Pre-service Mathematics Teachers' Understanding of Rate of Change Throughout a Model Development Sequence

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Abstract

This study examined a cohort of middle school pre-service mathematics teachers' understanding of the rate of change as they engaged in a model development sequence. By adopting a design-based research perspective, a model development sequence on the concept of rate of change has been designed and implemented as part of a mathematical modeling course for pre-service teachers. The data were collected from twenty senior year middle school pre-service mathematics teachers (PSTs) through questionnaires, modeling activities, reflection papers, and semi-structured interviews. The data analysis showed that PSTs have difficulties conceptualizing the rate of change and conceiving it as a multiplicative comparison of changes in two covarying quantities. As they frequently employed its percentage interpretation, PSTs experienced additional difficulty conceiving the conventional meaning of rate of change in a population growth context. PSTs generally used motion context as a reference while explaining the rate of change in different non-motion contexts. In general, PSTs developed their conception of the additive rate of change throughout the model development sequence. However, for some PSTs, difficulty in ratio-based reasoning on the rate of change in different non-motion contexts prevailed. We provided some arguments concerning the teaching and learning of rate of change.

Keywords Mathematical modeling · Pre-service mathematics teachers · Quantitative reasoning · Rate of change

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Introduction

The concept of rate of change is indispensable for understanding the foundational differential calculus concepts such as the derivative (Confrey & Smith, 1994; Thompson, 1994a). Nevertheless, not only that student difficulties in understanding the derivative are often rooted in their weak understanding of the concept of function and rate of change (Bezuidenhout, 1998; Tall, 1992; Zandieh, 2000), but also the concept of rate of change is often underemphasized in curricular documents (Bezuidenhout, 1998; Bingolbali, 2008; Herbert & Pierce, 2012; Teuscher & Reys, 2012). The concept of rate of change can be considered in three different forms: a directly experienced quantity as fastness, a quantitative operation producing a new quantity resulting from the multiplicative comparison of two quantities, or a formal mathematical interpretation (e.g., slope, the quotient of difference) (Stroup, 2002). The ability to form coordination between these three views has been identified as critical for a robust conceptualization of the rate of change (Confrey & Smith, 1995; Stroup, 2002; Thompson, 1994a).

As extensively explained by Kertil (2014), various studies on students' understanding of rate of change (e.g., Bezuidenhout, 1998; Byerley & Thompson, 2017; Coe, 2007; Confrey & Smith, 1994, 1995; de Beer et al., 2015; Herbert & Pierce, 2008; Teuscher & Reys, 2012; Thompson, 1994a, 1994b; Wilhelm & Confrey, 2003) point out that students from different grade levels have the following difficulties or misconceptions regarding the rate of change concept.

- Difficulty giving meaning to the term "rate of change" (e.g., confusion between "average rate of change," "average value of a continuous function," and "arithmetic mean") (Bezuidenhout, 1998; de Beer et al., 2015; Orton, 1983; White & Mitchelmore, 1996)
- Difficulty interpreting the rate of change in non-motion contexts (Gravemeijer & Doorman, 1999; Herbert & Pierce, 2008, 2012; Stroup, 2002; Wilhelm & Confrey, 2003; Zandieh & Knapp, 2006)
- Conceiving the rate of change as the amount of change in the dependent variable (Byerley & Thompson, 2017; Rowland & Jovanoski, 2004; Thompson, 1994b)
- Difficulty interpreting the rate of change in non-linear functional situations and reasoning with negative rates (Ärlebäck & Doerr, 2018; Feudel, 2017; Mkhatshwa & Doerr, 2018; Teuscher & Reys, 2012)
- Difficulty making connections among the rate of change, slope, and difference quotient (Byerley & Thompson, 2017; Herbert & Pierce, 2008; Teuscher & Reys, 2012; Zandieh, 2000)

The studies mentioned above indicate that for a conceptual understanding, non-motion real-life contexts should be used explicitly in teaching and learning of derivative and rate of change. Additionally, researchers assert that students' difficulties with the rate of change are closely related to their covariational reasoning ability, which should be considered in any learning activity that aims at

developing learners' understanding of the rate of change (e.g., Carlson et al., 2002; Kertil et al., 2019; Thompson & Carlson, 2017). Thus, the present study aimed to contribute to the literature by describing prospective middle school mathematics teachers' (PSTs) existing and developing ways of thinking regarding the concept of rate of change throughout a model development sequence involving activities from non-motion real-life contexts. The research questions that guided the study were (*i*) How do PSTs reason in situations involving the rate of change? and (*ii*) How does PSTs' understanding of the rate of change develop throughout a model development sequence?

Research on Rate of Change

The contextual meaning of derivative and rate of change is frequently confused with the amount of change in the output variable (Rowland & Jovanoski, 2004; Thompson, 1994a, 1994b; Zandieh & Knapp, 2006). However, students might not necessarily need a ratio-based understanding to determine the variation in the intensity of change between the covarying quantities (Johnson, 2012; Kertil, 2014; Zandieh & Knapp, 2006). Byerley and Thompson (2017) reported that among 251 secondary school mathematics teachers, only a few could conceive the rate of change as comparing the relative sizes of two covarying quantities.

As reported in various studies, the rate of change is generally introduced in a motion context (e.g., distance vs. time), and students have difficulties in transferring these concepts into non-motion contexts (Gravemeijer & Doorman, 1999; Herbert & Pierce, 2008; Kertil, 2014; Mkhatshwa & Doerr, 2018; Wilhelm & Confrey, 2003; Yoon et al., 2010). Herbert and Pierce (2011) observed that 10th-grade students might see speed "as a single entity with little emphasis on the covariance of the variables of time and distance" (p. 476). Motion context seems to be not cognitively demanding for students to think about the rate of change resulting from the simultaneous covariation between distance and time (e.g., Gravemeijer & Doorman, 1999; Herbert & Pierce, 2008; Thompson, 1994b).

As extensively reported in the study by Kertil (2014), various intervention studies using design research perspectives on teaching the rate of change have reported some effective practices (Ärlebäck et al., 2013; Doerr & O'Neil, 2011; Doorman & Gravemeijer, 2009; Herbert & Pierce, 2008; Hoffkamp, 2011). Doorman and Gravemeijer (2009) demonstrated that after a teaching experiment based on Realistic Mathematics Education, the majority of the 10th-grade students showed the ability to use the difference quotient and interpret the rate of change on the graph by considering the dimensions of the variables. This was also observed by Hoffkamp (2011), in which 10th-grade students worked on interactive activities related to calculus concepts. In both studies, students developed a mathematical sense for the qualitative aspect of the rate of change and formed connections between the amount and rate graphs. In a study by Herbert and Pierce (2012), secondary school students clarified the meaning of speed as a rate of change in distance with respect to time. Still, their difficulty in transferring this idea to the non-motion contexts continued. The results of these studies are concurrent with the qualitative calculus notion introduced by

Stroup (2002) in terms of evidencing the possibility of developing students' intuitive understanding of rate and amount.

Conceptual Framework

From a quantitative reasoning perspective, the concept of rate of change is a quantitative multiplicative relationship comparing the changes in two functionally related quantities (Thompson, 1994a, 1994b). According to Thompson (1994b), the rate and ratio concepts are products of mental operations by which people comprehend situations. That is, the mental operations of a person who, for instance, tries to quantify distance traveled within the duration of the trip (time) can be characterized as a ratio or rate independent of the quantities having like or unlike nature. Thompson (1994b) characterized the rate as a multiplicative comparison of changes in two simultaneously changing quantities.

According to Thompson (1994b), rate and rate of change concepts are intensive quantities, and they can be quantified in two different ways: *gross quantification* and *extensive quantification*. Gross quantification of the rate of change in different situations involves some perceptual and experiential criteria. In contrast, extensive quantification involves a ratio-based conception of it as a multiplicative relationship between two simultaneously changing quantities. While perceiving speed directly as "seems faster" is an example of gross quantification, perceiving it as a new quantity resulting from the ratio between distance and time and stating it with the measured values, such as 50 km per hour, involves an extensive quantification (Kertil, 2014; Kertil et al., 2019).

Gross quantification of the rate of change coined by Thompson (1994b) has been more comprehensively explained by Stroup (2002) with the notion of *qualitative calculus* that concerns calculus-related ideas of young learners emerging in real situations, but not linked to ratio-related ideas (e.g., slope, ratio) of standard calculus curriculum. The intensification of rate and forming reversibility between "how much" (amount) and "how fast" (rate) in different situations might be considered examples of qualitative understanding. Stroup (2002) argues qualitative calculus as being "cognitively significant and structural in its own right" (p.170) rather than being transitional to more formal ratio-based ways of thinking. According to Stroup, qualitative aspects of calculus ideas are underestimated or assumed to be developed by formal instruction. However, this is not the case in practice, as evidenced by many research studies (e.g., Kertil & Küpcü, 2021; Wilhelm & Confrey, 2003).

The rate of change may be conceptualized differently in various functional situations. By investigating students' way of thinking in exponential situations, Confrey and Smith (1994) determined three possible ways of interpreting the rate of change: *(i) an additive rate of change, (ii) a multiplicative rate of change, and (iii) a "proportional new to the old" rate of change*. The additive rate of change is the conventional understanding of the concept and is obtained as the ratio between differences in succeeding values of independent and dependent variables. The multiplicative rate of change is obtained by taking the ratios between the succeeding values of the dependent variable for every unit change in the independent variable.

Confrey and Smith (1994) reported that most of the students preferred the multiplicative interpretation of the rate in the population growth context. They described the rate as "a unit per unit comparison" of simultaneously changing quantities where the unit is defined as an "invariant relationship between a successor and its predecessor" (p. 142)—e.g., taking the difference between succeeding values of time yields 1, an invariant additive unit. With the 1 unit change in time, there is also a change in the population. The ratio between the two differences is the additive rate of change. Moreover, if we consider a tabular population data with 1 unit time increments, and if the ratio between subsequent cells in the population column is nine, this ratio is an invariant multiplicative unit (Kertil, 2014). In the multiplicative unit, we look at the ratio between the succeeding values from the same measure space. So, the unit can be additive or multiplicative, and one can prefer to use one of them according to the situation.

Method

Design-based studies usually involve designing and implementing innovative learning artifacts and generating domain-specific arguments related to more effective ways of teaching and learning (Cobb & Gravemeijer, 2008; Cobb et al., 2003, 2017). In this study, a model development sequence (Lesh et al., 2003) on the rate of change was designed, experimented with, and evaluated from a design-based research perspective in a small-scale classroom experiment.

The current study was part of a broader investigation of pre-service teachers' developing conceptions of derivative. The more extensive study involved designing, implementing, and refining a model development unit within a mathematical modeling course for pre-service teachers (PSTs) in two iterations over two consecutive semesters. The implementation of a model development unit on the derivative concept lasted eight-week and consisted of three model development sequences (MDS), each focused explicitly on the (*i*) covariational reasoning, (*ii*) rate of change, and (*iii*) graphical connections between a function and its derivative (see Fig. 1). The current study reports on the data collected through the second implementation of the model development sequence designed to reveal and foster PSTs' understanding of the rate of change.
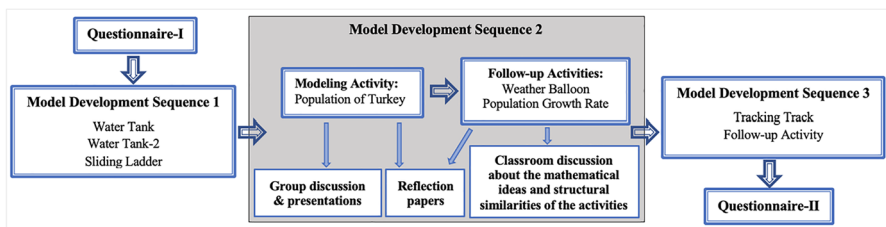

Fig. 1 Implementation of the model development unit consisting of three MDS

The Model Development Sequence

A model development unit involves developing the modeling and follow-up activities, sequencing the activities, and organizing group presentations and classroom discussions (Lesh et al., 2003). Using group work, classroom discussions, and contextual mathematical tasks, designing a model development sequence requires highlighting cognitive and social aspects of students' conceptual development to design an effective learning environment. According to Cobb et al. (2017), classroom design studies involve specifying goals for students' learning, determining the instructional starting points, deciding a hypothetical learning trajectory, and placing the study in a theoretical context to develop domain-specific instructional arguments.

To develop the instructional unit in this classroom design study, based on the theoretical perspectives and the relevant literature (e.g., Stroup, 2002; Thompson & Carlson, 2017; Thompson, 1994b), we first identified that covariational reasoning, rate of change, and graphical connections between a function and its derivative are the key concepts for understanding the derivative. Next, we specified the design principles (see Table 1), which guided the planning of a hypothetical learning trajectory (HLT) (Cobb & Gravemeijer, 2008; Simon, 1995).

We then developed an HLT by reviewing the studies on learners' understanding of and difficulties with the concept of derivative. As the current study focused on the pre-service mathematics teachers' understanding of the rate of change, Table 2 shows HLT consisting of the big ideas, learning goals, and the corresponding mathematical tasks (the Population of Turkey and follow-up activities) for the second model development sequence (MDS-2).

MDS-2 involved a main task, the population of Turkey, and two follow-up activities, the Weather Balloon and the Population Growth Rate (see the Appendix). Based on the relevant literature, we decided to use a non-motion real-life context in the instructional sequence. Thus, the model development sequence began with the Population of Turkey task, which asked PSTs to focus on and make sense of the effect of the change in one variable on change in another variable and coordinate the changes in two variables within a context. In this task, a tabular data with unequal increments in the independent variable was also provided. We expected that by providing these data, PSTs would see the pattern of change in the dependent variable without considering the change in the independent variable (Cooney et al., 2010).

The first follow-up activity provided tabular data showing the altitude of a weather balloon with the corresponding time and atmospheric pressure values and asked PSTs to find the *rate of change in altitude with respect to time* and *rate of change in pressure with respect to altitude* for various intervals. With this task, we primarily intended to have PSTs reinterpret the meaning of the rate of change across different representations. We also anticipated that PSTs would further their understanding of the "rate of change" through this activity that comprised a temporal and non-temporal context. In this task, we also expected that PSTs would interpret the meanings of the units through answering specific questions asked to be answered in the given context.

Table 1 Design principles that guided the model development unit

Design principles	Source
• Covariational reasoning ability is a critical prerequisite in conceptualizing the rate of change and derivative. Real-life situations involving dynamically and simultaneously changing quantities can be useful for supporting students' covariational reasoning	The literature on covariational reasoning (e.g., Thompson & Carlson, 2017)
• "Rate of change" is a more inclusive interpretation of the derivative, but it is generally introduced only limited to the motion context. The rate of change should be studied within non-motion real-life contexts	Review of literature & Quantitative reasoning (e.g., Thompson, 1994b)
• Qualitative aspects of calculus concepts are critical, as well as their formal mathematical representations. Therefore, students should be provided opportunities to cover qualitative aspects of calculus concepts in different real-life contexts (Stroup, 2002)	Qualitative calculus (Stroup, 2002)
• Students' contextual understanding of the ideas involved in derivative seems problematic. Mathematical modeling activities can elicit and support students' meaningful conceptions of these ideas	Mathematical modeling (e.g., Lesh & Doerr, 2003)

Table 2 Hypothetical learning trajectory for MDS-2

Big ideas	Students are expected to be able to…	Main task
Change and coordination of changes	focus on the change in one quantity or coordinate the changes in two quantities by using the numerical values provided in tables	Population of Turkey
Verbal expressions	use and discriminate the meanings of "change," "average rate of change," and "rate of change" expressions	Population of Turkey
Transformation between different representations (i.e., graphical and/or algebraic)	reinterpret the meaning of "rate of change" across different representations	Weather Balloon
Units of the quantities	focus on the units of quantities and interpret the meanings of the units in the given context	Weather Balloon
Multiplicative and additive rate of change	distinguish the additive rate of change and multiplicative rate of change	Population Growth Rate
Context diversity	use and apply the "rate of change" interpretation of derivative in different real-life contexts	Across all tasks

The second follow-up activity, the Population Growth Rate, was added to the instructional sequence based on the first iteration after observing PSTs' difficulties in discriminating additive and multiplicative rates of change. In this task, PSTs were asked to think about two situations: (i) what it means in terms of annual population changes if the annual population growth in percentage is constant, and (ii) what it means in terms of annual growth in percentage if the population grows annually with equal increments. With this task, we conjectured that PSTs would see the distinction between additive and multiplicative rates of change and construct an idea regarding when to use each way of understanding.

Participants

The study took part in an undergraduate elective course focusing on the knowledge and pedagogy of mathematical modeling. Seventeen of the 20 students (seven male, 13 female) enrolled in the course were in their senior year of the middle school (grades 5–8) mathematics teaching program. Three of the participants were master's students who graduated from the same undergraduate program in the previous semester but had not yet started teaching.

PSTs enrolled in the course worked in groups of three to four while working on the modeling activities and stayed in the same groups throughout the semester. Using a purposeful sampling method, four of the PSTs enrolled in the course [Halis (M), Betül (F), Renan (F), and Nazan (F), all names are pseudonyms] were selected to be interviewed after the implementation of each modeling task based on the following criteria: (i) being volunteered to assume additional data collection tasks, (ii) having high communicative abilities, and (iii) being a member of different small groups. All participants were from similar backgrounds in terms of subject-matter knowledge. By the time of data collection, all participants had completed courses in differential calculus, integral calculus, linear algebra, and differential equations offered as part of the program.

Data Collection

The data collection tools used in this study were (*i*) Questionnaire-I and Questionnaire-II for evaluating PSTs' conceptual understanding of the rate of change (see the electronic supplementary material [ESM] for the tasks in Questionnaire-I and Questionnaire-II, and the table of specifications); (*ii*) written group reports for the modeling activities and individual written solutions to follow-ups of MDS-2 and MDS-3. Although it was not part of the MDS-2, data from a follow-up activity in the MDS-3 was also included because of its alignment with the focus of MDS-2 as it revealed PSTs' conceptions of the rate of change in a height-volume context (see "Follow-up of Tracking Track" in the Appendix); (*iii*) 30-min semi-structured interviews; (*iv*) individual reflection papers about modeling process written after solving the modeling task; and (*v*) video-recorded classroom observations. The video recordings of two groups and the whole classroom were used as supportive data sources.

Prior to and after implementing the MDS, a pre-test (Questionnaire-I) and a post-test (Questionnaire-II), each consisting of 11 open-response items prepared in parallel forms, were implemented to measure PSTs' procedural and conceptual knowledge of derivative, including the rate of change (see ESM). For example, inspired by Bezuidenhout (1998) and Goerdt (2007), the sixth item in both questionnaires involved contextual interpretation of average and instantaneous rates of change (see Fig. 2): the fuel efficiency of a car as a function of its speed in Questionnaire-I and the temperature-solubility as a matched item in Questionnaire-II. With such contexts, we aimed to observe PSTs' understanding of rate change and derivative in non-motion contexts. All the PSTs who attended the course completed both questionnaires.

Two semi-structured interviews were conducted with the selected four PTSs. The first interview was conducted after implementing the Population of Turkey task and its follow-up activities. The second interview was conducted after completing Questionnaire-II. Before conducting the interviews, we prepared a set of interview questions personalized to each participant by carefully examining their written solutions and reflection papers.

Data Analysis

As multiple data collection methods and sources are involved in the current study, the constant comparative method (Strauss & Corbin, 1998) was followed in analyzing the data. The data analysis started simultaneously with the experimentation process. The ongoing data analysis was to get a first impression of the data collected in the first language. All audio-recorded interviews were transcribed, and the written solutions for the questionnaires, modeling, and follow-up activities provided by the participants were scanned and stored electronically. As PSTs' understanding of mathematical concepts and their ways of reasoning were the focus of this study, the data analysis was executed at the cognitive level (Collins et al., 2004). The unit of analysis was the incidents such as a sentence, an episode from a dialogue, or a graph reflecting a mathematical idea.
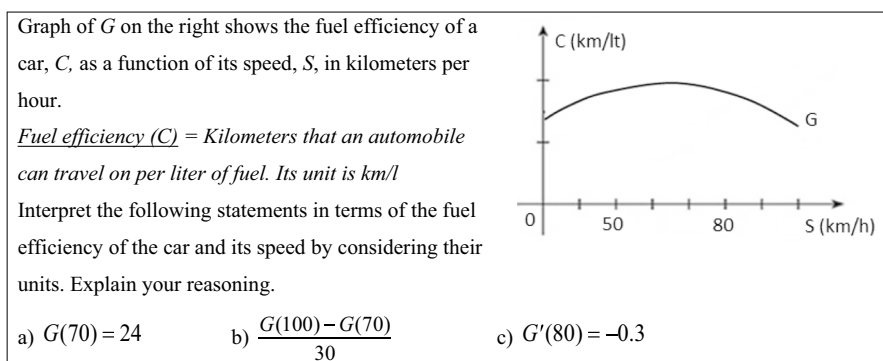

Graph of G on the right shows the fuel efficiency of a car, C, as a function of its speed, S, in kilometers per hour.

Fuel efficiency (C) = Kilometers that an automobile can travel on per liter of fuel. Its unit is km/l

Interpret the following statements in terms of the fuel efficiency of the car and its speed by considering their units. Explain your reasoning.

a) $G(70) = 24$ b) $\dfrac{G(100) - G(70)}{30}$ c) $G'(80) = -0.3$

Fig. 2 Item 6 in Questionnaire-I

Table 3 shows the coding schema used to analyze data. The data sets were analyzed by considering the chronological order, as the second research question was about the developmental understanding of the PSTs. First, we obtained a tentative list of codes by comparing the incidents in the PSTs' answers to the rate of change items in Questionnaire-I. Next, we compared the incidents in the written group solutions to the modeling and follow-up activities. Then, the reflection papers and transcribed interviews were analyzed. We used a confirmatory auditor, a mathematics education Ph.D. student, to control the inferences drawn from the data. After the data analysis was completed, the analysis and quotes were translated from the first language to English when preparing the findings for publication.

Findings

The analysis of data regarding the PSTs' conceptions of the rate of change revealed six categories: (*i*) lexical difficulties, (*ii*) amount of change, (*iii*) additive rate of change, (*iv*) multiplicative rate of change, (*v*) slope, and (*vi*) other. Table 4 and Table 5 show a descriptive analysis of the PSTs' conceptions of the rate of change for the items in Questionnaire-I and Questionnaire-II, the tasks in MDS-2, and a follow-up task in MDS-3, respectively.

Table 4 shows that PSTs demonstrated different conceptions across the items in the Questionnaires. Lexical difficulties were observed in the context-free average rate of change items (i.e., Item-2) in both Questionnaires despite a noticeable decrease in Questionnaire-II. The frequencies of the amount of change conception in Item 6(b) and Item-6(c) decreased while the additive rate of change conception increased from Questionnaire-I to Questionnaire-II. No multiplicative rate of change conception was observed across both Questionnaires.

As shown in Table 5, lexical difficulties in group reports were also observed for the Population of Turkey task in almost all groups. A multiplicative rate of change interpretation was observed in all groups, just in the Population of Turkey and Population Growth tasks. In the follow-up tasks, PSTs generally used additive rate of change interpretation. PSTs rarely used the slope of the tangent or secant line interpretation independent of the other interpretations. Therefore, we prefer to present results regarding slope interpretation as the occasion arises within the other interpretations—the amount of change, additive, and multiplicative rate of change. However, we observed that the slope interpretations prevailed in situations involving graphical representations. For example, in the Population of Turkey task, some PSTs interpreted the rate of change as the slope of the secant line after transforming the tabular data into a graphical representation. Similarly, we did not devote a separate section to reporting PSTs' ways of reasoning labeled "Other" since they were rarely observed. For example, the following explanation by a PST for Item-6(c) in Questionnaire-I involving the interpretation of a negative derivative at a point was categorized as such: "This expression shows the decrease in the engine power of a car because of a sudden decrease in its speed."

Table 3 Coding schema used for analyzing PSTs' conceptions of the rate of change

Categories	Abbreviation	Explanation
Amount of change in the variable	AC	Focusing only on the change in the output variable without considering the input variable, additively comparing the changes in the output variable while keeping changes in the input variable constant e.g., $x_2 - x_1 = \Delta x_i$; and $x_3 - x_2 = \Delta x_k \rightarrow \Delta x_i \leq \Delta x_k$
Additive rate of change	ARC	Considering the rate of change as a *ratio* in the form of additive rate of change. Indicating a comparison of relative sizes of changes in two quantities. Imagining two quantities covarying. It involves multiplicatively comparing the change in the dependent and change in the independent variable in an interval (relative size); Relating it to the Difference Quotient Rule
Multiplicative rate of change	MRC	Considering the rate of change as a *ratio* in the form of a multiplicative rate of change. Using the percentage interpretation by taking the ratio between the successive values of a variable e.g., $\frac{x_2}{x_1}$ or $\frac{x_2 - x_1}{x_1}$
Lexical difficulty	LD	Difficulty in giving meaning to the Turkish expression for the term "rate of change." For instance, computing the average of derivative values at the endpoints or using some other operations such as the average of values of a function in the interval $\left(\frac{f(x_1)+f(x_2)+....+f(x_n)}{n}\right)$
The slope of the tangent line	SL	Considering the rate of change as slope of tangent or secant line
Other	O	No answer or irrelevant information

Across the first three tasks (Cassette Player, Water Tank, and Sliding Ladder) related to the covariational reasoning used in MDS-1, PSTs used some ways of reasoning about the rate of change while deciding the intensity of change. In the initial phases, they frequently used gross quantification (perceptual) while determining the rate of change without an explicit quantitative operation. As the study progressed, they started to quantify the rate of change extensively by additively comparing the successive changes in the dependent variable while keeping the changes in the independent variable constant. Thus, PSTs frequently used the amount of change in the dependent variable while deciding the variation in the rate of change (Kertil et al., 2019).

Table 4 Frequencies of PSTs' conceptions of the rate of change across the items in questionnaires

	Questionnaire-I			Questionnaire-II		
	Item-2	Item-6(b)	Item-6(c)	Item-2	Item-6(b)	Item-6(c)
Lexical difficulty	17	–	–	8	–	–
Amount of change	–	11	10	–	9	7
Additive rate of change	3	3	1	11	10	11
Multiplicative rate of change	–	–	–	–	–	–
Slope	–	1	5	2	–	1
Other	–	3	3	1	1	–
Omits	–	3	1	–	1	2
Total	20	21	20	22	21	21

In some cases, individuals/groups demonstrated more than one way of thinking about a particular item. Thus, the total frequencies might be greater than 20

Table 5 Frequencies of PSTs' conceptions of the rate of change across the tasks ($n=20$ for individuals, and $n=6$ for groups)

	Population of Turkey	Population Growth	Weather Balloon	Follow-up of Tracking Track
Lexical difficulty	G1, G3, G4, G5, G6	–	3	–
Amount of change	G1, G3, G5, G6	–	–	8
Additive rate of change	G2	16	19	9
Multiplicative rate of change	G1, G2, G3, G4, G5, G6	17	–	–
Slope	G1, G2, G3	–	1	2
Other	–	3	2	1
Omits	–	–	–	–
Total	19	36	25	20

(*i*) In some cases, individuals/groups demonstrated more than one way of thinking for a particular task. (*ii*) For the Population of Turkey, the group names are directly given ($n = 6$). (*iii*) G# denotes group name

Lexical Difficulties with the Term "Rate of Change"

In Turkish, the term "oran" is used when referring to both "rate" and "ratio." We observed that the PSTs had difficulty giving meaning to the Turkish phrase "değişim oranı" for describing "rate of change." PSTs' responses to Item 2 in Questionnaire-I showed their lexical difficulties with the rate of change. This item required PSTs to find the average rate of change of a context-free function. PSTs used various irrelevant procedures as they did not understand the meaning of the term "average rate of change." They applied a variety of operations that can be related to the term "average." Halis answered Item 2 in Questionnaire-I by finding the average rate of change of the derivative function in the given domain (see Fig. 3).

To find the average rate of change of $f(x) = 3x^2 - 4$ between $x = 2$ and $x = 5$, Halis computed the values of the derivative function for these two endpoints and applied the difference quotient rule to the derived function as $\frac{f'(5)-f'(3)}{5-3}$. Betül also used the values of the derivative function for these points in answering the same item (see Fig. 4).

Some PSTs indicated that they did not know what the expression "rate of change" means. Only three PSTs (e.g., Renan) correctly applied the difference quotient rule to compute the average rate of change. Lexical difficulties were also observed in the Population of Turkey task. In his reflection paper, Halis indicated that they (as a group) had difficulty interpreting "the average rate of change in population with respect to time."

> After answering the first question using MS Excel, we continued with the second question. We had difficulty with this question because we could not understand the meaning of the expression "…the average rate of change in population with respect to time." (Halis, Reflection Paper, Population of Turkey)

Other groups had similar difficulties in the Population of Turkey task. As we documented in the following sections, PSTs' preference for percentage interpretation in this task can also be related to their difficulty giving meaning to the "average rate of change" expression.

Fig. 3 Halis's answer for item 2 in Questionnaire-I

Fig. 4 Betül's answer for part (a) of item 2 in Questionnaire-I

Amount of Change Conception of the Rate of Change

The amount of change in the dependent variable was the dominant conception that PSTs demonstrated related to the rate of change. Such a way of reasoning first appeared in Questionnaire-I while interpreting the symbolic expressions of average and instantaneous rates of change. For example, Betül interpreted the expression $G'(80) = -0.3$ in part-c of Item 6 as the decrease in fuel efficiency at the given speed. She only focused on the change in the dependent variable, as appeared in her statement: "When our speed is 80 km/h, fuel efficiency decreases." Halis interpreted $G'(80) = -0.3$ as a decrease in fuel efficiency (see Fig. 5).

Most of the PSTs perceived the rate of change in the Population of Turkey task as the percentage (multiplicative rate of change). As various other groups did, Halis's group consistently used the ratio $\frac{P_2 - P_1}{P_1} \times 100$ for calculating the rate of change, where P_1 and P_2 stand for populations for two consecutive years. The following episode from the interview shows how Renan and her groupmates interpreted the Turkish expression used for the rate of change in the population context.

> Researcher: How did you conceive the expression "rate of increase in population with respect to time"?
> Renan: We interpreted this as a percentage. I mean, we did not use the annual population changes... Let me explain by a simple example. The change in population from 1000 to 1500 and the change in population from 500 to 1000 are not the same. Although the amounts of change are equal for both, the population increased by 100% for the second while it increased by 50% for the first. I mean, for the rate of increase, we considered the percentage of change in population with respect to its initial value.

Renan argued her reasoning by pointing out the difference between the amount and percentage of change. She indicated that they considered the rate of change as the percentage of change in the population with respect to the previous year's population. When asked about the other alternative, she indicated the annual population change, but they considered it an amount of change. Later in the interview, Renan was asked to transform the given tabular data into a graph and reinterpret the expressions given in the problem context.

> Researcher: Well, you thought with percentage interpretation. Now, if we draw such a graph (drawing a graph), this axis (x-axis) shows years, and this one

Fig. 5 Halis's answer for part (c) of item 6 in Questionnaire-I

(*y*-axis) shows the population. This point shows the population for 1990, and this point shows the population for 2000. Then, what does the slope of the line give us?

Renan: It gives the rate of change in population with respect to years.

Researcher: Well, what had we asked in the question?

Renan: Hmmm… with respect to time… This is the slope because we divide it by 10, we find the annual population increase. Yeah, this is the same thing you asked.

Renan: Hmmm, OK, this is actually what we already did… I got it; *f*(*a*) minus *f*(*b*) divided by *a* minus *b*. 2007 and 1990 will be assigned in places of *a* and *b*, respectively…

Researcher: Very well; what is the unit?

Renan: Divided by year… It refers to an increase in population per year: the number of people in a year.

After representing the tabular data as a graph, Renan could explain the meaning of the slope of the secant line on the "population vs. year" graph as "the rate of change in population with respect to years." She recognized the similarity when asked about the verbal expressions used in the problem text. She stated that the slope of the secant line indicated the annual population change for a given interval. Until now, Renan considered annual population change as the amount of change. Similarly, Betül and her group interpreted the rate of change as per year change in the population but did not accept it as a rate.

Researcher: In your group, have you discussed the expression "Average rate of change in population with respect to time?" What did you understand from this expression?

Betül: From this expression, we thought we needed to find the change in population per unit of time. I mean, it is computing the annual population change. However, when we talk about the rate, it seems different…

Researcher: How is it different?

Betül: When we say the rate of change in the population, it may involve dividing the change in population by the value of the population. Then it can be a rate of increase. I mean, the rate can be obtained because of dividing something by another thing.

Researcher: Did you discuss these issues in your group?

Betül: We thought like that … [thinking for a while]. The rate of increase in population will be different because the population values are different each year. Changes in the population and the population values are different. We did not calculate it for each year separately. We thought in a way that we computed the annual population values for 30 years.

Researcher: Was it for the annual population change?

Betül: Yes, we computed as annual population change, not as the rate of change.

As observed in this episode, Betül and her groupmates did not accept annual population change as a rate of change; instead, they considered this value an amount of

change. Betül indicated the need for computing percentage in the given population context. Although the same terminology was used in the Weather Balloon task, none of the participants demonstrated thinking with the percentage interpretation. Generally, they applied the difference quotient rule for calculating the average rate of change in the air pressure with respect to the elevation. Halis indicated their previous experience with the percentage interpretation of population change as a possible reason.

In a follow-up activity conducted after the "Tracking Track" task, as shown in Fig. 6a, Halis explained the variable on vertical axes of the derivative graph of height-volume as an "increase in height." Like Halis, Nazan also used the amount of change in height to label the *y*-axis. On the other hand, Renan and Betül answered this question by demonstrating the ratio-based conception of the rate of change (see Fig. 6b for Renan).

In Item 6 in Questionnaire-II, most of the PSTs continued considering the rate of change as an amount of change in the dependent variable. For example, Betül's ratio-based conception of the rate of change was evidenced in her written solutions and during the interviews. Her interpretation of symbolic expressions of the rate of change was still in the form of an amount of change. She interpreted the symbolic expression of the average rate of change as "what is the change in the solubility in the interval of 8–40 °C?" Likewise, she interpreted the symbolic expression $S'(16) = -0.25$ as "Change in solubility decreases at the temperature of 16 °C and its value is 0.25."

Halis also used "amount of change" to explain the symbolic expression of the average rate of change in the solubility-temperature context while answering Item 6 in Questionnaire-II (see Fig. 7). To explain the symbolic expression of the average
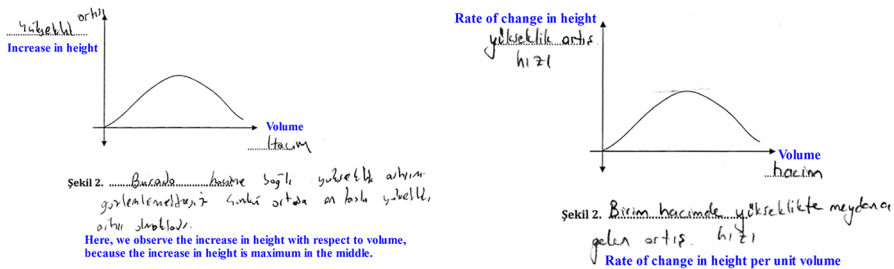

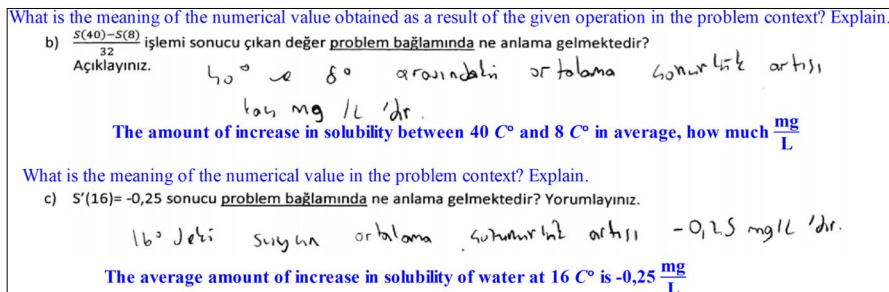Fig. 6 **a** Halis's and **b** Renan's answers for the follow-up problem of the "Tracking Track" task

Fig. 7 Halis's answer for part (b) and part (c) of item 6 in Questionnaire-II

rate of change, he used the expression "average increase in solubility." Similarly, Nazan's explanation for the same symbolic expression also included the amount of change.

Multiplicative and Additive Conceptions of the Rate of Change

For Item 6 in Questionnaire-I, only a few PSTs demonstrated ratio-based reasoning on the rate of change. Most of the PSTs explained the symbolic expressions involving the rate of change as the amount of change in the dependent variable or used other irrelevant explanations. In the Population of Turkey task, most of the PSTs (in all groups) interpreted the rate of change as the percentage of change in the population (see Table 4). However, this task also helped PSTs realize the relationship between the multiplicative and the additive rate of change. As they worked on this task, they started to see the expression "change in population per year" as a rate. To exemplify, Betül and her group calculated annual population change for every interval. However, they used "amount of annual population change" in their verbal explanations without seeing it as a rate. In the interview, when asked about her understanding of the rate of change in the population context, Betül expressed that at first, "we did not find the rate; we only found the amount." Regarding the meaning of the expression "771 people per year" that they used, she conceived this expression as speed by transferring from the description of speed that is "distance covered per unit time." She refined her understanding of annual population change as a rate by using speed as a reference context. In the same way, Halis also clarified his understanding of the rate of change in the interview.

> Researcher: What mathematical ideas were covered in this task [Population of Turkey]?
> Halis: Percentage, slope.
> Researcher: What does slope mean in this context?
> Halis: Average rate of change in population with respect to time. Increase in population with respect to unit second or unit year. There is instantaneous velocity, average velocity in this question that we remember from physics. However, when the population is asked, the annual percentage of change comes to my mind.

According to Halis, the increase in percentage and slope were the basic ideas in the Population of Turkey task. When asked about the meaning of slope in the population context, he expressed that "slope is the average rate of change in population with respect to time." Also, he connected it with physics concepts. In the solution of the Weather Balloon task implemented after the Population of Turkey task, almost all of the PSTs correctly interpreted the rate of change in the height-temperature context (see Table 4).

When they worked on the follow-up activity asking about the names of axes on the derivative graph of the volume-height graph, nine out of 20 of the PSTs explained the name of the vertical axis as the rate of change in height. The rest

of the PSTs either used the amount of change in height or some other irrelevant explanations (see Table 4). For example, Renan identified the vertical axis of the derivative graph of volume-height as the "rate (speed) of increase in the height" (see Fig. 6 (b)).

Regarding Item 6 in Questionnaire-II, while about half of the PSTs demonstrated ratio-based reasoning, the other half reasoned either by the amount of change or using some other irrelevant expressions. For instance, using the idea of the rate of change properly, Renan interpreted the symbolic expressions given in the form of the difference quotient rule and the instantaneous rate of change as the rate of change (see Fig. 8).

Throughout the model development sequence, PSTs occasionally demonstrated the ratio-based conception of the rate of change. Nevertheless, it is difficult to argue that they all developed a robust ratio-based, additive conception for the rate of change, since some of them continued considering the rate of change as the amount of change in the dependent variable.

Discussion

Developing PSTs' understanding of the rate of change was one of the main goals of the model development sequence implemented in the current study. In the initial phases, PSTs had difficulties giving meaning to the rate of change or explained the situations involving it by using the amount of change in the dependent variable. Although critical developments were observed in PSTs' conceptual understandings in subsequent phases, they showed particular difficulties and weaknesses. Different rates of change conceptions appeared in the Population of Turkey task. The frequency of the additive rate of change conception increased across the tasks used in the model development sequence. Most of the PSTs confused the rate of change with the amount of change. Although some progress was observed, it is difficult to argue that the PSTs formed a robust conception of the rate of change (Kertil, 2014).

Fig. 8 Renan's answer for parts (b) and (c) of item 6 in Questionnaire-II

A noteworthy observation in the initial phases of the current study was PSTs' difficulty in giving meaning to the Turkish phrase "değişim oranı" used for the "rate of change." Previous studies have also reported difficulty in giving meaning to the term "rate of change" (e.g., Bezuidenhout, 1998; Orton, 1983; White & Mitchelmore, 1996). In the Population of Turkey task, most of the PSTs interpreted the Turkish expression for the term "rate of change" as the percentage of change in the dependent variable. Considering the rate of change as the percentage of change involving the ratio between successive values of the dependent variable is a prototypical example of the multiplicative rate of change introduced by Confrey and Smith (1994). As noted by Confrey and Smith (1994), the conventional understanding of the rate of change is additive, which is difficult to interpret in some contexts involving exponential growth, where the multiplicative rate of change is easier to interpret.

PSTs in the current study frequently considered the rate of change as the amount of change in the dependent variable, as observed in many studies (e.g., Byerley & Thompson, 2017; Herbert & Pierce, 2012; Rowland & Jovanoski, 2004; Zandieh & Knapp, 2006). Similarly, Rowland and Jovanoski (2004) reported that most of the calculus students interpreted differentiation as the amount of change in a function. Some calculus students in Zandieh and Knapp (2006) thought that "derivative is a change" (p. 12). In the current study, PSTs also used the amount of change, for example, change in the height (i.e., the dependent variable) in place of the rate of change in the height with respect to the volume. Many of the PSTs demonstrated ratio-based reasoning on the rate of change in the later phases of the model development unit. The ratio-based conception of the rate of change involves the multiplicative comparison of changes in two quantities and being aware that the quantity obtained is the ratio between two changes (Thompson, 1994b). PSTs in the current study used ratio-based reasoning while explaining the symbolic expressions of average and instantaneous rates of change. For instance, most of the PSTs considering "the change in population per year" as an amount at first (Herbert & Pierce, 2012; Zandieh & Knapp, 2006) started to conceive it as a rate as they progressed. The important point here was that although there have been long in-class discussions and emphasis on distinguishing between rate and amount conceptions, considering derivative as an amount of change was persistent for some PSTs and appeared in some subsequent tasks (e.g., in the solubility-temperature context).

In the Population of Turkey task, PSTs did not interpret the rate of change in population in an interval as the slope of the secant line; instead, they conceived it as the percentage of change in population with respect to the previous year. Confrey and Smith (1994) explained this way of reasoning as the multiplicative rate of change. Furthermore, most of the PSTs did not consider "annual population change" as a rate, ratio, or slope. They realized the relationship between "annual population change," the slope of the secant line, and the average rate of change after converting the tabular data into the graphical form. Such result was also reported by Zandieh (2000) and Zandieh and Knapp (2006), who found that to answer the question of approximating the derivative at a particular point by using the values in a table, students used the slope of the tangent line interpretation after transferring the given tabular data to the graphical representation. The current study's data also revealed PSTs' inexperience in interpreting the rate of change, slope, or derivative when provided as a tabular representation. This finding

shows the need to benefit from multiple representations in teaching derivative (Santos & Thomas, 2001; White & Mitchelmore, 1996).

Furthermore, this study shows the need to use a variety of real-life contexts in introducing the derivative concept as its meaning gains distinctive characters in different contexts. PSTs generally used and visited the definition of "speed" when they had difficulty interpreting the meaning of derivative or rate of change in non-motion contexts. In explaining the meaning of slope or rate of change in the height-volume context, they borrowed from the "distance covered per unit time" definition of speed and transferred it to the height-volume context as "change in height per unit volume." This could result from the frequent use of motion context for explaining the contextual interpretation of derivative (Bingolbali, 2008; Gravemeijer & Doorman, 1999; Herbert & Pierce, 2008; Wilhelm & Confrey, 2003; Yoon et al., 2010). Students' conceptions of the rate of change are generally rooted in motion context, and they have difficulties in projecting these concepts into non-motion contexts (Bezuidenhout, 1998; Gravemeijer & Doorman, 1999; Herbert & Pierce, 2008; Wilhelm & Confrey, 2003; Yoon et al., 2010; Zandieh & Knapp, 2006). The frequent use of motion context may also foster the idea that "time" is always an independent variable that results in confusion in non-temporal situations. Findings of the current study indicate that students need long-term experiences to conceive the meaning of derivative in non-motion contexts.

As also being the focus of the HLT in this study, forming connections between the rate of change, slope, difference quotient rule, and the symbolic derivative is vital for the conceptual understanding of derivative (Herbert & Pierce, 2008; Sahin et al., 2015; Zandieh, 2000; Zandieh & Knapp, 2006). However, the current study's findings evidenced that PSTs may not easily realize the connections between the slope, difference quotient rule, derivative, rate of change, and the daily or verbal expressions used for describing them. We also realized that context of the tasks affects students' representational preferences. For example, in the Population of Turkey task, while PSTs interpreted the rate of change as a percentage of change, they conceived it as the slope of the secant line or a ratio of changes after transforming the tabular data into a graphical representation. Knowledge of derivative as the slope of the tangent line, the ability to apply the rules of derivative in symbolic or idealized real-life contexts, and the ability to solve symbolic procedural derivative problems do not bring about the conceptual understanding of derivative (Byerley & Thompson, 2017; Kertil & Küpcü, 2021). As pointed out by Byerley and Thompson (2017), forming connections among different interpretations of the rate of change should be an explicitly stated learning objective in curricular documents, and textbooks should contain many examples directly aiming at those connections.

Conclusions and Implications

Although the amount of change conceptions continued to appear in PSTs' ways of reasoning, the model development sequence contributed to them developing the conception of the additive rate of change. Throughout the study, PSTs demonstrated various conceptions of the rate of change. They frequently used the definition of speed while giving meaning to the derivative in other contexts. As also

elaborated in Kertil (2014), these findings are concurrent with the literature suggesting that introducing the rate of change as limited to the physics concepts does not necessarily support students understanding of it in non-motion contexts (Bingolbali, 2008; Herbert & Pierce, 2008; Wilhelm & Confrey, 2003; Yoon et al., 2010).

In this study, the Population of Turkey task, with a pivoting role in the MDS-2, aimed to reveal and develop PSTs' conceptions of the rate of change. In general, this task worked well for pointing out the difference between "amount of change" and "rate of change." Also, this task and its follow-up activities contributed PSTs to realizing the connections between different representations of the derivative. On the other hand, it also revealed PSTs' difficulties in interpreting the difference between multiplicative and additive rates of change, resulting in some confusion. Therefore, we recommend using a different modeling activity for further iterations if the main aim is to develop a conventional idea of the rate of change. The current study showed that PSTs had a strong tendency to use the multiplicative rate of change in the population growth context and had difficulty interpreting the difference between the multiplicative and additive rate of change. Therefore, rather than supporting students' understanding of the rate of change, the MDS-2 implemented in this study seemed more appropriate to study the distinctions and mathematical connection between additive and multiplicative rate of change. To do this, the meaning of symbolic derivative, slope, or difference quotient in population contexts should be emphasized by proper use of the units.

The role of curricular materials cannot be underestimated in guiding teachers' practices. As revealed by various studies, most textbooks ignored the rate of change interpretation or only used the motion context (Bingolbali, 2008; Herbert & Pierce, 2012; Teuscher & Reys, 2012). However, it seems that the reasoning about the meaning of the rate of change as a quantity should be contextualized, and thus providing students with opportunities to study and reflect on the rate of change in diverse contexts would promote a more robust understanding of it. Accordingly, the concept of rate of change should be comprehensively dealt with from different perspectives in the curricular materials by providing diverse contexts (Kertil & Gülbağcı-Dede, 2022). In doing so, the meaning of derivative as a rate of change in various non-temporal contexts should be clearly emphasized with context-specific terms such as marginal cost (or revenue) in economics, fuel efficiency in driving, and flow rate in physics (Feudel, 2017; Kertil & Gülbağcı-Dede, 2022; Mkhatshwa & Doerr, 2018). Discussing the meaning of marginal cost (i.e., change in cost for producing one additional product) as appear in economics textbooks and then interpreting this expression with the slope interpretation or symbolic difference quotient with a limiting process would contribute to students' understanding of derivative from a broader perspective. As also indicated by Byerley and Thompson (2017), trying to interpret derivative as the rate of change in non-temporal contexts can support students' understanding of derivative and rate of change as a new quantity and the relative size between two simultaneously changing quantities. Therefore, in further research, we recommend designing and implementing long-term HLTs involving non-temporal real-life situations in various contexts to investigate students' understanding of derivative as a rate of change.

Appendix. The tasks used in the current study

1. **Population of Turkey**

State Planning Organization (SPO) makes strategic plans for the next century. In this context, how the population of Turkey changes and the demographic structure would differ within the next 100 years are decisive for managing investment plans. Because investments are made according to the current conditions would become useless because of the changes in the population structures. SPO officials asked you to prepare a report answering the following questions. In the report, you need to investigate the changes in the population in the past and the formation of the population structure in the future.

- How has the population of Turkey changed over the years?

    – What is the amount of change in the population growth between 1980 and 1985 and in other year intervals?
    – What are the year intervals where the average rate of change in population growth with respect to time is maximum and minimum?
    – What is the average rate of change in population growth with respect to time in the year 2000 approximately?
    – What is the average rate of change in population growth with respect to time in the year 2004 approximately?
    – How do the senior and young populations change in years?
    – Some experts claim that the population of Turkey will become stationary in the future. Do you think this is possible? If yes, when do you think this will happen?

Years	Population*($\times 1000$)	Age groups			
		0–19	20–39	40–59	60 or more
1980	45 586	22 556	13 745	6 756	2 529
1985	50 660	24 467	15 015	7 922	3 256
1990	55 971	25 981	17 500	8 456	4 034
2000	67 800	27 438	22 458	12 215	5 689
2007	70 786	24 938	23 604	15 176	7 068
2008	71 557	24 996	23 927	15 551	7 083
2009	72 641	25 133	24 225	15 832	7 451
2010	73 724	25 155	24 482	16 265	7 822

[*]Mid-year population data (e.g., data for each year belongs to the sixth month of that year)

    2.   **Follow-Up (Weather Balloon)**

Weather balloons are launched daily to collect various meteorological data. As the balloon ascends, data is collected on pressure, humidity, and temperature through sensors at certain time intervals, and altitudes are transmitted to the mission control. The following table shows six weather balloon measurements collected on a mission.

Measurement #	Time (min)	Altitude (m)	Pressure (mbar)
1	0	Ground level	1000
2	360	260	925
3	650	440	850
4	1400	890	700
5	2750	1790	500
6	3600	2240	400

- What is the rate of change in the altitude with respect to time between the 2nd and the 3rd measurements? Do not forget to state the unit of measurement.
- What is the rate of change in the pressure with respect to altitude? Do not forget to state the unit of measurement.

    – Between the 4th and the 5th measurements
    – Between the 1st and the 6th measurements

3. **Follow-up (Population Growth Rate)**
    The population of a country is 50 million and 61 million in the years 2000 and 2010, respectively, as shown in the following graph.
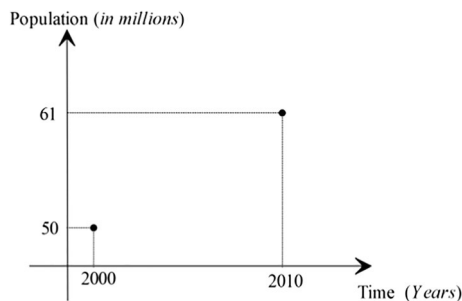
- Draw and interpret the population-time graph assuming that this country's population increased at the same rate (in percent) every year between 2000 and 2010.
- Draw and interpret the population-time graph assuming that this country's population increased in equal amounts every year between 2000 and 2010. In this case, how does the annual population growth (in percent) change?

4. **Follow-up of Tracking Track**

    The population of a country is 50 million and 61 million in the years 2000 and 2010, respectively, as shown in the following graph.

A water tank is being filled with water at a constant flow rate. Figure 1 shows the height of the water as a function of volume. Figure 2 shows the derivative graph of the height-volume graph. What should be the units of the axes in Fig. 2? Write your answers into the given boxes in Fig. 2.
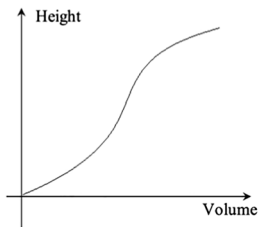
Figure 1. Height-Volume Graph

Figure 2. ....

Declarations

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