attributes of rational numbers that constitute Definition 2 (including the statement that *b* should not be zero). As noted in the literature review, definitions of rational numbers that relate to rational numbers as a set of numbers include a statement that rational numbers are numbers, and thus adding that *b* should not be zero is redundant and violates the minimal requirement of definition. Yet, as exemplified in the literature review, some definitions of rational numbers include the statement that *b* should not be zero. In fact, the issue of whether mathematical definitions should be minimal is a debatable issue, especially when considering didactical aspects. The mathematical and the pedagogical issues related to the minimality requirement are discussed in several articles (e.g., Avcu, 2023; Haj-Yahya, 2022; Leikin & Winicky-Landman, 2000; Leikin & Zazkis, 2010; Torkildsen et al., 2023; Ulusoy, 2021; Usiskin et al., 2008; Zazkis & Leikin, 2008).

In addition to the attributes of rational numbers that are included in at least one of the definitions of rational numbers, and were mentioned by the prospective teachers, one prospective teacher (Peter) raised a critical attribute that is not part of the definitions of rational numbers. He claimed that *a* and *b* should be "normal numbers". He probably employed one of the daily connotations of the notion rational, that of being normal, to the concept rational numbers and provided only natural numbers (and perhaps also zero but certainly not negative numbers) the status of normal numbers, and thus rational numbers. One possible source of the observed linkage that Peter made between rational numbers and natural numbers is that rational numbers as a topic, in various school systems, including in Israel, are widely populated only with positive whole number examples and definitely not with negative numbers. Additionally, a glance into the history of the development of the concept of number reveals that the concept of negative numbers was resisted by non-professionals and by mathematicians until the seventeenth century and accepting zero as a number was also a long, debatable issue (Fischbein, 1987; Korry, 2015).

Students often experience similar difficulties to those reported in the historical development of mathematical concepts while studying them (e.g., Anthony & Walshaw, 2004; Fischbein, 1987; Wilson, 2001). We also note that while the responses of Nancy, Ana and Ken to the question: "What is a rational number?" are quite frequent among elementary prospective teachers, the assertion that *a* and *b* are normal numbers is not a common response to this question.

Prospective Teachers' Example Sets of Rational Numbers

The three other research questions related to the example set of rational numbers that was inferred by the prospective teachers' responses to the rational number assignments. The answer to the question: "Are the prospective teachers' example set of rational numbers consisted with their definitions?" Is that all the examples and all the non-examples of rational numbers that the prospective teachers provided were in accordance with their own definitions of rational numbers. The manner in which two prospective teachers (Peter and Ken) categorized the mathematical expressions in the classification item was consistent with their definitions of rational numbers.

Yet, the two other prospective teachers (Nancy and Ana) missed out some expressions, according to their own definitions. One possible explanation for these inconsistencies between the definitions and the classification is via the phenomenon of compartmentalization, that is, that Nancy and Ana believed in the correctness of two incompatible images. Nancy argued, on the one hand, that rational numbers can be written as $\frac{a}{b}$ and that *a* and *b* could be any integers– but that $\frac{4}{0}$ is not a rational number although both 4 and 0 are integers. Ana claimed that rational numbers can be written as $\frac{a}{b}$, b is not zero. She did not put any restrictions on *a* and only one constraint on *b* ("b is not zero"), and yet she did not include negative numbers, irrational numbers and algebraic expressions in her list of rational numbers. Another possible explanation is related to the issue of what should be included in a definition of a mathematical concept (e.g., Gilboa et al., 2023). Nancy and Ana did not write, in their individual responses to the question "What is a rational number?", that a rational number is a number. Nancy did not state that $b \neq 0$ (Ana did), and thus, according to Nancy's response, $\frac{a}{0}$ (*a* is integer) is a rational number. In their Pair Response, Nancy clarified that there is no need to mention that a rational number is a number because "it [being a number] is written in their names" and Ana accepted Nancy's position. Thus, the definitions of rational numbers that they constructed do not follow the criterion that a new concept should be described as a specific case of a more general concept, and that the more general concept should be mentioned in the definition.

The two, additional research questions related to mis-out and mis-in examples of rational numbers. At this stage, we ask the reader to return to the issue that was raised in the Prolog. According to Definitions 1 and 2, the number 2 is a rational number, However, according to Definition 3, it is not a rational number. A similar situation was evident with the classification item. Some expressions in this list are categorized as rational numbers according to Definitions 1 and 2, which are equivalent definitions, but not according to Definition 3, a definition that is not equivalent to the other two definitions. Moreover, if we enlarge the realm of definitions of rational numbers to include not only those relating to rational numbers as a type of numbers but also as an "equivalence classes of ordered pairs of integers, any two pairs, (*a, b*) and (*c, d*) being equivalent if ad = bc" (Borowski & Borwin, 1991), the integers are not a subset of the rational numbers but are equivalent to this set. Thus, the identification of mis-out and mis-in examples depends on the specific definition of rational numbers.

The situation of the dependency of the example set of rational numbers (and thus, the mis-out and mis-in examples) on the chosen, mathematical definition is evident not only when addressing the mathematical definitions of rational numbers but also when referring to the rich literature on the learning and teaching of rational numbers in mathematics education. In mathematics education, rational numbers are often represented as a multidimensional construct consisting of several subconstructs. Behr et al. (1983), for instance, described six subconstructs of rational numbers: a part-to-whole comparison, a decimal, a ratio, an indicated division (quotient), an operator, and a measure of continuous or discrete quantities. Notably, various expressions are identified as rational numbers according to some of these constructs but not according to others. For instance, the expression $\frac{5}{3}$ is naturally identified as a rational number when implementing the ratio construct, but not the part to-whole comparison construct. Thus, in mathematics education, different subconstructs define different sets of examples (and non-examples).

Conclusions and Implications

In this section, we first refer to the contribution of this paper to our knowledge and understanding of elementary, prospective teachers' conceptions of rational numbers and to possible implications of this knowledge to a design of effective education. Then we comment on the methodology that was used in the study.

Prospective Teachers' Conceptions of Rational Numbers and Related Instruction

In this paper, we highlighted some issues regarding elementary prospective teachers' conceptions of rational numbers. The reaction to the question: "What is a rational number?" exposed a tendency to provide one, specific response to this question, to identify the term "rational" with "natural", not to include a clarification that a rational number is a number, and a controversy regarding including (or not including) a statement that $b \neq 0$ in the response to this question. Other observations related to a tendency not to categorize negative numbers (and perhaps also zero) as rational numbers and an inconsistency between the responses to the question "What a rational number?" and the sorting of examples of rational numbers. Teacher educators could benefit from considering these observations when designing their instruction. They could use the prospective teachers' ideas about rational numbers that were described in this paper as a springboard to address specific issues related to rational numbers and more general issues regarding mathematics and mathematics instruction. We shall point at some possible directions.

The tendency to provide one, specific response to the question "What is a rational number?" could lead to a discussion on various, mathematical definitions and subconstructs of rational numbers, to the importance of attending to a specific context when dealing with the definition of the concept of rational numbers, and with equivalent and non-equivalent definitions. The issue of equivalent and non-equivalent definitions is relevant not only to rational numbers but more generally to mathematics and mathematics education. It is dealt with in various articles, some addressing specific concepts and others presenting general views. In the parentheses we listed some of the articles that could contribute to a discussion on equivalent and non-equivalent definitions (Fujita, 2012; Kontorovich et al., 2021; Leikin & Winicki-Landman, 2000; Usiskin et al., 2008; Winicki-Landman et al., 2000). Such discussions could naturally lead to exploring various definitions and discussing the dependency of the example sets of rational numbers (and thus of the mis-out and mis-in examples) on the specific definition (or construct) of rational numbers and of similar situations with other mathematical concepts.

Another issue (exemplified by the response of Peter, in this paper) was to argue that a rational number is a number written as $\frac{a}{b}$, and that a and b are normal numbers (i.e., natural numbers and possibly also zero, but not negative numbers). The identification of "rational number" with "normal numbers" could lead to a discussion on the discrepancies between daily usages and mathematical usages of certain notions (Fischbein & Baltsan, 1999, could serve as one source for discussing this issue). It also could call for a description of the resistance, for decades, to grant zero and negative numbers the status of numbers and more generally, to the possible roles of the history of mathematics in the learning and teaching of mathematics (e.g., Charalambous et al., 2009; Jankvist, 2009; Tzanakis & Arcavi, 2000).

Hesitations about the status of minimality in the definition of rational numbers was raised in the responses of Nancy and Ana to the "Issue of Concern". We suggest to raise the prospective teachers awareness of the related debate. The articles that were mentioned in the discussion could form the basis for such conversation. Similarities and differences between mathematical definitions and everyday definitions could be dealt with in this discussion (Kotsopoulos, 2007 and Vinner, 1991 are two related sources).

Last but not least –The inconsistencies between the responses of Nancy and Ana to the question "What is a rational number?" and their classification of examples of rational numbers, could lead to presenting the concept image-concept definition theory, mis-in and mis-out examples and the phenomenon of compartmentalization to prospective teachers. Acquaintance with this framework could provide the prospective teachers with lenses for observing, identifying, analyzing and designing ways of reacting to various occurrences in their future, mathematics classes (Tall & Vinner, 1981; Tirosh & Tsamir, 2022; Tsamir & Tirosh, 2023; Vinner, 1991 are related sources).

Prospective Teachers' Conceptions of Mathematical Notions – Some Methodological Issues

We would like to end this article by sharing with the readers some thoughts about conducting research that aims at learning about prospective teachers' ways of thinking about mathematical notions. The first relates to the items that we chose to include in the rational number assignments. The information that we gathered from the chosen items raises the issue of which items in an assignment are valuable for this purpose, given that the number of items that can be included in any assignment is limited. Regarding the personal concept definition – we chose to ask: "What is a rational number?" Our impression from the participants' responses is that they aimed at providing definitions of the notions. Still, would asking, directly, to write a definition of a rational number result in a different response? What are the pros and cons of asking, directly, to write a definition?

Another decision was to include in the rational number assignments two types of items, aiming to expose the participants' example set of rational numbers: offering their own examples and non- examples and sorting out examples of rational numbers from a given list of expressions. We realized that asking to provide several examples and several non-examples might result in a list of prototypical examples and of obvious non-examples. However, asking to provide three examples (and three non-examples) opened a window to the personal example spaces of rational numbers of the participants (Watson & Mason, 2005). Asking to sort out examples from a given list of items provides valuable information about the concept images of the participants in the study. However, creating worthwhile lists of entities in a sorting list presupposes a body of knowledge of typical, mis-out and mis-in examples of the concept at hand.

A third decision was to include two specific stages in the research: individual work and pair work. The combination of these two stages supplies information that could not be gathered from employing one of these stages. The pair work, after working alone on the assignment, raises hesitations and dilemmas, leading to pose relevant issues of concerns. Three words of caution are in place here. First, it might happen that two individuals choosing to work together will provide very similar responses to their individual work. Our experience, however, is that even if it happened, issues of concern are almost always raised by each pair of prospective teachers after the pair stage. Second, in some cases (such as that of Ken and Peter) the members of the pair may choose to adapt a position that does not match any accepted mathematical convention. Such occurrences should be dealt with in a class discussion following the activity. Third, we are not arguing that these two stages are the only possible approach to study prospective teachers' personal concept definitions, and concept images (including mis-in and mis-out examples) and to elicit and discuss relevant, related issues. There are of course, other ways to encourage the participants to reflect on their own responses (e.g., searching in encyclopedias, dictionaries, sites).

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Declarations

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