International Journal of Science and Mathematics Education (2022) 20:1573–1593
https://doi.org/10.1007/s10763-021-10201-1

Design of Classroom Discussions and the Role of the Expert in Fostering an Effective and Aware Use of Examples as a Means of Argumentation

Annalisa Cusi[1] · Shai Olsher[2]

Received: 30 November 2020 / Accepted: 5 July 2021 / Published online: 5 October 2021

Abstract
Tasks that require students to construct examples that meet certain constraints are frequently used in mathematics education. Although examples do not serve as proofs for general statements, they have a supporting role in the preliminary stages of making sense of a certain mathematical phenomenon as well as in the development of argumentation. We hypothesize that examples of the limit-confirming type could also support the initiation of arguments for refuting an existential claim. Although students may be able to construct this type of example, they rarely use it effectively in their argumentation. In this qualitative study, we analyze how teachers could scaffold students' awareness of the potential role of limit-confirming examples as tools for supporting argumentative processes and reflections on methods of construction of effective examples. We analyzed teacher's actions to explain and generalize this process by identifying and categorizing key moments that could characterize an approach fostering students' aware and effective use of examples to develop argumentations.

Keywords Argumentative processes · Limit-confirming examples · Reasoning with examples · Teacher's role

✉ Annalisa Cusi
annalisa.cusi@uniroma1.it

Shai Olsher
olshers@edu.haifa.ac.il

[1] Department of Mathematics, La Sapienza University of Rome, Piazzale Aldo Moro 5, 00185 Roma, Italy

[2] Department of Mathematics Education, University of Haifa, 199 Abba Hushi Blvd, Haifa, Israel

Rationale

Examples are basic entities in mathematics and serve as manifestations of abstract concepts used for showing, communicating, and explaining mathematical ideas. The construction and use of examples play an important role in the process of argumentation and proof, first and foremost, in refuting general statements or confirming existential ones, but not restricted to these functions (e.g. Buchbinder & Zaslavsky, 2013). We hypothesize that limit-confirming examples (LCEs) can also support the initiation of argumentation for refuting an existential claim, serving as an initial step toward a proof (Cusi & Olsher, 2019). In our previous study, we showed that although students may be able to construct this type of examples, they rarely use them effectively in argumentation as an initial stage of proving. Our hypothesis is that teachers can scaffold students' awareness of the potential role of LCEs and of their methods of construction. In this paper, we focus on a teaching experiment aimed at qualitatively analyzing how an expert would design and implement a classroom discussion aimed at refuting an existential statement, starting from examples and arguments developed by students. By analyzing the design and implementation of expert's class discussion, we explain and generalize the process. To this end, we identify and categorize key moments that have the ability to support the teachers' design of classroom discussions aimed at fostering students' aware and effective use of examples to develop argumentations and proofs.

Theoretical Background

Use of Examples for Conjecturing, Developing Arguments, and Proving

Students' construction and use of examples have been a central topic of research in mathematics education for the last decades (see, for instance, Bills & Watson, 2008; Antonini et al., 2011; Zaslavsky & Knuth, 2019). Studies that focused on design of settings for evoked example use (Zaslavsky, 2014), and on student's internal structures of mathematics objects (Goldenberg & Mason, 2008) as they manifested through their example space (Watson & Mason, 2005), stressed that "teaching effectively includes making use of tasks and interactions through which learners gain access to examples, to construction methods" (Goldenberg & Mason, 2008, p. 190). The objective in the use of these tasks is to support students' development of "the knowledge of when and how to use examples productively for conceptualization and critical thinking" (Zaslavsky, 2019, p. 254). Recognizing the limitations of the use of empirical examples in constructing proofs (Zaslavsky, 2018), researchers have proposed reconceptualizing example-based reasoning as a necessary and critical foundation in learning to prove (Zaslavsky & Knuth, 2019). Arguing why characteristics of certain examples would work for any other example could lead from inductive example–based arguments to example-based generic arguments (Dreyfus et al., 2012), providing an initial step in the proving process. This process requires moving from the use of empirical examples to making sense of conjectures focusing on the specifics of a generic example use, which implies seeing the general case through the specifics (Aricha-Metzer & Zaslavsky, 2017).

Another fundamental element that characterizes an effective construction and use of examples in proving-related activities is the awareness of the logical status of examples in determining the validity (or lack thereof) of the mathematical statements being explored. The logical status of examples may change based on whether they are used to prove or disprove universal or existential statements, as it has been shown by Buchbinder and Zaslavsky (2013). In their comparison between how mathematicians and students (from middle school to university) select and use examples, Lynch and Lockwood (2019) highlighted two main differences in the strategies being used. First, students' and mathematicians' strategies differed in the attention (or inattention, in the case of students) paid to the relationship between the intended purpose of an example and the strategy used to choose it. Second, whereas students usually did not consider the logical form of a given conjecture, metacognitive awareness of where in a logical argument an example may be used has been a hallmark that characterized mathematicians' example-related activity.

The research studies reported in this section have clearly stressed the importance of focusing on the design and implementation of activities that can foster the learners' development of the metacognitive awareness necessary to support their sophisticated use of examples. Yet, the literature on this theme has stressed the need to deepen the investigation on the efficacy of interventions aimed at fostering students' effective construction and use of examples for reasoning, producing argumentations, and proving (Stylianides et al., 2016). The study documented in this paper is aimed at contributing to this investigation. The teaching intervention on which it is focused has been designed around the notion of limit-confirming examples.

Limit-Confirming Examples as a Means to Support Argumentation

Experts use different criteria and strategies to select examples for the purpose of exploring conjectures (Lockwood et al., 2016): examples based on their familiarity with a particular domain or with particular mathematical properties, examples that increase in complexity or generality, and examples that serve as extreme cases (Clement, 1991) or boundary cases (Ellis et al., 2013). This last strategy of generating extreme or boundary cases may be carried out by creating an auxiliary problem in which the condition of the initial problem is taken to the limit (Balk, 1971). Research also refers to this strategy as "generating limit cases." Inspired by this idea, Cusi & Olsher (2019) introduced a theoretical construct aimed at identifying a certain category of examples that can represent effective means of argumentation, referred here as limit-confirming examples (LCE). This construct was developed by studying criteria that could support the design of tasks aimed at fostering students' exploration of examples to refute the existence of given types of mathematical objects in which certain characteristics coexist. Refuting an existential statement is equivalent to proving a universal statement; therefore, the tasks we designed led students to the creations of LCEs for a universal statement. We defined these as specific-limit or boundary examples that incorporate all other possible confirming examples. The characteristic of incorporating all possible confirming examples makes LCEs support the construction of complete argumentations about the truthfulness of a universal statement. This is because they set in motion a sort of domino effect, enabling students to show why all other examples that can be constructed confirm the statement. Similarly, we define non-

LCEs as confirming examples that do not satisfy this definition. An unambiguous outcome of the study by Cusi and Olsher (2019) showed that when a task aimed at fostering students' construction of LCEs was proposed to students in a 10th grade class in Italy, most students were able to spontaneously identify LCEs as supporting examples. This result confirms the notion that these examples may be an easier starting point in the argumentation process, opening the way for the construction of a complete argumentation consistent to some extent with example-based arguments (Dreyfus et al., 2012) as means of argumentation (Stylianides et al., 2016). Nevertheless, Cusi and Olsher (2019) also reported students, in their written explanations, did not motivate their choice of LCEs. This suggests that because most of the students use LCEs spontaneously but are not aware of the reasons why these types of examples are effective, the work that they do with LCEs is in their zone of proximal awareness (Mason et al., 2007). This term, introduced by Mason et al. (2007), refers to the awareness that is imminent or available to learners, but might not reach their attention or consciousness without specific interactions with mathematical tasks, cultural tools, colleagues, teachers, or some combination of these.

The Key Role Played by Teacher in Fostering Students' Awareness of the Construction and Use of Examples for Developing Argumentation

The previous section stressed the importance of guidance by an expert in fostering students' reflections on the use of specific examples to justify or confute statements and in making them focus their attention on the reasons underlying the effectiveness (or lack thereof) of their choice of examples as means of argumentation. These ideas are discussed by Mason (2019), who noted the importance of focusing students' attention on the structural relationships that underlie the chosen examples and of supporting them in recognizing sophisticated relationships and in deliberately shifting between the particular and the instantiation of a generality. He suggested that the teacher should focus on what students are attending to when they use examples to make conjectures and construct proofs, and on how they attend to it, stimulating students in being explicit about the generality behind the examples they use.

In the last two decades, researchers have explored the role played by the teacher in supporting students' development of argumentative processes, investigating purposeful interventions by the teacher to encourage students to verbalize their ideas, make them public, and explain them (Mueller et al., 2014). They also investigated the teachers' use of questions aimed at promoting effective argumentative interactions (Bova, 2017). Conner et al. (2014) identified categories of teachers' support for collective argumentation, distinguishing between teacher's direct contribution of argument components, questions posed to prompt the formulation of argument components, and other supportive actions used to facilitate the development of arguments.

The teacher's role in supporting students' construction and use of examples has been less investigated. Most of the research on this topic discusses general pedagogical implementations, stressing the importance of designing interactions between the teacher and students to turn sets of examples into didactic objects (Watson & Chick, 2011). Some of the research focuses on strategies that could support teachers' effective task design aimed at promoting such students' actions on examples as recalling, tinkering and gluing, complexifying, varying, and generalizing (Watson & Mason, 2005).

Arzarello et al. (2011) have studied in detail the teacher's role during classroom discussions about the creation of examples of mathematical objects satisfying certain constraints within a given mathematical domain (elementary calculus). They identified a variety of teacher actions that promote students' development of sophisticated awareness of the theoretical and logical background of the examples, which are necessary to grasp the meaning of the examples and to organize them into a web of relationships that structure the example space. Examples of these actions, framed within the cognitive apprenticeship paradigm (Collins et al., 1989), are those aimed at stimulating the students in making their thinking visible by means of various kinds of signs, at providing the correct linkages with the theoretical aspects of the activity, and at discussing the selection or rejection of examples from a logical point of view.

The study documented in this paper is aimed at combining these two foci by investigating the expert's role in supporting the students' effective and aware construction of examples as a means to develop argumentative processes.

Research Aim and Research Questions

Starting from our hypothesis about the fundamental role that an expert could play in supporting students working with LCEs in their zone of proximal awareness, we developed a teaching experiment aimed at qualitatively identifying and categorizing characteristics of a classroom discussions designed and implemented by an expert to foster students' aware and effective use of LCEs to support argumentative processes. We attempted to advance toward this goal by answering the following questions: How could a classroom discussion be designed with the aim of supporting students in an effective and aware use of LCE as a means of argumentation? Specifically, what are the key moments that characterize the structure of this discussion? And how can the expert's interventions in the discussion be characterized?

Methodology

Analytical Framework

The analytical framework needed to answer our research questions is made up of two main components: (a) theoretical tools that we identified for interpreting the students' use of examples as means to support argumentation and (b) a theoretical construct for analyzing the role played by the expert guiding classroom discussions aimed at making students reflect on their use of examples to support argumentative processes.

First Component of the Analytical Framework: Focus on Argumentation

We used the definition of argumentation introduced by Stylianides et al. (2016) as "the discourse or rhetorical means (not necessarily mathematical) used by an individual or a group to convince others that a statement is true or false" (p. 316). The tool we chose to interpret students' use of examples (in particular, LCEs) as means to support argumentation and to model and analyze students' argumentative processes was a simplified

Toulmin model of argumentation (Fig. 1), which represents a fundamental reference to the study of argumentative processes in mathematics (Knipping & Reid, 2019).

Toulmin (2003) distinguished between the claim or conclusion being sought and the facts used as a foundation for the claim, in other words, the data. When the ground on which an argument is constructed is strong enough, the new task is to show that the step toward the original claim or conclusion is appropriate and legitimate. This requires considering hypothetical statements in the form "if D, then C," which can act as bridges and legitimize the step. These types of propositions are called warrants. The task of a warrant is to "register explicitly the legitimacy of the step involved and to refer it back to the larger class of steps whose legitimacy is being presupposed" (Toulmin, 2003, p. 92). The support consists of the assurances that stand behind the warrants and enable the author to answer why in general a warrant should be accepted as having authority. The diagram in Fig. 2 models the students' use of LCEs to support the development of argumentation, based on the Toulmin model.

Classroom discussions represent a fruitful context in which students can be guided to reflect about their argumentations and to analyze, compare, or contrast other argumentations, and to collectively construct complete argumentations, in which all the steps needed to reach the conclusion are made explicit (Cusi, Morselli & Sabena, 2017, 2019).

Second Component of the Analytical Framework: the M-$_{AE}$AB Construct

We chose the M-$_{AE}$AB (acronym for "Model of Aware and Effective Attitudes and Behaviors") construct (Cusi & Malara, 2009, 2013, 2016) to highlight the key roles played by the expert in mediating classroom interactions that support students' reflections on their argumentative processes and their development of metacognitive awareness about the role played by examples in supporting these processes. The M-$_{AE}$AB construct is consistent with Mason's characterization of the approach of a teacher who is "mathematical with and in front of learners" (Mason, 2008), with the aim of educating their awareness.

M-$_{AE}$AB seeks to highlight the key roles played by a teacher who deliberately behaves with the objective of "making thinking visible" (Collins et al., 1989). The objective is to guide students in focusing not only on syntactic aspects but also on the effective strategies developed during classroom activities and on meta-reflections on the actions being performed. These key roles are subdivided into two main groups: (a) those that a teacher plays posing as a learner who faces problems, in order to make the

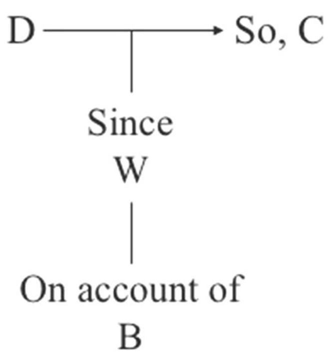


Fig. 1 Simplified Toulmin's model of argumentation (based on Toulmin, 2003)

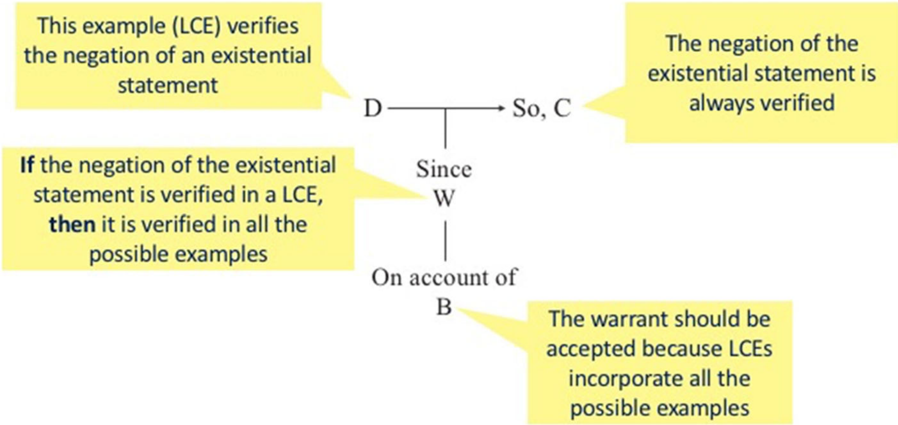


Fig. 2 Argumentation showing aware use of LCEs to refute existential statements

hidden thinking visible and to share the objectives, strategies, and interpretation of the results; and (b) those that the teacher plays to guide students to reflect on the approaches adopted during the activities, and to become aware of the relationships between the activities in which they are involved and the knowledge they had previously developed. Because we focus here only on the classroom discussion concerning the task, we refer to the second group of roles, which are presented in Table 1, together with indicators to support the coding process.

Participants and Research Setting

Participants were 22 secondary Italian students, from a 10th grade class (students aged 15–16) of a scientific lyceum, in Italy, together with their teacher. We focused on the classroom discussion developed following the students' resolution of a task that was part of an online activity in analytical geometry, specifically, lines intersecting a segment. The online activity was proposed to the group of students in the middle of the school year, when they had already studied some basis of analytic geometry (in particular, coordinates of points, equations of lines, and conditions of perpendicularity and parallelism).

The activity comprised three tasks designed as interactive diagrams describing a geometrical context on a Cartesian plane, using the STEP platform (Olsher, Yerushalmy & Chazan, 2016). The interactive diagrams, built using GeoGebra, enabled participants to construct or drag a set of elements in the diagram. Participants needed to submit examples satisfying different conditions. To explore students' spontaneous construction and use of examples, we did not explain what LCEs were, nor did we ask them to find examples with specific characteristics.

Students were asked to complete the entire activity within 1 h, working in the STEP platform in their school computer lab. Following completion of the task, at the next lesson, a researcher (one of the authors) conducted a face-to-face discussion based on the students' answers. In this paper, we analyze the part of the discussion focusing on the students' answers to the third task of the activity, which was designed to foster students' construction of LCEs. We present this task in the next section.

Table 1 The second group of roles within the M-$_{AE}$AB construct

Roles of a M-$_{AE}$AB	Characterization of each role	Indicators to code each role
Guide in fostering a harmonized balance between the syntactic and the semantic levels	Helps students control the meaning and the syntactic correctness of the mathematical expressions they construct, and at the same time, the reasons underlying the correctness of the transformations they perform.	Poses questions or intervenes to make students reflect on the correctness of given transformations being performed, and highlights connections between the processes that characterize the resolution of a problem and the corresponding meanings. For example: "Is this transformation correct?" "Is it legitimate to simplify this expression?" "Why did you make this transformation?" "How have we obtained this result?" "Why have we obtained this result?"
Reflective guide	Stimulates reflections on the effective approaches carried out during class activities to make students identify effective practical and strategic models from which they can draw their inspiration in facing problems.	Poses questions or intervenes to support students in making the meaning of effective strategies and approaches explicit. For example: "Could you explain your reasoning to your classmates?" "Is there someone who could explain your colleague's reasoning?", "She reasoned as follows: 'Since I want to obtain this kind of result, I could…'" (The teacher speaks as if she were the student, repeating the words of the students or reformulating their argument.) "Is it clear what your colleague said? She observed that…" (The teacher repeats what a student said referring to her in the third person singular).
"Activator" of both reflective attitudes and metacognitive acts	Stimulates and provokes meta-level attitudes, with particular focus on the control of the global sense of processes.	Poses questions or intervenes to support students in highlighting strengths or weaknesses of specific arguments and strategies, and to foster the sharing and comparison of different arguments and strategies. For example: "Do you agree with what your colleague said?" "Do you think it is an effective choice or strategy? Why?" "What do you think about what's written here?" "What are the differences between these answers? What do they have in common?" "Was this task difficult for you? Why?" "Would you adopt the same strategy if the problem was …?"

Data sources included classroom discussions that have been audio recorded and transcribed. Additional data sources were the student submissions for the task, the supporting examples they constructed and attached, and the verbal explanations.

The Task

The task (Fig. 3) required students to explore an existential statement, looking for examples to justify their choice.

The existential statement on which the task is focused (the claim in bold in Fig. 3) could be formulated as follows: "There are two lines that satisfy all of the following three properties: (a) the two lines belong to the family *y*=mx; (b) the two lines are perpendicular to each other; (c) the two lines intersect the segment AB." Therefore, it could be represented as $A \wedge B \wedge C$, that is, an intersection of three conditions. To prove that this statement is false requires proving that the universal statement $\neg(A \wedge B \wedge C)$ is true, which is logically equivalent to each of the following statements: $(A \wedge B) \rightarrow \neg C$; $(B \wedge C) \rightarrow \neg A$; $(A \wedge C) \rightarrow \neg B$.

Thus, to prove that the existential statement in Fig. 3 is false, it is enough to prove one of the following universal statements:

1) "If two lines belong to the family *y*=mx and are perpendicular to each other, then they do not both intersect segment AB" ($(A \wedge B) \rightarrow \neg C$)
2) "If two lines belong to the family *y*=mx and both intersect segment AB, then they are not perpendicular to each other" ($(A \wedge C) \rightarrow \neg B$)
3) "If two lines both intersect segment AB and are perpendicular to each other, then they do not both belong to the family *y*=mx" ($(B \wedge C) \rightarrow \neg A$)

Students should be aware of the fact that identifying a confirming example for each of the universal statements 1, 2, and 3 is not enough to prove them. Yet, students' choice of the examples that should support their claim that an existential statement is false could be an important clue of their awareness of the type of examples that could be the starting point for constructing the argumentations.

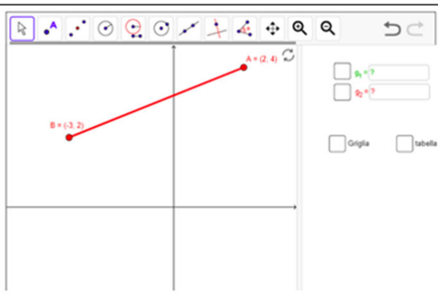
The text of the task	Applet accompanying task
Consider the segment connecting the points A(2,4) and B(-3,2) and the family of lines y=mx. Claim: ***There are two lines of the family perpendicular to each other and both intersecting the segment AB.*** Is this claim true? If you think that this claim is true, submit the equations of two lines that satisfy it. If you think that this claim is not true, explain and submit a screenshot that supports your choice.	

Fig. 3 Text of the task and the applet

The two LCEs for statement 1 ("If two lines belong to the family y=mx and are perpendicular to each other, then they do not both intersect the segment AB"), presented in Fig. 4a and b, are characterized by the fact that they were constructed to have two lines, perpendicular to each other, passing through the origin, with one of the two lines intersecting the segment at one of its edges. They are LCEs because in the case of other examples constructed in the same way (with two perpendicular lines passing through the origin, if one line intersects the segment in a point other than the edges for AB, the other line certainly does not intersect AB) (Fig. 4c).

Regarding statement 2 ("If two lines belong to the family y=mx and both intersect the segment AB, then they are not perpendicular to each other"), one LCE is possible (Fig. 5a). Its characteristic is the fact that the two lines are the ones that intersect the segment in its extreme points.

We can consider 5a an LCE because all the other pairs of lines intersecting the segment and belonging to the family y=mx (Fig. 5b) form an angle that is smaller than the one in Fig. 5a, which means that the two lines are not perpendicular. There is also an LCE for statement 3, which is described in Cusi and Olsher (2019), but we do not present it here because it did not appear in the students' work in this study.

Methodology of Analysis

To answer our research questions, we used qualitative methods to analyze the role that the expert in our teaching experiment played in the classroom discussion. We coded the classroom discussion to focus on how it was structured by the expert. In particular, we investigated how the expert planned her interventions, on one hand, to foster students' reflections on their use of examples and on the characteristics of their argumentations, and on the other, to help students develop an effective and aware use of LCEs as means of supporting argumentation.

The classroom discussion was coded using the analytical tools introduced above, from three main perspectives: (a) the ways in which the expert scaffolded awareness of the potential role of examples and of their methods of construction; (b) the ways in which the expert organized the main phases of the classroom discussion to foster reflection on the characteristics of the constructed argumentations (consisting of both examples and written texts) and on the individual components of argumentative texts, as per Toulmin's model; and (c) the nuances of the roles played by the expert, in particular at the metacognitive level, with reference to the M_{AE}AB analytical

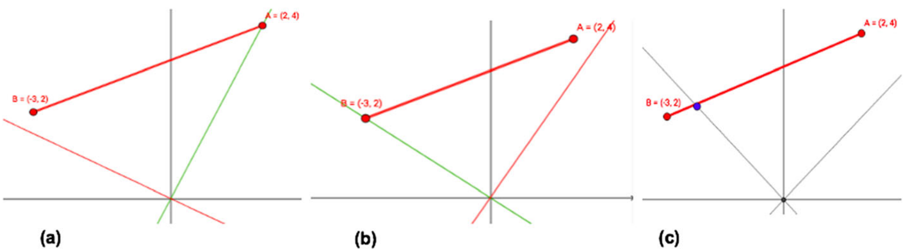


Fig. 4 LCEs (**a**, **b**) and a non-LCE (**c**) for statement 1

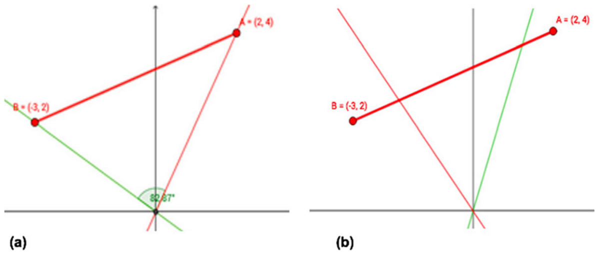


Fig. 5 LCE (**a**) and non-LCE (**b**) for statement 2

framework; the expert played these roles consciously. We used the coding to identify and categorize key aspects of the teacher's guidance of the students.

Results

Our analysis revealed six key aspects of the teacher's guidance during the classroom discussion: (a) the logical structure of the statement being analyzed; (b) the structure and characteristics of the constructed examples, in light of the structure of the statement; (c) the structure and characteristics of the constructed argumentative texts, in light of the structure of the statement; (d) comparison between examples and the corresponding argumentative texts; (e) characteristics of examples that make them effective tools in supporting the construction of a complete argumentative text (or fail to do so); and (f) the design of new effective examples for the construction of complete argumentative texts.

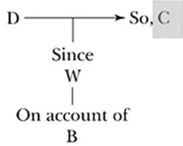
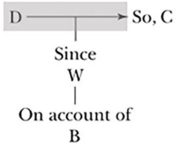
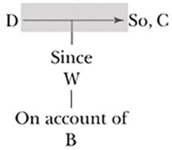
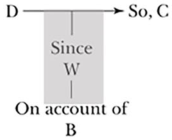
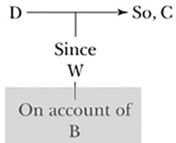
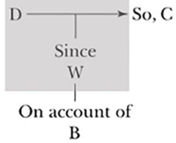
These key aspects correspond to key moments in the outline of the classroom discussion. The aspects in question concern the aim of fostering the development of awareness about the role of the constructed examples (perspective a, summarized in the first column of Table 2), and the characteristics of the examples and their corresponding argumentative texts (perspective b, summarized in the second column of Table 2). For each key moment, the roles played by the expert are described (perspective c, summarized in the third column of Table 2), together with her aims in relation to perspectives a and b.

Key Moment 1: Logical Structure of the Statement Under Analysis

Key moment 1 represents the process of identification of the logical properties involved in the formulation of the statement. It is therefore focused on the claim, with reference to Toulmin's model. According to this structure, students can finalize the claim as true or false.

Students found the last task (task 3) difficult, and only eight (of 22 students) submitted answers to this task. Of the eight students who worked the task, three answered that the statement was true. Figure 6 shows the three examples submitted by these students. It is clear that the pairs of lines proposed in these examples do not satisfy the statement because they do not satisfy the property (a) (they do not both belong to the family).

Table 2 Results of our analysis according to the three adopted perspectives

Key aspects on which the discussion focused	Key elements according to the Toulmin model	Key roles of the expert
(1) Logical structure of the statement under analysis	Focus on the claim 	Guidance in fostering a harmonized balance between syntactic and semantic aspects
(2) Structure and characteristics of the constructed examples, in light of the structure of the statement	Focus on the data in relation to the claim	Reflective guidance and activation of metacognitive acts
		
(3) Structure and characteristics of the constructed argumentative texts, in light of the structure of the statement	Focus on the way in which the connection between data and claim is explained 	Activating metacognitive acts and reflective attitudes
(4) Comparison between examples and corresponding argumentative texts	Focus on how to construct and finalize a warrant that is coherent with the chosen data 	Activating metacognitive acts and guidance in fostering a harmonized balance between syntactic and semantic aspects
(5) Characteristics of examples that make them effective tools in supporting the construction of a complete argumentative text (or fail to do so)	Focus on the backings 	Reflective guidance
(6) Design of new effective examples for the construction of complete argumentative texts.	Focus on the data as coherently constructed from the warrant 	Guidance in fostering a harmonized balance between semantic and syntactic aspects, and reflective guidance

Key moment 1 occurred at the beginning of the classroom discussion, when the expert made students reflect on the examples submitted by the students who answered that the statement was true (Fig. 6). In the following excerpt, the expert (R), after having examined the examples in Fig. 6 and found that they do not satisfy the statement, guided the students in the identification of the three properties (a, b, c) of the pairs of lines that satisfy the statement.

4) R: Which features must the two lines have to satisfy the statement?
5) S1: The coefficients [referring to the slope] should be anti-reciprocal [one slope is the opposite of the reciprocal of the other slope]
6) R: They [the lines] should be perpendicular, which corresponds to the condition you mentioned. Then?
7) S1: They should belong to the family.

In the next part of the discussion, the students, with some difficulties, identified the third property, stating that the two lines should also intersect the segment. The expert went on to schematize the statement to enable more efficient work and communication about it. Guided by the students, she summarized the three properties on the board: (a) they both belong to the family of lines; (b) they are perpendicular to each other; and (c) they both intersect the segment.

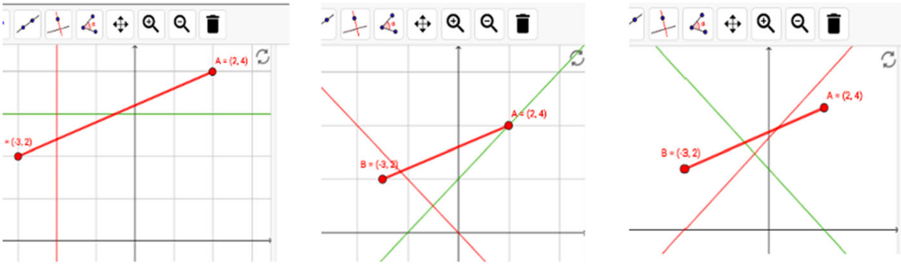
The main role R played was that of a guide in fostering a harmonized balance between the syntactic and semantic aspects. Her aim was to make students notice and attend to the logical structure of the statement, as a prerequisite to being able to systematically assess the characteristics of the examples submitted by their classmates.

Key Moment 2: Structure and Characteristics of the Constructed Examples in Light of the Structure of the Statement

This key moment represents the phase in which the expert, referring to the logical structure of the statement, directed students' reflections on the examples they constructed to make them evaluate the effectiveness of these examples in supporting the claim about the truthfulness/falseness of the statement. At this stage, the focus was not on the argumentative texts produced by the students, but on the data in relation to the claim.

The key moment occurred immediately after key moment 1 in the classroom discussion, when R and the students analyzed the three examples in Fig. 6 in light of the logical conditions they had just enunciated. S1 observed that the three examples did not satisfy property (a) because one or both lines did not belong to the family. R relaunched this observation to make the students analyze also the other properties.

12. R: S1 said that these three examples propose lines that do not belong to the family. What do you think? Do you agree?
13. (In chorus): Yes
14. R: Do the lines in the screenshots satisfy the other conditions?
15. S2: Not all of them
16. R: Explain



(a) (b) (c)

Fig. 6 Examples submitted by the three students who answered that the statement was true

The students commented on the properties that each pair of lines in the three examples did or did not satisfy. Next, R summarized their main observations:

22) R: So, they are perpendicular to each other and they were drawn to intersect the segment AB. But…
23) (In chorus): They do not belong to the family.
24) R: Who sent this screenshot or a similar one?
25) S3: I made a similar drawing, in which the lines don't pass through the origin. I didn't realize that that equation represented a family of lines going through the origin ...

At the beginning of the excerpt (line 12), R played the role of reflective guide, reformulating S1's observation to make other students reflect about his assessment of the examples on display. Next, she guided students in recalling the characteristics of the examples to help them analyze the examples according to the logical aspects noted earlier, and assess their effectiveness in light of these aspects.

After establishing which characteristics were not met in these examples, R turned to the students who submitted these answers (line 24), to make them reflect on their examples. Here, the role of the expert was that of an activator of metacognitive acts, because her aim was to make the authors of the incorrect answers reflect on their misinterpretation of the examples. As a result, S3 (line 25) stated his renewed awareness of the problematic aspects of the example he chose.

Key Moment 3: Structure and Characteristics of the Constructed Argumentative Texts, in Light of the Structure of the Statement

This key moment refers to the phase of the discussion in which the expert guided the students in observing how their written arguments were structured, with reference to the logical aspects previously discussed. It had to do with the way in which the connection between data and claim is explained. The reflections fostered in key moments 2 and 3 formed the necessary basis on which a reflection about the coherence between the chosen examples and the corresponding written argumentations could be developed (the focus of key moment 4).

Key moment 3 occurred in the part of the discussion in which R started focusing on the five answers stating that the statement was false. Although all five answers were discussed, because of space limitations, in this paper, we focus only on three of them (a, b, and c in Table 3), which contained an LCE for statement 1. The written arguments submitted by the first two students (a and b in Table 3) were not characterized by the same logical structure of the chosen examples, that is, $(A \wedge B) \rightarrow \neg C$. Indeed, the structure of the proposed argumentations evoked statement 3: $(B \wedge C) \rightarrow \neg A$. Moreover, the written argument of the third student who proposed an LCE (c in Table 3) was not in agreement with the chosen example because it can be interpreted as a reformulation of the statement: the two real numbers that are introduced in the arguments correspond to the slope of the two lines, which should belong to the set $]-\infty, -2/3] \cup [2, +\infty[$ to satisfy the properties A and C and should be antireciprocal to also satisfy property B.

In the following excerpt, after having displayed answers a and b in Table 2, R made students reflect on the common structure of the two written arguments.

40) 40) R: I'd look at the first two [answers] together. Why do you think I suggest looking at the first two together?
41) 41) S5: Because in all of them it's written that we would need the "known term."
42) 42) R: In all of them it's written that to make them perpendicular, the "known term would be needed." What does this sentence mean? How can we interpret it?
43) 43) S6: That they don't pass through the origin.
44) 44) S7: That they cannot pass through the center.
45) 45) R: Do you mean the center of the family?
46) 46) S7: Yes.
47) 47) R: At least one of them should not pass through the center of the family, which is the origin. At least one of the two does not pass through the origin. What do they mean by "we need the known term?"
48) 48) S7: That, therefore, the statement would no longer be true because we would need the known term.

From the beginning of the excerpt, the focus of this discussion is at the meta-level, as R plays the role of activator of metacognitive acts (line 40), fostering students' reflections

Table 3 Examples and the written arguments submitted by the students who answered "no, the statement is false"

Submitted examples	Corresponding written arguments
(a) The student submitted an LCE for statement 1, similar to those in Fig. 4.	No, because the lines perpendicular to the intersections of the vertices do not intersect segment AB. To intersect AB, the point of intersection of the perpendiculars should be over the vertex, so [in the equation] there should be the "known term" [one of the lines should have an equation in the form y=mx+q, with q different from 0].
(b) The student submitted an LCE for statement 1, similar to those in Fig. 4.	No, because to make them perpendicular we should know the "known term."
(c) The student submitted an LCE for statement 1, similar to those in Fig. 4.	No, because two real numbers that are anti-reciprocal and one of them is smaller than -2/3 and the other greater than 2 do not exist.

on the reasons on which the grouping and display of the answers is based. Making students compare these two written arguments, she sought to make them highlight their common structure.

When the students focused on a key sentence used in the two written answers (line 41), that is, "there should be the known term" and "we should know the known term," R immediately played the role of activator of reflective attitudes (line 42), fostering students' interpretation of this key sentence to make them reflect on the characteristics of these arguments. Students correctly interpreted the sentence, stressing that in both written arguments, it was said that if two lines satisfy properties (b) and (c), they cannot also satisfy property (a), that is, they do not belong to the family. In this way, they highlighted the structure underlying both arguments: $(b \wedge c) \rightarrow \neg a$.

Key Moment 4: Comparison Between Examples and Corresponding Argumentative Texts

This key moment occurred during the phase of the discussion in which the expert encouraged students' reflections on the coherence between the examples they chose and the corresponding argumentative texts, in light of the analysis previously developed during key moments 2 and 3. From the point of view of Toulmin's model, the aim was to make students become aware of the importance of coherently connecting the data and the corresponding warrant in producing a complete argumentation. Therefore, the focus was on how to construct and finalize a warrant (the written text) that was coherent with the chosen data.

This key moment occurred in the part of the discussion in which R, after having fostered students' reflections on the structure of the argumentative texts proposed by the students who submitted answers a and b in Table 2 (the focus of key moment 3), asked them to compare the two examples and to reflect on the coherence between the written texts and the examples associated with them. The main roles she played were those of activator of metacognitive acts, because she made students reflect at a meta-level, causing them to investigate whether or not there were effective connections between the structure of the examples and that of the corresponding arguments, and to achieve this objective, the role of guide in fostering a harmonized balance between syntactic and semantic aspects, helping students observe that while the structure of the two written arguments was $(b \wedge c) \rightarrow \neg a$, the structure of the examples was $(a \wedge b) \rightarrow \neg c$.

Key Moment 5: Characteristics of Examples that Make Them Effective Tools When Constructing Argumentations

This key moment focused on reflections on the effectiveness of given examples as tools that support the construction of complete argumentations. Through these reflections, the characteristics of effective examples are made explicit, enabling students to become aware of the criteria that can guide both the identification and the construction of effective examples. For this reason, key moment 5 was aimed at identifying the backing in Toulmin's model.

The following excerpt is taken from the moment of the discussion when the expert enabled students' reflections on the examples proposed in answers a and b (Table 3), asking them to explain the strategy underlying the choice of considering lines that intersect segment AB in its extreme point (A or B).

61) R: Why did they choose to work with the extreme points of the segment? What strategy can be behind this choice? Is it a good strategy to consider a line passing through one extreme point of the segment, as a limit case?
62) S11: To be as close as possible to the other extreme point
63) S12: It is the maximum point of intersection with the segment.
64) S13: I did so.
65) R: Did you give one of these two answers? Explain your reasoning.
66) S13: I thought that, since the perpendicular to the vertex didn't intersect the segment, consequently all the others wouldn't have intersected it because the slope would have gradually increased and the perpendicular would have moved away from the segment.

In the next part of the discussion, R reformulated S13's sentence to make the meanings underlying the choice of this particular LCE explicit, stressing the fact that it encompassed all the other examples. In this excerpt, R played the role of reflective guide, as she encouraged students to make the strategic choice behind the two analyzed examples explicit (line 61), so that the role of the LCE would become part of the shared understanding of the entire class. In line 65, she played this role again, asking S13 to explain his choice to make the other students reflect on the effectiveness of working with LCEs.

Key Moment 6: Designing New Effective Examples for the Construction of Complete Argumentative Texts

This key moment made students display the fact that they internalized the reflections developed in key moment 5, through the collective construction of new examples that have the effective characteristics previously identified. In Toulmin's model, key moment 6 focused on the data as coherently constructed from the warrant, because the reflections developed were about the coherence between (a) the data and (b) the way in which the connection between data and the claim was explained.

This key moment occurred toward the end of the classroom discussion, when during the analysis of answer c (Table 3), students were led to develop a comparison between the example submitted and the corresponding written text, highlighting the incoherence between them by investigating their logical structures. The expert played two main roles. First, she served as a guide in fostering a harmonized balance between semantic and syntactic aspects, as she made students reflect on the fact that the choice of the example should be in agreement with what was written, and when she stimulated students' investigation, asking them to construct another example whose structure was coherent with that of the written text. Next, she also served as a reflective guide, when she asked the students to explain this new construction to enable them to share the meanings underlying the choice of this example and the criteria for the construction of LCEs.

Conclusions

In this article, we deepened the investigation of how classroom discussions can be designed and implemented with the aim of scaffolding an aware and effective construction and use of LCEs to develop complete argumentations. We identified six key moments around which discussions may be structured starting from students' answers. Each key moment was aimed at making students aware of one of the following fundamental key aspects in the use of LCEs as means of argumentation: (a) the logical structure of the statement being analyzed; (b) the connections between this structure and that of the examples constructed by the students; (c) the connection between this structure and the corresponding argumentative texts; (d) the coherence between examples and the corresponding argumentative texts; (e) the reasons that could guide the identification of effective examples to support the development of complete argumentative texts; and (f) the construction of such examples.

Adopting multiple perspectives has strengthened our qualitative analysis. It also enabled us to focus on how each key moment supports students' development of awareness about given aspects related to the construction and use of effective examples (perspective a), and on the corresponding construction of a coherent argumentation, whose components were gradually analyzed and reconstructed during the discussion (perspective b).

We investigated the classroom discussions designed according to this structure by analyzing the expert's role with reference to the M-$_{AE}$AB construct (perspective c), which enabled us to identify the actions and interventions that an expert could deploy to assist students in developing awareness about the key aspects in the use of LCEs as means of argumentation. These actions and interventions conformed to the individual key moment on which the discussion was focused, as shown in the third column of Table 2. In particular, our analysis showed the importance of the expert assuming roles that bring the discussion to a meta-level. Assuming such roles as reflective guide, activator of reflective attitudes and metacognitive acts, guide in fostering a harmonized balance between the syntactic and semantic levels, has promoted students' reflections, enabling them to fill possible gaps in their awareness. Activation of these roles is often combined with the use of certain techniques, such as exposing students' answers to make them become a direct object of reflection, and making students compare and contrast different examples and written arguments.

Our results are consistent with the findings of Arzarello et al. (2011), which have stressed the importance of aiming the teacher's actions toward developing the students' awareness of the meaning of examples at both the theoretic and logical levels. We believe that our study marks a step forward in the investigation started by Arzarello, Ascari and Sabena because we combined the perspective of supporting students building their example spaces with that of making students create and use examples to develop argumentative processes. In this way, our work contributes to the identification of the main characteristics of the classroom context advocated by Mason (2019), in which students are "in the presence of teachers and peers talking about how they perceive examples" and "immersed in a discourse that is explicit about the perceived scope of generality" (Mason, 2019, p. 345).

Identification of the six key moments according to which classroom discussions can be structured, and of the roles that an expert can play at each key moment, enabled us to outline a methodology for designing and implementing classroom discussions that scaffold students' aware and effective use of examples to develop argumentative processes. This methodology, which we call the "six key moments (SKM) approach," offers a model to which teachers can refer for designing and conducting classroom discussions focused on the construction and use of examples as means of argumentation, not only when the focus is on LCEs.

This idea opens up new directions for future research. Our next step is to explore the use of our methodology as a tool for teachers to structure the design and implementation of argumentative processes connected to other uses of examples, for example, to refute universal statements or to support the development of generic arguments. A possible critical point of our study is that during the analyzed discussion we did not capture the students' voices and interpretations in their entirety. Therefore, we will deepen the investigation of the effect of this approach on the students' development of awareness of examples as means of argumentation, focusing also on students who do not participate in classroom discussions.

Funding Open access funding provided by Università degli Studi di Roma La Sapienza within the CRUI-CARE Agreement.

References

Antonini, S., Presmeg, N., Mariotti, M. A., & Zaslavsky, O. (2011). On examples in mathematical thinking and learning. Special issue "Examples in mathematical thinking and learning from an educational perspective". *ZDM Mathematics Education, 43*, 191–194.

Aricha-Metzer, I., & Zaslavsky, O. (2017). The nature of productive and non-productive example-use in proving. *Journal of Mathematical Behavior, 53*, 304–322.

Arzarello, F., Ascari, M., & Sabena, C. (2011). A model for developing students' example space: the key role of the teacher. *ZDM Mathematics Education, 43*, 295–306.

Cusi, A., & Malara, N.A. (2009). The Role of the Teacher in developing Proof Activities by means of Algebraic Language. In M. Tzekaki et Al. (Eds.), *Proceedings of PME 33*, vol. 2 (pp. 361-368). Thessaloniki (Greece).

Cusi, A., & Malara, N.A. (2013). A theoretical construct to analyze the teacher's role during introductory activities to algebraic modelling. In B. Ubuz et al. (Eds.), *Proceedings of Cerme 8* (pp. 3015-3024). Antalya (Turkey).

Cusi, A., & Malara, N. A. (2016). The Intertwining of Theory and Practice: Influences on Ways of Teaching and Teachers' Education. In L. English, & D. Kirshner (Eds.), *Handbook of International Research in Mathematics Education 3rd Edition* (pp. 504–522). Taylor & Francis.

Cusi, A., & Olsher, S. (2019). Task design fostering construction of limit confirming examples as means of argumentation. In U.T. Jankvist, M. Van den Heuvel-Panhuizen & M. Veldhuis (Eds.). (2019). *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics*

Education. Utrecht, the Netherlands: Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.
Cusi, A., Morselli, F., & Sabena, C. (2017). Promoting formative assessment in a connect- ed classroom environment: design and implementation of digital resources. *ZDM Mathematics Education, 49*(5), 755–767.
Cusi, A., Morselli, F., & Sabena, C. (2019). The Use of Polls to Enhance Formative Assessment Processes in Mathematics Classroom Discussions. In G. Aldon & J. Trgalova (Eds.), *Technology in Mathematics Teaching. Selected Papers of the 13th ICTMT Conference* (pp. 7–30). Springer.
Balk, G. D. (1971). Application of heuristic methods to the study of mathematics at school. *Educational Studies in Mathematics, 3*(*2*), 133–146.
Bills, L., & Watson, A. (2008). Editorial introduction. Special issue "The role and use of examples in mathematics education". *Educational Studies in Mathematics, 69*, 77–79.
Bova, A. (2017). The role of the teacher in promoting argumentative interactions in the learning contexts of higher education. In F. Arcidiacono & A. Bova (Eds.), *Interpersonal Argumentation in Educational and Professional Contexts* (pp. 75–93). Springer.
Buchbinder, O., & Zaslavsky, O. (2013). A holistic approach for designing tasks that capture and enhance mathematical understanding of a particular topic: The case of the interplay between examples and proof. In C. Margolinas (Ed.), *Task Design in Mathematics Education. Proceedings of ICMI Study 22*. United Kingdom.
Clement, J. (1991). Non-formal reasoning in experts and in science students: The use of analogies, extreme cases, and physical intuition. *Informal Reasoning and Education, 1*, 345–362.
Collins, A., Brown, J. S., & Newman, S. E. (1989). Cognitive apprenticeship: Teaching the crafts of reading, writing and mathematics! In L. B. Resnick (Ed.), *Knowing, Learning, and Instruction: Essays in Honor of Robert Glaser* (pp. 453–494). Lawrence Erlbaum Associates.
Conner, A., Singletary, L. M., Smith, R. C., Wagner, P. A., & Francisco, R. T. (2014). Teacher support for collective argumentation: A framework for examining how teachers support students' engagement in mathematical activities. *Educational Studies in Mathematics, 86*(3), 401–429.
Dreyfus, T., Nardi, E., & Leikin, R. (2012). Forms of proof and proving in the classroom. In G. Hanna & M. de Villiers (Eds.), *Proof and proving in mathematics education* (pp. 191–214). Springer.
Ellis, A. E., Lockwood, E., Williams, C. C. W., Dogan, M. F., & Knuth, E. (2013). Middle school students' example use in conjecture exploration and justification. In L. R. Van Zoest, J. J. Lo, & J. L. Kratky (Eds.), *Proceedings of the 34th annual meeting of the North American chapter of the psychology of mathematics education* (pp. 135–142). Western Michigan University.
Goldenberg, P., & Mason, J. (2008). Shedding light on and with example spaces. *Educational Studies in Mathematics, 69*, 183–194.
Knipping, C., & Reid, D.A. (2019). Argumentation analysis for early career researchers. In G. Kaiser and N. Presmeg (Eds.), *Compendium for Early Career Researchers in Mathematics Education* (pp. 3-31). ICME-13 Monographs.
Lockwood, E., Ellis, A. B., & Lynch, A. G. (2016). Mathematicians' example-related activity when exploring and proving conjectures. *International Journal of Research in Undergraduate Mathematics Education, 2*(*2*), 165–196.
Lynch, A. G., & Lockwood, E. (2019). A comparison between mathematicians' and students' use of examples for conjecturing and proving. *Journal of Mathematical Behavior, 53*, 323–338.
Mason, J. (2008). Being mathematical with and in front of learners. In B. Jaworski & T. Wood (Eds.), *The mathematics teacher educator as a developing professional* (pp. 31–55). Rotterdam: Sense Publishers.
Mason, J. (2019). Relationships between proof and examples: Comments arising from the papers in this issue. *Journal of Mathematical Behavior, 53*, 339–347.
Mason, J. Drury, H. & Bills, E. (2007). Explorations in the zone of proximal awareness. In J. Watson & K. Beswick (Eds.) *Mathematics: Essential research, essential practice: Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia*, vol. 1 (p. 42-58). Adelaide: MERGA.
Mueller, M., Yankelewitz, D., & Maher, C. (2014). Teachers promoting student mathematical reasoning. *Investigations in Mathematics Learning, 7*(*2*), 1–20.
Olsher, S., Yerushalmy, M., & Chazan, D. (2016). How might the use of technology in formative assessment support changes in mathematics teaching? *For the Learning of Mathematics, 36*(3), 11–18.
Stylianides, A. J., Bieda, K. N., & Morselli, F. (2016). Proof and argumentation in mathematics education research. In A. Gutierrez, G. C. Leder, & P. Boero (Eds.), *The second handbook of research on the psychology of mathematics education* (pp. 315–351). Sense Publishers.
Toulmin, S.E. (2003). *The use of arguments. Updated edition*. Cambridge University press.

Watson, A., & Chick, H. (2011). Qualities of examples in learning and teaching. *ZDM Mathematics Education, 43*, 283–294.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. Routledge.

Zaslavsky, O. (2014). Thinking with and through examples. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Vol. Eds.), *Proceedings of the 38th conference of the international group for the psychology of mathematics education and the 36th conference of the North American chapter of the psychology of mathematics education*, Vol. 1, (pp. 21–34). Vancouver, Canada: PME.

Zaslavsky, O. (2018). Genericity, Conviction, and Conventions: Examples that Prove and Examples that Don't Prove. In Stylianides, A. J., & Harel, G. (Eds.), *Advances in mathematics education research on proof and proving* (pp. 283-298). Cham : Springer.

Zaslavsky, O. (2019). There is more to examples than meets the eye: Thinking with and through mathematical examples in different settings. *Journal of Mathematical Behavior, 53*, 245–255.

Zaslavsky, O., & Knuth, E. (2019). The complex interplay between examples and proving: Where are we and where should we head? *Journal of Mathematical Behavior, 53*, 242–244.