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Mathematics Teachers' Perceptions of Diagrams

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Abstract
External representations such as diagrams often appear in teachers' repertoire of pedagogical tools to support students' conceptual learning and problem-solving activities. The study of diagrams and diagrammatic representations has received growing interest from diverse disciplines. Diagrams remain a central feature of science, technology, engineering, and mathematics (STEM) learning areas, in schools and beyond. This study examined teachers' perceptions of diagrams in mathematics, focusing on two aspects. The first was teachers' perceptions on the utility value of diagrams and how they incorporate them in their instructional practice to communicate and make connections between mathematical ideas. The second was teachers' perceptions about their students' use of diagrams in mathematics. An open-ended survey was administered to 20 secondary school (grades 7 to 10) teachers, with at least 3 years of mathematics teaching experience. The findings show that teachers perceived diagrams primarily as a tool for communication of mathematical ideas during instruction. To a lesser extent, they used diagrams to link concepts within and between topics, and as a tool in any part of problem-solving. Some of the challenges they perceived that students may encounter with diagrams were the lack of diagram-specific skills, cognitive demand of constructing diagrams, and less than proficient visuospatial abilities to decode and extract pertinent information. The possible reasons for these findings are discussed with the support of existing literature and inform potential teacher learning programmes and serve as a springboard for future research.

Keywords Diagrams · Big Ideas · Perceptions · Mathematics teachers · Singapore

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Introduction

The study of diagrams and diagrammatic representations has received growing interest from diverse disciplines. Diagrams remain a central feature of science, technology, engineering, and mathematics (STEM) fields in schools and beyond. Diagrams are used as a means of communication, both formally and informally in mathematics. They may be considered a Big Idea in mathematics which is "a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole" (Charles, 2005, p. 10). Mathematics as a discipline often deals with objects and entities that are not immediately tangible which Arcavi (2003) termed the "unseen". Through visualisation, a disciplinary practice, by means of graphs, diagrams, and models, one may begin to "see" the "unseen".

Research has documented how diagrams play distinct roles and take on different utility value in the teaching and learning of mathematics. Diagrams help to externalise important relations between quantities and operations, and aid in extracting pertinent information from a given problem (Larkin & Simon, 1987). They also function as external representations of interconnected pieces of information that can be put together and relieve working memory (van Essen & Hamaker, 1990). Therefore, it hardly comes as a surprise that they often appear in a teacher's repertoire of pedagogical tools to support conceptual learning and problem-solving activities. However, in our review of literature, we found very few studies specifically addressing how teachers use diagrams in the teaching and learning of mathematics (e.g. Bautista et al., 2015; Stylianou, 2002; Stylianou & Silver, 2004; Sunzuma et al., 2020). Comparatively, most of the existing research on visual representations focuses greatly on students' learning and conditions that give rise to achievement (e.g., Chu et al., 2017; Mudaly, 2012). Our study intends to contribute to this limited area of research by exploring teachers' perceptions of what diagrams are and how they use them in their mathematics instruction.

In response to diagrams as a Big Idea in the Singapore school mathematics curriculum since 2018, this study attempts to ascertain Singapore mathematics teachers' perceptions of diagrams and their use in the teaching and learning of mathematics. In so doing, our intend is two-fold. Firstly, to document Singapore mathematics teachers' perceptions of diagrams. Mathematics teacher education, both preservice and in-service, at the National Institute of Education — the sole teacher education institution in Singapore — has no explicit focus on diagrams as part of mathematics instruction other than the use of it as a problem-solving heuristic. Secondly, to explore the different roles and utility values of diagrams in Singapore secondary mathematics classrooms. This would provide a window to teachers' Mathematics Pedagogical Content Knowledge (MPCK) that is necessary to "provide students with the lens to look at diagrams" (Ministry of Education [MOE], 2018, p. 8) in a holistic and coherent manner.

Literature Review

Definitions of Diagrams

Winn (1987) defined diagrams as an abstract, visual representation that exploits spatial layout in a meaningful way, enabling complex processes and structures to be represented holistically. He distinguished between graphic forms, such as charts and graphs, and diagrams. In this position, the function of charts and graphs is to simply display relationships between variables. On the contrary, the function of diagrams is to "describe whole processes and structures often at greater levels of complexity" (p. 153).

According to Purchase (2014), a diagram is "taken to mean a composite set of marks (visual elements) on a two-dimensional plane that, when taken together represent a concept or object in the mind of the viewer" (p. 59). She broadly classified diagrams into two categories — abstract and concrete. An abstract diagram requires a clear set of rules that the viewer must be aware of to interpret the diagram accurately. They are symbolic representations that do not possess a perceptual relationship to what they represent. An example of one such diagram is an Euler diagram, a diagrammatic representation of set memberships where the type of relationship is depicted by the overlapping geometric shapes. Concrete diagrams are governed by fairly simple rules of interpretation as they have a direct relationship between an object and its diagrammatic form. They would include sufficient and accurate visual elements to enable the viewer to interpret them. Concrete diagrams are further categorised into three broad areas — schematics, arrangement of geometric shapes, and digital images. Schematics closely depict the physical attributes such as a drawing of the respiratory system in humans, or a simplified line-drawing of architectural diagrams.

The focus of our study is to document how teachers harness the powerful ability diagrams possess to represent information, allow one to make connections, and provide an external representation of a learner's mental model. Hence, we adopt Winn's (1987) definition of diagrams for the purpose of this study.

Diagrams as Visual Representations

Diagrammatic representations have been shown to be superior to exclusively employing verbal or sentential representations when solving problems (Larkin & Simon, 1987). They indicate a substantial connection between diagram use and problem-solving, specifically, when diagrammatic forms are representative of a cognitive process or are schematic in nature. They are useful in providing a complementary language to sentential representations of the same knowledge. Moreover, they make apparent those quantifiable relationships that define a problem (Sunzuma et al., 2020). De Toffoli (2018) distinguished between the distinct types of representations and purported that choosing the right one is essential. By considering an example of the circle, De Toffoli (2018) illustrated how different representations serve different purposes as they can be reasoned with in differing ways. Varying properties and features of the circle are exhibited in the different representations which can be manipulated differently to produce new information. This is ascribed to the capability of diagrams to externalise the relevant properties, the “seen” and “unseen” (Arcavi, 2003), the known and unknown quantities (Stylianou, 2011), to be manipulated. It can serve as an intermediate step between a mental representation and a physical representation of a concept that can be seen and comprehended externally. They highlight important relations between quantities and operations in a given problem and assist students to extract pertinent information (Larkin & Simon, 1987). In geometry, diagrams play a vital role in the construction, argumentation, and understanding as diagrams are employed as methods to visualise geometric concepts as well as study the meaning of it (Dimmel & Herbst, 2015). This illustrates the idea of whole processes and structures as noted by Winn (1987) in his definition.

Diagrams as Connectors

Visual representations support understanding where various aspects of a problem can be combined to observe how they interact. In particular, diagrams are useful to the extent that the learner can interpret the symbols and information illustrated, supported by their prior knowledge, to engage in the process of meaning construction (Mudaly, 2012). Furthermore, they facilitate connections between concrete and abstract representations. They allow both teachers and students to view a problem in its entirety as all parts are displayed on the diagram at the same time (Sunzuma et al., 2020). When presented with a problem, a student needs to invest cognitive load to process the interrelated elements. A visual representation assists the student in activating their schema of the related topic and draws on their prior knowledge to determine the underlying problem structure (Chu et al., 2017; Larkin & Simon, 1987; Ngu et al., 2014). The presence of a diagram boosts germane cognitive load as students use less of their working memory to process the information in the given problem (Ngu et al., 2014). The versatility of diagrams and their capacity to make connections discernible in a holistic manner makes them an integral component of doing and understanding mathematics (Samkoff et al., 2012).

Diagrams as Tools for Thinking and Problem-Solving

Visual representations take on different utility value as the objective of a problem-solving activity changes. Employing an alternate form of representing the same knowledge facilitates the process of sense-making and knowledge construction (Mudaly, 2012). They are flexible exploration devices that allow a solver to generate new information about the problem or reveal information that is not immediately obvious. Additionally, they support monitoring and assessing progress in problem-solving (Stylianou, 2002). A diagram acts as an external sketch where interconnected pieces of information can be put together and therefore reduces the cognitive load imposed on a learner and in turn alleviates working memory (Murata, 2008; Ngu et al., 2014; van Essen & Hamaker, 1990). The diagram becomes a communication tool for students to explain their thinking and for teachers to assess their progress and provide feedback. Research suggests that students need to be trained to acquire the strategies that will enable them to solve problems (Stylianou, 2002). Effective representations allow a learner to manipulate the relevant mathematical functions that make it conducive for both discovery and understanding, thereby simplifying a given problem (Stylianou & Silver, 2004).

Zazkis et al. (1996) argued that in the process of problem-solving, visualisation and analysis are not dichotomous. Rather, there exists a mutual dependence between the two. Building on the visualisation and analytical (VA) model by Zazkis et al. (1996), Stylianou (2002) offered a refined version to further describe the explicit steps of visualisation and analysis specified in the VA model. The results from her study provide empirical evidence of the actions taken during the stages of visualising and analysing diagrams in problem-solving. The author suggests four analytic activities, namely, (a) inferring additional consequences from visual representation, (b) mathematical elaboration and further investigation of the new information, (c) setting new goals with respect to visual representation use, and (d) monitoring statements of one's own problem-solving.

Cellucci's (2019) position of diagrams as a heuristic tool in mathematical proofs draws parallels with the influential work by Polya (1945). Drawing on his original definition, heuristics is the "study of means and methods of problem solving" (Polya, 1962, p. x) and relates to experience-based techniques of discovery and problem-solving. The role of heuristics and his 4-step model for problem-solving, namely, (1) understand the problem; (2) devise a plan; (3) Carry out the plan; and (4) look back, inspired the teaching of problem-solving in schools though with varying degrees of success. Particularly, the heuristics of draw a diagram can be used in various stages of the model. It can assist at the start of a problem where a solver is required to understand and or used when engaged in formulating a plan to be then carried out. Drawing diagrams is in fact a significant problem-solving heuristic and many mathematicians employ visual imagery when tackling problems (Wong, 1999). Likewise, research shows that the heuristic of visual representation through a model or diagram is the most effective amongst others for problem-solving (see meta-analysis by Hembree, 1992; Uesaka & Manalo, 2012).

Challenges Related to Diagrams

Diagram literacy refers to knowing about diagram use and being able to use the knowledge appropriately (Diezmann & English, 2001, p. 77) and this is part of visual literacy. Visual literacy is "the ability to understand and use and to think and learn in terms of images" (Hortin, 1994, p. 25). Diezmann and English (2001) state that students may have difficulties in the use of diagrams due to a lack of the concept of a diagram, inability to generate an appropriate diagram for a particular problem, and inability to reason with diagrams thereby not making sense of the inherent problem structure. Lowrie and Diezmann (2007) reported that encoding and decoding visual representations can prove to be challenging as they are influenced by a student's age and the complexity of the diagram.

Students are required to tap into their visuo-spatial skills to decode a diagram in which the mathematical information is embedded. This mapping of information to symbols is a two-way street that is effective only when the learners are able to communicate more efficiently with it and are able to identify and make sense of what is being communicated (Zazkis & Liljedahl, 2004). However, some students may be predisposed to high performance in such tasks and are more successful in subsequent problem-solving while others are unable to navigate through this task successfully. Similarly, Zodik and Zaslavsky (2008) postulated that students may find it difficult to move their attention intentionally to alternate between parts of a diagram and viewing the diagram as a whole. This can be attributed to limited diagram comprehension skills, especially in lower ability mathematics students (Booth & Koedinger, 2012; Chu et al., 2017). Specifically, for problems involving algebraic equations, Ng (2003) noted that beginning algebra students in Singapore demonstrated a preference of algebraic over diagrammatic methods of solving. It may be unnecessary in such cases to construct diagrams as an equation is also a representation of the problem structure, albeit symbolic in nature. Ngu et al. (2014) suggested that although a diagram highlights the structural features of the problem, producing one may impose extraneous cognitive load as students are required to search and match relevant information in their diagrams. Furthermore, poorly designed diagrams and superficial diagrams which do not denote vital concepts render them ineffective (Sunzuma et al., 2020).

Goldin (1998) noted the consensus amongst researchers that proficient problem solvers employ effective heuristic strategies, but these strategies have proven to be difficult to teach explicitly. Mousoulides and Sriraman (2014), in examining Polya's (1945) contribution to the role of heuristics in problem-solving, identified some constraints related to the teaching and learning of heuristics. The heuristics of drawing a diagram is often taught using textbook problems presenting a strategy which students use to solve problems. They argued that this approach deviates from Polya's notion of heuristics and is no longer realised in the sense intended by Polya. In fact, research has consistently shown that this traditional approach to teaching heuristics is minimally useful in improving students' problem-solving abilities (Mousoulides & Sriraman, 2014; Schoenfeld, 1992). The authors also identified that teachers' skills related to teaching heuristics is another constraint. In citing Burkhardt (1988), the authors stated that "teachers should be equipped with the experience, confidence and self-awareness, in order to work well with problems without knowing all [that] the answers require" (p. 255).

Chen and Herbst (2013) claimed that students' interactions with diagrams in geometry vary considerably, and their knowledge of mathematical objects is mediated by their representations. Hence, building their knowledge likely necessitates "more than simple engagement in deductions from definitions and axioms" (p. 286). Similarly, Herbst (2004) in his investigation of interactions with diagrams in geometry maintained that the matter is not just about identifying the types of interactions that promote rich and authentic mathematical activity in which students use diagrams to conjecture and prove theorems. The conditions in which such activity is viable need to be considered. Particularly, recognising what is manageable by a teacher within the constraints of a classroom. Some, although few, studies attempt to contribute to the examination of how and why teachers use visual representations in their instructional practices.

Bautista et al. (2015) examined how teachers used graphs to teach mathematics as part of a professional development (PD) programme. Before the PD programme, the teachers involved made no explicit references regarding how or why they used graphs to help students learn the mathematical content at hand. Subsequently, after participating in the PD programme, teachers displayed a greater awareness of the vital role graphs played in fostering students' learning about the mathematical topics addressed. The PD programme further helped teachers to articulate deep justifications of the learning that might occur as a result of students' interactions with diagrams. This study provided empirical evidence that as a result of targeted PD, "the use of graphs was qualitatively more sophisticated than the beginning of lessons" (p. 103).

By examining mathematicians' (mathematics professors) problem-solving behaviour, Stylianou (2002) identified that their visual representations of problems were built in stages to systematically include more information which led to a richer visual image. That is, mathematicians engaged in visualisation and analysis to illuminate the path to solving problems. This process appeared to be automated in mathematicians who possessed a rich schema of the possible operations to be applied to the visual representations (Stylianou, 2002). The authors raised the question of how to develop this automated response in students as well. The challenges associated with diagrams partially explain the reluctance of many students to use them (Stylianou & Silver, 2004; Uesaka et al., 2010). It is then necessary for teachers to find applicable teaching methods and develop the skills of students to self-regulate their learning to fully benefit from the use of diagrams. Teachers are required to engage in a multi-faceted, complex process of selecting, crafting and facilitating problem-solving all while helping students develop the appreciation to see that diagrams facilitate problem-solving.

Teachers' Knowledge of Diagrams and Instructional Practice

It is well-documented that MPCK (Ball, 2000; Hill et al., 2005; Shulman, 1986; Turnuklu & Yesildere, 2007) directly impacts teachers' instructional practices. It includes "knowledge of procedures and techniques, fundamental mathematical concepts, common student misconceptions and mistakes, specific techniques, questions and problems that can support students in making sense of ideas and provide opportunities for reasoning and sense-making" (Sheppard & Wieman, 2020, p. 4). Shulman (1987) and Hill et al. (2005) noted that experienced teachers have a greater depth of knowledge for mathematics teaching. Marshall et al. (2010) also noted that teachers' purposeful selection of representations when designing and selecting problem-solving tasks was important. This is attributed to a teacher's pedagogical content knowledge (PCK) to support learners. Turnuklu and Yesildere (2007) rightly noted that teachers' knowledge of representations is significant in developing a clear understanding of mathematical concepts. This in turn develops students' conceptual understanding and boosts their problem-solving skills. If teachers are ill-equipped to translate mathematical abstractions into forms that enable learners to relate mathematics to their pre-existing knowledge, then they will fail to learn with understanding (Fennema & Franke, 1992). Furthermore, teachers having mathematics content knowledge alone is insufficient to develop students' mathematical concepts (Kahan et al., 2003). The ways in which teachers relate subject matter to their PCK and promote mathematical thinking are integral components of effective teaching and learning mathematics.

External representations such as diagrams are commonly used pedagogical tools to support students' conceptual learning and problem-solving activities (Belenky & Schalk, 2014). Schoenfeld (1985) purported that explicit teaching which focuses on the use of diagrams might be helpful for learners to understand what to read from a diagram and to make inferences that assist problem-solving. It calls attention to the ability and proficiency of teachers to utilise diagrams effectively in their instructional practices. The extent to which teachers draw on their MPCK varies from one individual to another, and it propels them to make important decisions when facilitating students' problem-solving activities.

Despite the ample research available to characterise students' interactions with diagrams and explain their inclination to or lack of spontaneity to use diagrams in problem-solving, we found that there is little said about teachers' views on diagrams specifically and the processes involved in the use of diagrams. This further reiterates the compulsion to understand teachers' orientation towards diagrams and identify how they harness their benefits to facilitate problem-solving for their students.

The Study

The study reported in this paper explored the perceptions of mathematics teachers in Singapore secondary schools about diagrams and the use of diagrams in their instruction. It adopted a qualitative research design and the phenomenological approach. This approach attempts to describe the essence of a phenomenon by exploring it from the perspective of those who have experienced it (Creswell, 2013; Patton, 2002). Through this qualitative research method, the study sought to answer the following research questions:

a) What do teachers perceive diagrams represent in mathematics?
b) When and how do teachers use diagrams in the teaching of mathematics?
c) What are teachers' perceptions of their students' interactions with diagrams? and
d) What are teachers' perceptions of the challenges that students encounter with diagrams in mathematics?

Instrument

The instrument used for the study was a questionnaire with open-ended prompts. The choice of an open-ended questionnaire as an instrument stems from the intent of the researchers to keep an open mind about teachers' perspectives and experiences in response to the prompts in the questionnaire (Tasker & Cisneroz, 2019). In addition, it also supports the aim of capturing insights into teachers' perceptions of diagrams which shape their instructional practices. Gay et al. (2011) noted the benefit of such questionnaires which allow for more depth of response that may permit insights into the reasons for the responses. In addition, teachers responding anonymously to a questionnaire are more likely to give frank responses. The questionnaire also made it possible to reach out to more teachers compared to conducting in-person interviews by the researcher within a stipulated time period.

The questionnaire items were developed drawing on theoretical and empirical work as explicated in the literature review of this study. The first section focused on eliciting teachers' perceptions of diagrams in the teaching of mathematics. The second section focused on teachers' perceptions of their student engagement with diagrams. Feedback on a draft of the instrument was sought from a group of two mathematics educators and three secondary school mathematics teachers. Their feedback shaped the instrument, shown in the Appendix, which was used in the study.

The Participants and Data Collection

The participants of the study were 20 (10 male and 10 female) mathematics teachers heading their respective departments in Singapore secondary schools. At the time of the study, they were attending in-service courses (higher degree courses or professional development sessions) at the National Institute of Education (NIE) in Singapore. Prior to administering the survey, individual meetings were held with the participants to brief them about the study and obtain their formal consent for participation. After obtaining consent, they were given the questionnaire to complete at their own time. They were encouraged to answer the questions as completely as possible and to illustrate with examples where necessary. The participants' names were anonymised and given teacher codes T1 through T20.

The participants had a minimum of three years of teaching experience at their respective secondary schools. As all teachers in Singapore schools attend their pre-service and in-service courses at the NIE, a review of the mathematics curriculum studies courses the teachers would have taken as part of their pre-service teacher education shows that there was no emphasis on diagrams as a Big Idea in mathematics. Nevertheless, a key heuristic that was always emphasised when solving problems was "draw a diagram". Mathematics textbooks used in Singapore secondary schools are laden with diagrams, particularly for ideation of concepts (Velayutham, 2020) for topics, such as geometry, trigonometry, etc. However, there is no explicit focus in the textbooks or accompanying teachers' instructional guides about the "why, what, and how" of diagrams in mathematics teaching and learning. It appears that, how teachers perceive diagrams in mathematics and their use in their instruction is largely an outcome of their first-hand and subjective experiences as learners of mathematics and MPCK that may have developed over the years of their mathematics teaching.

Data Analysis

To provide a comprehensive understanding of the participants' perceptions of diagrams in mathematics, a hybrid process of deductive and inductive thematic analysis was used to facilitate the coding process. A thematic analysis, as noted by Braun and Clarke (2006) is a useful and flexible method that can potentially provide a rich and detailed account of data for qualitative research. A hybrid approach employed for this study incorporated both the "bottom-up" inductive approach of Boyatzis (1998) and the "top-down" deductive approach outlined by Crabtree and Miller (1999). The process of data extraction, coding, and categorisation was divided into two stages:

– Stage 1: The deductive approach produced a priori set of codes. This approach drew on the literature review to produce a list of codes as shown in Table 1. It allowed for the coding process to be structured and grounded in existing theories as an initial coding cycle (Linneberg & Korsgaard, 2019).
– Stage 2: During the exploratory coding cycle, it became apparent that not all the responses could be captured by the priori set of codes. So, the codes were refined to be more succinct in identifying the prevalent trends in the data. This resulted in the posteriori codes also shown in Table 1. The codes derived from the data were italicised for reference. Swain (2018) notes that this approach of encoding the data results in theory being a precursor to, and an outcome of, data analysis. The set of codes derived from both literature consulted and the actual data serves as a conceptual framework which guided the process of analysis. This approach complemented the research questions by allowing the literature consulted to be integral to the process of deductive thematic analysis while allowing for the emergence of themes directly from the data using inductive coding. The final coding of the responses was done by both authors, using the posteriori set of codes. After this coding cycle, the inter-rater agreement was tabulated using a simple percent agreement (McHugh, 2012). A total of 174 responses were coded (31 for item 1, 38 for item 2a, 18 for item 2b, 24 for item 3, 8 for item 4, 33 for item 5, and 22 for item 6). Both the coders agreed on the codes for 155 items (28 for item 1, 35 for item 2a, 15 for item 2b, 22 for item 3, 7 for item 4, 30 for item 5, and 18 for item 6). The percent agreement between the two coders was 89%. As this percent agreement was greater than 75%, it was within the acceptable range according to McHugh (2012). The disagreements were clarified and both coders agreed on a code during the resolution phase of the final coding cycle.

Table 2 shows an example of the data extraction and coding for survey item 3. During the exploratory coding cycle, it was evident that some code descriptions could be merged to illustrate an overarching trend. This resulted in three broad strokes by which the data for survey item 3 was encoded. Feasibility (C4) was included in the posteriori (final) set of codes to capture the essence of the responses as far as possible. This process was repeated for each category of codes to reflect, as closely as possible, the perceptions of diagrams in mathematics that teachers held.

Table 1 Priori and posteriori set of codes created for analysis

Priori category code description		Posteriori category code description	
[Survey Item 1] Category code: representations (R)			
R1	External representations (Winn, 1987)	R1	External representations (Winn, 1987)
R2	Source of information (Stylianou, 2011)	R2	Source of information (Stylianou, 2011)
R3	Means to record information (Stylianou, 2011)	R3	A heuristic used in any part of problem-solving (Cellucci, 2019; Polya, 1945; Uesaka & Manalo, 2012)
R4	A heuristic used in any part of problem-solving (Cellucci, 2019; Polya, 1945; Uesaka & Manalo, 2012)		
R5	Simplifying tool — known and unknown quantities can be represented. (Stylianou, 2011)		
[Survey Item 2a] Category code: affordances (A)			
A1	Facilitate understanding (Cellucci, 2019; Stylianou, 2011)	A1	Facilitate understanding (Cellucci, 2019; Stylianou, 2011)
A2	Reduce cognitive demand (Murata, 2008; Ngu et al., 2014; Stylianou, 2011)	A2	Reduce cognitive demand (Murata, 2008; Ngu et al., 2014; Stylianou, 2011)
A3	Diagrams act as a means to connect between existing knowledge and skills (Mudaly, 2012)	A3	Assist with visualisation (Polya, 1945)
A4	Exploration device (Stylianou, 2011)	A4	Exploration device (Stylianou, 2011)
A5	Monitoring and assessing students' understanding (Stylianou, 2011)	A5	Monitoring and assessing students' understanding (Stylianou, 2011)
[Survey Item 2b] Category Code: Making Connections (MC)			
MC1	Connect between existing and new knowledge (Mudaly, 2012)	MC1	Connect between existing and new knowledge (Mudaly, 2012)
MC2	Boost germane cognitive load (Ngu et al., 2014)	MC2	Boost germane cognitive load (Ngu et al., 2014)
MC3	Connect between different representations (De Toffoli, 2018)	MC3	Connect between different representations (De Toffoli, 2018)
[Survey Items 3 and 4] Category code: considerations (C)			
C1	Age of students (Chu et al., 2017)	C1	Complexity (Chu et al., 2017)
C2	Complexity of problem (Chu et al., 2017)	C2	Readiness (prior knowledge Mudaly, 2012; visuo-spatial abilities Lowrie & Diezmann, 2007)
C3	Prior knowledge (Mudaly, 2012)	C3	Suitability (Diezmann, 1995)
C4	Suitability (Diezmann, 1995)	C4	*Feasibility*

Table 1 (continued)

Priori category code description		Posteriori category code description	
[Survey Item 5] Category code: students' usage of diagrams for problem-solving (PS)			
PS1	Understanding the problem (Polya, 1945)	PS1	Understanding the problem (Polya, 1945)
PS2	Unpacking the given problem (Mudaly, 2012)	PS2	Sense-making (Mudaly, 2012)
PS3	Sense-making (Mudaly, 2012)	PS3	*Suitability of topic*
[Survey Item 6] Category code: challenges (CL)			
CL1	Cognitive demand in constructing diagrams (Chu et al., 2017; Larkin & Simon, 1987; Ngu et al., 2014)	CL1	Cognitive demand in constructing diagrams (Chu et al., 2017; Larkin & Simon, 1987; Ngu et al., 2014)
CL2	Interpreting diagrams (Lowrie & Diezmann, 2007; Mudaly, 2012)	CL2	Interpreting diagrams (Lowrie & Diezmann, 2007; Mudaly, 2012)
CL3	Superficial comprehension (Sunzuma et al., 2020)	CL3	Superficial comprehension (Sunzuma et al., 2020)
CL4	Diagram-specific skills (Booth & Koedinger, 2012; Chu et al., 2017)	CL4	Diagram- specific skills (Booth & Koedinger, 2012; Chu et al., 2017)
CL5	Poorly designed diagrams (Sunzuma et al., 2020)	CL5	*Distraction*
CL6	Confusions (Sunzuma et al., 2020)		

Table 2 Example of analysis of responses for survey item 3

Teacher	Response to Q3: list the factors you would consider when using diagrams in the teaching of mathematics	Codes
T1	• The topic that I am teaching • What am I trying to teach my students • Will diagram help to reinforce the explanation to aid in students' understanding?	
	• The **suitability/accuracy** of diagrams	C3
	• **Convenience/availability** of diagrams (Self-created/Hand-drawn/using diagrams in textbook/online resources)	C4
T12	• **Effectiveness** of the diagrams in relaying the concept/idea	C3
	• The **readiness of students** in working with the diagrams (Much time has to be devoted to students observing details of the diagram before jumping straight into using the diagrams to answer questions)	C2

Findings and Discussion

The findings are presented according to the emergent themes identified in the posteriori codes shown in Table 1.

What Diagrams Represent (R)

The responses to survey item 1, asking what diagrams represent, revealed three key perceptions. Consistent with the definition of diagrams of this study, 65% ($n=13$) of them identified the powerful ability of diagrams to provide a visual representation of real-world or mathematical objects. T1 stated that diagrams may "represent statistical information" and are also able to "summarise the problem" given. Furthermore, 60% ($n=12$) recognised that diagrams provide a concise method of presenting information. Many respondents indicated that diagrams "communicate information" such as mathematical "concepts and properties" (T2, T3, T4, T20). However, only 30% ($n=6$) perceived diagrams as a heuristic to be used in any part of problem-solving to facilitate understanding and consequently guide problem-solving activities. This finding is of concern as diagrams are a significant heuristic for problem-solving (Hembree, 1992; Polya, 1945; Uesaka & Manalo, 2012; Wong, 1999). Furthermore, mathematical problem-solving is the primary goal of mathematics instruction in Singapore schools. Despite the prevalent use of visual elements in mathematics textbooks and instructional materials in the Singapore mathematics curriculum (Velayutham, 2020), there appears to be no unified perception shared by all teachers. This diminishes the potential of diagrams to function as communication tools to holistically view diagrams as tools to be used in any stage of problem-solving.

Diagrams in Instructional Practices

The responses from four survey items 2a, 2b, 3, and 4 were analysed to uncover the espoused conceptions of diagrams that determine when and how teachers incorporated diagrams in their instructional practices. It forms the core of the findings to elucidate characteristics of teachers' instructional practices involving diagrams. The findings can be sub-categorised into three main aspects — affordances of diagrams, making connections, and considerations when using diagrams in mathematics.

Affordances (A)

Survey item 2a asked the respondents when and why they used diagrams in mathematics to elicit aspects of their pedagogical knowledge which informs their instruction. To that end, 80% ($n=16$) of respondents identified that diagrams assist with visualisation (A3). Respondents explained that they would use diagrams when "explaining concepts as it helped students visualise" (T20) and guided them when solving problems (T17). Half of the respondents noted that diagrams facilitate understanding (A1) by serving as external representations of interrelated concepts and properties. T11 mentioned that students having difficulty understanding the problem will benefit from a diagram as it "will provide a clearer picture". Less than a third of the respondents placed emphasis on the following during their mathematics instruction: (i) diagrams reduce cognitive demand (A2), (ii) diagrams allow one to monitor and assess the understanding of mathematical ideas (A5), and (iii) diagrams as an exploration device (A4). In studies investigating the problem-solving behaviours of mathematicians (Stylianou, 2002; Stylianou & Silver, 2004), it was observed that they used diagrams for many purposes including exploring the problem space, introducing new information and monitoring theses explorations. It appears that most respondents did not regard these cognitive sub-tasks of problem-solving as benefits diagrams afforded.

Making Connections (MC)

Survey item 2b, asked respondents if they used diagrams to make connections and to instantiate with examples. All the respondents affirmed that they used diagrams to make connections between mathematical ideas in their teaching. Most respondents only referred to specific topics and chapters of the mathematics curriculum such as trigonometry, geometry, and statistics in which they would use diagrams to make connections between concepts and methods of solving. About a third of the respondents, 35% ($n=7$), referred to the use of diagrams to connect between existing and new knowledge (MC1) and that they used diagrams to relate one representation to another (MC3). The respondents mostly associated making connections within specific topics and chapters that they believed required the use of diagrams. While the results from prior empirical work present diagrams as useful in engaging prior knowledge in the process of meaning construction (Mudaly, 2012) and facilitating the connections between concrete and abstract representations (Sunzuma et al., 2020), only about a third mentioned these in their responses. Similarly, only 10% ($n=2$) were aware that the use of diagrams boosts germane cognitive load (MC2). It appears that in both instances of reducing cognitive load (A3) and boosting germane cognitive load (MC2) most teachers were not cognisant of them.

While 20% ($n=4$) respondents were aware of diagrams serving as tools to teach the concept of functions across mathematics topics, that is characterising diagrams as a Big Idea, T11 was the only respondent to mention Big Ideas of mathematics in their response. They related that in the teaching of arc and sector length of circles, it could be linked to the Big Idea of Proportionality. T11 mentioned that using the diagrams shown in Fig. 1 students would be guided to comprehend the proportion of arc length over the circumference of circle. This example draws on students' prior knowledge of fractions and circle properties, which are topics taught prior in the Singapore mathematic curriculum, to teach the new concepts of arc and sector lengths.

Considerations (C)

Survey items 3 and 4 asked respondents for their considerations when deciding to use diagrams in their instruction and the extent to which they placed emphasis on the use of diagrams. The responses allowed us to gain insights of how teachers negotiated the use of diagrams. The findings revealed that their considerations were noticeably varied and there was no predominant consideration reflected in their responses. Forty percent ($n=8$) reflected that the complexity of the problem (C1) was a determining factor. In the topic of coordinate geometry of circles, T3 mentioned that it would be "difficult to understand without a diagram, to make sense of what are the relationships and how to go about solving it". A diagram would then be useful to explicate key relationships. Thirty-five percent ($n=7$) reflected on the suitability (C3) of a diagram to the topic or concept being taught. T10 and T12 described that diagrams must carry relevance and be effective in relaying concepts to students. Here, a key factor is the degree of congruence between a diagram and the problem structure (Diezmann, 1995). T2 reflected that the purpose of using a diagram is to "unpack meaning of texts/symbols into representations that help to manipulate, reorganize, and interpret relationships visually". This description presents diagrams as flexible tools that facilitate the processes of visualising and analysing when solving problems which concurs with the VA model (Zazkis et al., 1996) in which problem-solvers engage in an iterative process of visualising and analysing using diagrams.

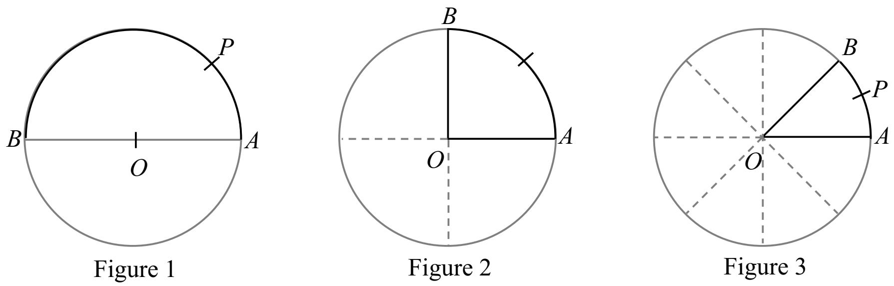

Fig. 1 Response from T11

Students' readiness (C2) was mentioned by 30% ($n=6$) of the respondents. They considered if their students were visual thinkers who would benefit from diagrammatic representation (T15) and if they were equipped with the skills necessary to encode and decode diagrams (T12). This finding coincides with Diezmann (1995) who reported that although the participant in her study displayed the apparent competence with the strategy *draw a diagram*, she was unable to use it as an effective problem-solving tool. The lack of or limited experience with diagrams reduced her ability to benefit from the use of a diagram. Feasibility (C4) emerged as a consideration from the data for 15% ($n=3$) of the respondents. They regarded convenience of constructing diagrams and the ease of accessing available online resources as determining factors. However, the respondents did not elaborate to specify reasons for raising this.

The varied responses of teachers' considerations illustrated by examples illuminate the complex nature of using visual representations in mathematics. The dynamic use of diagrams includes determining the appropriate use of diagrams in the solutions to problems (Diezmann, 1995). The considerations are also reflective of the multifaceted interaction of student abilities and mathematical problems which influences the development of successful use of diagrams (Booth & Koedinger, 2012; Larkin & Simon, 1987).

Diagrams in the Learning of Mathematics

The responses to survey item 5 attempted to identify teachers perceptions related to when their students used diagrams. From the responses, it was evident that a majority of 75% ($n=15$) stated that diagrams served as a tool in understanding a problem (PS1). This finding concurs with theoretical and empirical arguments established in the literature review that diagrams facilitate conceptual understanding as they plainly show how different elements of a problem structure interact with one another (Mudaly, 2012; Stylianou, 2011; van Essen & Hamaker, 1990). Respondents elaborated that students were inclined to transform a word problem into a diagram to understand the problem structure and draw out unknown quantities (T1, T8, T15). This resonates with Arcavi's (2003) notion of seeing the "unseen". However, less than half, 45% ($n=9$) felt that their students used diagrams for sense-making (PS2). T11 elaborated with an example of a kinematics problem requiring students to find the total distance travelled by a particle. An example of a possible diagram is shown in Fig. 2 where "with a diagram

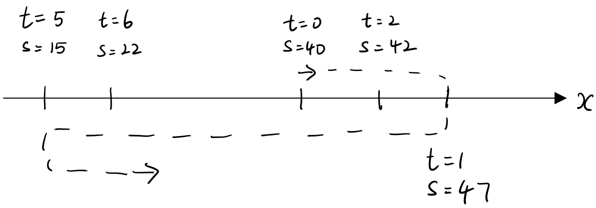

Fig. 2 Response from T11

of the travelling direction drawn out, students would find it easier to calculate the total distance" (T11). This concurs with the findings of Mudaly (2012) who established that diagrams that function as self-explanatory tools support better solutions. Figure 2 depicts such a diagram, where the key relations are reflected and the pathway to solving for the total distance travelled becomes apparent.

Forty-five percent ($n=9$) of the respondents indicated that suitability of diagrams (PS3) to topics was a determining factor in their students' use of diagrams. Respondents stated topics such as geometry, measurement, circles, and trigonometry as possible instances when students used diagrams. T4 perceived that there was little inclination for their students to self-regulate their learning by "scaffolding their own understanding using diagrams". This finding is similar to the consideration of topic suitability (C3) teachers raised in survey item 3. It appears that teachers generally regarded the relevance of diagrams to topics as a determining factor as opposed to viewing diagrams as a tool to be used in any part of problem-solving, regardless of topic. This finding is elaborated upon in the next section by reviewing the challenges identified.

Challenges with the Use of Diagrams (CL)

The responses to survey item 6 attempted to identify challenges that teachers perceived their students faced when interacting with diagrams. It was apparent from the responses that 60% ($n=12$) surfaced diagram-specific skills (CL4) as a common challenge. T4 described that students may be "lacking in knowledge or skills to use diagrams as problem-solving tools". Some mentioned that students "do not know how to start" (T1, T6, T12) and "do not know how to use" (T1) diagrams to support their problem-solving activities. The "lack of exposure and practice" (T13) results in students not fully comprehending the "features of a diagram and may misinterpret it" (T9). The extent to which students internalise diagrammatic conventions and their relation to domain-specific knowledge affect the extent to which they benefit from the use of it (Booth & Koedinger, 2012).

Fifty percent ($n=10$) raised the cognitive demand involved in constructing diagrams (CL1) and interpreting diagrams (CL2) as challenges. T1 described students being unable to "make additional inferences" when interacting with diagrams. Students are engaged in an assortment of cognitive actions to come to understand the information encoded in a diagram. This corresponds with findings from earlier works established by Uesaka and Manalo (2012). They investigated the influence of task-related factors in the use of diagrams and reported that constructing or interpreting diagrams may afford a high cognitive load which could prove to be challenging to some students.

Twenty-five percent ($n=5$) of the respondents mentioned superficial comprehension (CL3) of diagrams as a challenge and 10% ($n=2$) suggested that students might be distracted (CL5) by extraneous details of a diagram. T3 described that students simply followed a series of learned steps to produce a diagram without much conceptual understanding. T10 further elaborated that some students may ignore critical information in the diagram and attempt to solve without fully comprehending the problem. Some respondents also reflected that their students only engaged in the use of diagrams if they had first been used regularly by teachers (T3, T4, T7, and T9). T4 wrote that that their students "hardly used diagrams to communicate their thinking process". This alludes to a lack of spontaneity in diagram use which research has identified to be a widespread problem (see Dufour-Janvier et al., 1987; Uesaka & Manalo, 2008, 2012; Uesaka et al., 2010). Furthermore, students yet to acquire the adequate skills in using diagrams are unlikely to use them spontaneously even if they perceived benefits in their use (Uesaka et al., 2010). Here, diagram literacy was raised as a challenge which was also identified by Diezmann and English (2001). It also suggests that students might struggle with transferring their knowledge to novel situations and do not automatically benefit from diagrams as a problem-solving tool.

Conclusions

The findings from the present study add to our understanding of when and how teachers use diagrams in the teaching and learning of mathematics. It demonstrates that teachers indeed hold diagrams in high regard and use them in multiple ways and means in their instruction although to varying degrees. Of concern is that only a third were able to identify diagrams as an effective heuristic to be used in any part of problem-solving. Most teachers primarily associated diagrams with being external sketches of information but did not consider the more specific capacity of diagrams to facilitate the sub-tasks of problem-solving such as a tool to monitor and assess progress and an exploration device. Although research advocates that teachers should engage in explicit teaching of heuristics, including the how and why (Lowrie & Diezmann, 2007; Schoenfeld, 1985), this will remain a challenge if teachers themselves cannot identify the essential elements amongst visual representations.

It is apparent that teachers, enacting the revised mathematics curriculum that places emphasis on diagrams as Big Ideas, may need to rethink and refresh their ways of viewing and drawing on diagrams in their instructional practices. The VA model developed by Zazkis et al. (1996) and subsequently refined by Stylianou (2002) is a possible developmental tool to support teachers in using diagrams more effectively. It provides a structure and purposeful language through which teachers can utilise diagrams for their various purposes and not just as visual representations alone.

While the teachers in the study mentioned that they made connections using diagrams, they did not articulate diagrams as a Big Idea that facilitated this. An inconsistent grasp of diagrams as Big Ideas suggests that teacher knowledge requires further enhancement. Furthermore, it appears that the respondents lacked a shared vocabulary to talk about what diagrams represent and how they afford benefits in their instruction. Conversations about teaching towards Big Ideas and ways to enhance pedagogy will impact the extent to which teaching mathematics can progress towards a larger conceptual understanding (Woodbury, 2000). These include professional learning team activities where teachers can discuss the different uses of diagrams and explore ways in which it can be exploited. To do so, teachers may need to explicitly work with diagrams and discuss their potential in the teaching and learning of mathematics, as part of their pre-service and or continuing professional mathematics education.

The limitations of this study are, firstly, the relatively small number of participants. Secondly, the open-end questionnaire which results in teachers self-reporting may not necessarily be reflective of their actual enacted curriculum.

To extend the findings in this study, examining classroom interactions focusing on diagrams will be useful to evaluate the utility value students associate with diagram. It can be subsequently corresponded with how teachers conceive diagrams benefit problem-solving. Drawing on the work of Bautista et al. (2015), capturing teachers' spontaneous instructional approaches reflect the inherent conditions in which they enact the curriculum. The constraints and challenges they face include diverse student ability and the demand of standardised tests which contribute to how classroom instruction is conceptualised and realised by teachers. Teacher inventions focusing on developing competencies with diagrams, in a larger scale, will support in determining the effectiveness of diagrams in mathematics and would be better aligned to address the needs of individual teachers. Our study was motivated by the need to add to the little literature addressing teachers' perceptions and instructional practices involving diagrams. The findings suggest that the use of visual representations such as diagrams require more attention in teacher education and professional development programmes. They provide baseline data which serve as a springboard for further in-depth investigations.

Appendix. Survey

Section 1: Diagrams and the Teaching of Mathematics

1	What do diagrams in mathematics represent?
2a	In your teaching of mathematics when do you use diagrams and why?
2b	In your teaching of mathematics, do you use diagrams to make connections between mathematical ideas? Yes/No If Yes, please provide an example to illustrate If No, explain why
3	List the factors you would consider when using diagrams in the teaching of mathematics

Section 2: Diagrams and the Learning of Mathematics

4	Do you place emphasis on the use of diagrams when engaging your students in doing mathematical tasks? Yes/No If Yes, how and provide an example to illustrate If No, explain why
5	When are your students most likely to use diagrams when working on their mathematics problems? Explain your responses as completely as possible and illustrate with examples

6	What are some of the challenges that you perceive your students are likely to face when a) Drawing (sketch/construct) a diagram (encoding a diagram) b) Reading a diagram (decoding a diagram)

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Declarations

Ethical Approval and Consent to Participate The study reported in this paper had the Nanyang Technological University – Institutional Review Board (NTU-IRB) approval (IRB-2021–432). Informed consent was obtained from all teacher participants included in the study.

Conflict of Interest The authors declare no competing interests.

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